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The Type II Topp Leone Generalized Inverse Rayleigh Distribution

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Abstract

A new three parameters distribution called the type II topp leone generalized inverse Rayleigh (TIITLGIR) distribution is introduced. The reliability analysis of the new model is discussed. Several of its mathematical properties are studied. The maximum likelihood (ML) estimation are derived for TIITLGIR parameters. The importance and flexibility of the TIITLGIR is assessed using one real data set.

Keywords: Type II Topp Leone family, generalized inverse Rayleigh distribution, order statistics, maximum likelihood

1. Introduction

An appropriate comprehensive lifetime model is often of concentration in the analysis of data. Trayer (1964) introduced a distribution in order to model reliability and survival data sets, named inverse Rayleigh distribution. After that, *inverse Rayleigh (IR)* distribution was championed by Voda (1972). He discussed its properties and ML estimator of the scale parameter. Further, Gharraph (1993) provided closed-form expressions for the mean, harmonic mean, geometric mean, mode and the median of this distribution.
Lot of works have been studied in the literature on IR distribution. Gharraph (1993) and Hassan et al. (2010) estimated the parameters using classical and Bayesian estimation methods. Beta inverse

Rayleigh distribution was studied by Leão et al. (2013), Ahmed et al. (2014) introduced a generalization of the inverse Rayleigh distribution, modified inverse Rayleigh distribution studied by Khan (2014), Rehman and Dar (2015) studied exponentiated inverse Rayleigh distribution, Khan and King (2015) studied transmuted modified inverse Rayleigh distribution, Haq (2016a) introduced transmuted exponentiated inverse Rayleigh distribution, Kumaraswamy exponentiated inverse Rayleigh distribution was studied by Haq (2016b).

The probability density function (pdf) and *cumulative distribution function (cdf)* of GIR distribution are given

$$g(x;\alpha,\gamma)=\frac{2\gamma\alpha^2}{x^3}e^{-\gamma\left(\frac{\alpha}{x}\right)^2} \quad , \quad x,\alpha,\gamma>0, \tag{1}$$

and

$$G(x;\alpha,\gamma)=e^{-\gamma\left(\frac{\alpha}{x}\right)^2} \quad , \quad x,\alpha,\beta,\gamma>0. \tag{2}$$

Recently, Elgarhy et al. (2018) studied *Type II top leone generated (TIITL-G) family of distributions*. The cdf of *TIITL -G* is given by:

$$F(x:\theta,\xi)=1-[1-[G(x;\xi)]^2]^{\theta} \quad ,x\in R,\theta>0. \tag{3}$$

The corresponding pdf to (3) is given by

$$f(x:\theta,\xi)=2\theta g(x;\xi)G(x;\xi)[1-[G(x;\xi)]^2]^{\theta-1} \quad , \tag{4}$$

where $g(x:\xi)$ considers a pdf of baseline distribution.

In this paper, we define a new lifetime model called the *TIITLGIR* distribution. The cdf of *TIITLGIR* distribution with set of parameters $\varphi=(\theta,\alpha,\gamma)$ is obtained by substituting (2) in (3) as

$$F(x;\varphi)=1-\left[1-e^{-2\gamma\left(\frac{\alpha}{x}\right)^2}\right]^{\theta} \qquad x,\theta,\alpha,\gamma>0. \tag{5}$$

The corresponding pdf to (5) is given by inserting (1) and (2) in (4) as

$$f(x;\,\varphi) = 4\theta\gamma\alpha^2 x^{-3} e^{-2\gamma\left(\frac{\alpha}{x}\right)^2}\left[1 - e^{-2\gamma\left(\frac{\alpha}{x}\right)^2}\right]^{\theta-1}, \qquad x, \theta, \alpha, \gamma > 0. \tag{6}$$

Here α and γ are scale parameters and θ is a shape parameters. Note that when $\gamma = 1$ we get *TIITLGIR distribution.*
Figure 1 displays some plots of the *TIITLGIR* pdf for some different values of parameters.

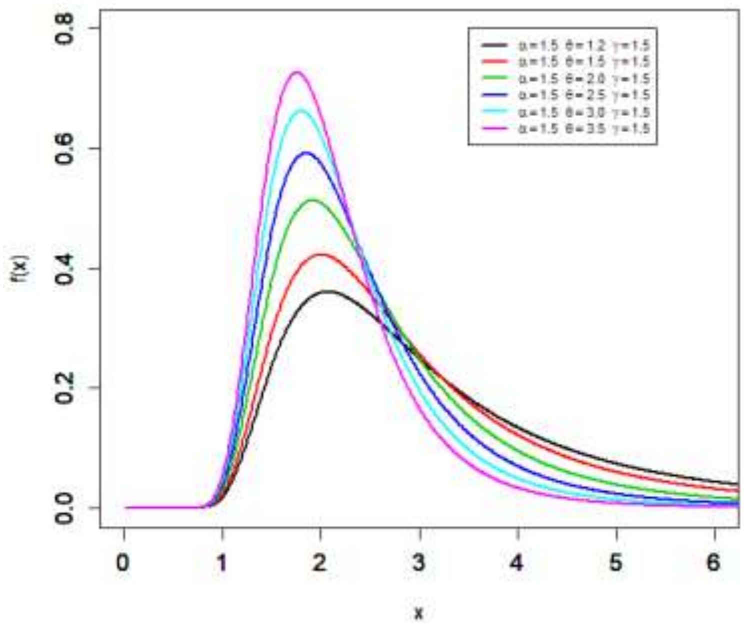

Figure 1: Plots of the pdf of the *TIITLGIR* distribution for different values of parameters

From Figure 1, we conclude that pdf of *TIITLGIR* distribution can be uni-model, symmetric and right skewed.
We aim that it will attract wider applications in engineering, medicine and other areas of research. This paper is organized as follows. In Section 2, reliability analysis is discussed. Section 3 studies the linear representation of the pdf for *TIITLGIR* distribution. Statistical properties is studied in Section 4. The maximum likelihood method of estimation is applied to calculate the estimates of the *TIITLGIR* parameters in Section 5. The analyses of one real data set is employed in Section 6. Summary is given in Section 7.

2. The reliability analysis

The survival function (sf), hazard rate function (hrf), reversed hrf and cumulative hrf of X are given, respectively, as follows:

$$R(x;\ \varphi) = \left[1 - e^{-2\gamma\left(\frac{\alpha}{x}\right)^2}\right]^{\theta} \qquad x, \theta, \alpha, \gamma > 0. \quad ,$$

$$h(x;\ \varphi) = \frac{4\theta\gamma\alpha^2 x^{-3} e^{-2\gamma\left(\frac{\alpha}{x}\right)^2}}{1 - e^{-2\gamma\left(\frac{\alpha}{x}\right)^2}} \quad ,$$

$$\tau(x;\ \varphi) = \frac{4\theta\gamma\alpha^2 x^{-3} e^{-2\gamma\left(\frac{\alpha}{x}\right)^2}\left[1 - e^{-2\gamma\left(\frac{\alpha}{x}\right)^2}\right]^{\theta-1}}{1 - \left[1 - e^{-2\gamma\left(\frac{\alpha}{x}\right)^2}\right]^{\theta}},$$

and

$$H(x;\ \varphi) = -\ln\left[1 - e^{-2\gamma\left(\frac{\alpha}{x}\right)^2}\right]^{\theta}.$$

Figure 2 displays some plots of the *TIITLGIR* hrf for some different values of parameters.

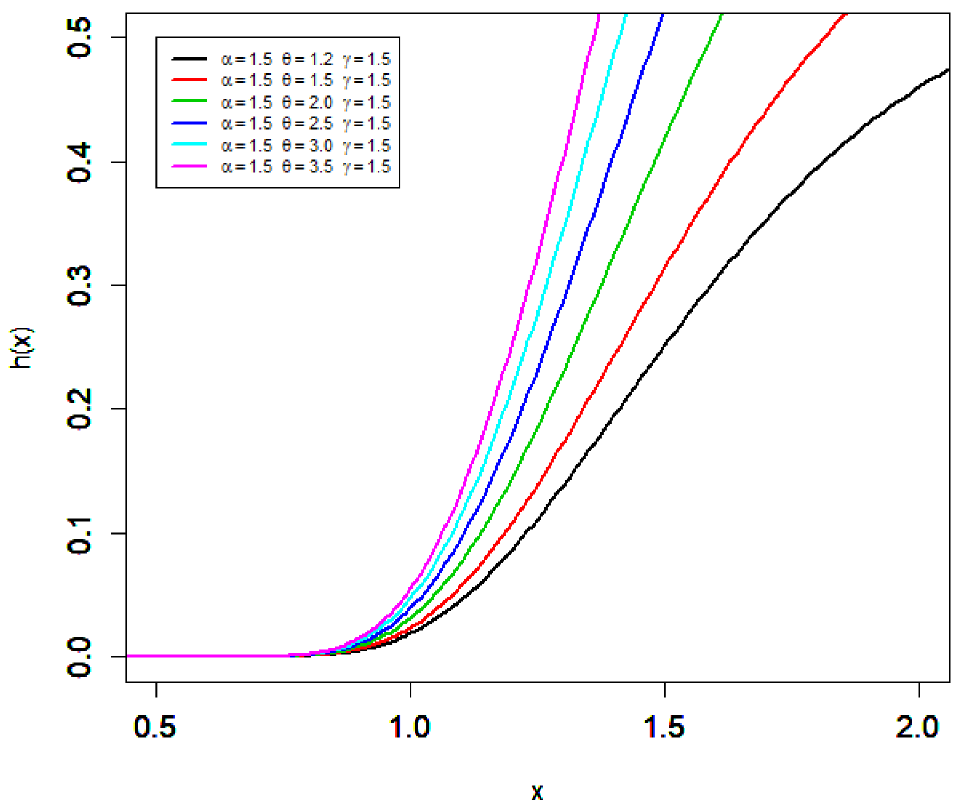

Figure 2: Plots of the hrf of the *TIITLGIR* distribution for different values of parameters

From Figure 2, we conclude that the hrf of *TIITLGIR* distribution can be J-shaped and unimodal.

3. Useful expansion

In this section we present the two useful expansions of pdf and cdf for *TIITLGIR* distribution. Now, consider the following well-known binomial expansions (for 0 < a< 1),

$$(1-a)^n = \sum_{k=0}^{\infty} (-1)^k \binom{a}{k} a^k. \qquad (7)$$

Thus, using (7), the following term in (6) can be expressed as

$$\left(1 - e^{-2\gamma\left(\frac{\alpha}{x_i}\right)^2}\right)^{\theta-1} = \sum_{i=0}^{\infty} (-1)^i \binom{\theta-1}{i} e^{-2\gamma\left(\frac{\alpha}{x_i}\right)^2}. \qquad (8)$$

Therefore, from (8) and (6) the pdf of *TIITLGIR* can be write as

$$f(x) = 2\alpha^2 \sum_{i=0}^{\infty} W_i x^{-\beta-1} e^{-2(i+1)\gamma\left(\frac{\alpha}{x_i}\right)^2} \quad , \quad (9)$$

Where $w_i = 2(-1)^i \binom{\theta-1}{i} \theta\gamma$.

By using binomial theory for $[F(x)]^h$, where h is an integer, leads to :

Since,

$$[F(x)]^h = \left[1 - \left(1 - e^{-2\gamma\left(\frac{\alpha}{x_i}\right)^2}\right)^{\theta}\right]^h.$$

Then,

$$[F(x)]^h = \sum_{k=0}^{\infty} W_k e^{-2k\gamma\left(\frac{\alpha}{x_i}\right)^2}. \qquad (10)$$

Where, $w_k = \sum_{j=0}^{h} (-1)^{j+k} \binom{h}{j} \binom{\theta j}{k}$.

4. Statistical properties

In this section some statistical properties of the *TIITLGIR* distribution are obtained.

4.1 Quantile function

The quantile function, say $Q(u) = F^{-1}(u)$ of X is given by

$$Q(u) = \alpha \left(\frac{-1}{\gamma} ln \sqrt{1 - (1-u)^{\frac{1}{\theta}}} \right)^{\frac{-1}{2}}, \qquad (11)$$

where, u is considered as a uniform random variable on the unit interval $(0,1)$.

In particuler, the median can be derived from (11) by setting $u = 0.5$. That is, the median (M) is given by

$$M = \alpha\left(\frac{-1}{\gamma} ln\sqrt{1-(0.5)^{\frac{1}{\theta}}}\right)^{\frac{-1}{2}}.$$

4.2 Moments

If X has the pdf (9), then its rth moment is given from the following relation

$$\mu_r^{'} = E(X^r) = \int_{-\infty}^{\infty} x^r f(x;\varphi)dx. \qquad (12)$$

Substituting (9) into (12) yields

$$\mu_r^{\backslash} = 2\alpha^2 \sum_{i=0}^{\infty} W_i \int_0^{\infty} x^{r-\beta-1} e^{-2(i+1)\gamma\left(\frac{\alpha}{x_i}\right)^2} dx.$$

Let $y = \left(\frac{\alpha}{x_i}\right)^2$ then,

$$\mu_r^{\backslash} = \alpha^r \sum_{i=0}^{\infty} W_i \int_0^{\infty} y e^{-2(i+1)\gamma y} dy.$$

Then, $\mu_r^{'}$ becomes

$$\mu_r^{\backslash} = \alpha^r \sum_{i=0}^{\infty} W_i \frac{\Gamma(1-\frac{r}{2})}{(2(i+1)\gamma)^{1-\frac{r}{2}}} \quad , \quad \frac{r}{2} < 1.$$

The moment generating function of *TIITLGIR* distribution is given by

$$M_X(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!} E(X^r) = \sum_{r,i=0}^{\infty} \frac{t^r}{r!} \frac{w_i \Gamma(1-\frac{r}{2})}{[2(i+1)\gamma]^{1-\frac{r}{2}}} \ , \ r/2 < 1.$$

4.3 Probability Weighted Moments

The PMWs can be calculated from the following

$$\tau_{r,s} = E[X^r \mathrm{F}(x)^s] = \int_{-\infty}^{\infty} x^r \mathrm{f}(x)(\mathrm{F}(x))^s dx. \qquad (13)$$

By inserting (9) and (6) into (10), replacing h with s, leads to:

$$\tau_{r,s} = 2\alpha^2 \sum_{i,k=0}^{\infty} W_i W_k \int_0^{\infty} x^{r-3} e^{-2(i+k+1)\gamma\left(\frac{\alpha}{x_i}\right)^2} dx.$$

Hence, the PWM of *TIITLGIW* distribution takes the following form

$$\tau_{r,s} = \alpha^r \sum_{i,k=0}^{\infty} W_i W_k \frac{\Gamma(1-\frac{r}{2})}{(2(i+k+1)\gamma)^{1-\frac{r}{2}}} \ , \qquad \frac{r}{2} < 1 \ .$$

4.4 Order statistics

Let $X_{1:n} < X_{2:n} < \ldots < X_{n:n}$ be the order statistics of a random sample of size n following the *TIITLGIR* distribution, with parameters α , θ and γ , then, the pdf of the kth order statistic, can be written as follows

$$f_{k:n}(x) = \frac{1}{B(k, n-k+1)} f(x) F(x)^{k-1} (1 - F(x))^{n-k} \ , \tag{14}$$

where, $B(.,.)$ is the beta function. By substituting (5) and (6) in (14), then

$$f_{k:n}(x) = \frac{4\theta\gamma\alpha^2}{B(k, n-k-1)} x^{-3} e^{-2\gamma\left(\frac{\alpha}{x}\right)^2} \left(1 - \left[1 - e^{-2\gamma\left(\frac{\alpha}{x}\right)^2}\right]^{\theta}\right)^{k-1} \left(1 - e^{-2\gamma\left(\frac{\alpha}{x}\right)^2}\right)^{\theta(n-k+1)-1} . \tag{15}$$

When we put $k=1$ in (15) we get the pdf of the smallest order statistics as

$$f_{1:n}(x) = 4n\theta\gamma\alpha^2 x^{-3} e^{-2\gamma\left(\frac{\alpha}{x}\right)^2} \left(1 - e^{-2\gamma\left(\frac{\alpha}{x}\right)^2}\right)^{\theta n-1} ,$$

when we put $k=n$ in (15) we get the pdf of the largest order statistics as

$$f_{k:n}(x) = 4n\theta\gamma\alpha^2 x^{-3} e^{-2\gamma\left(\frac{\alpha}{x}\right)^2} \left(1 - \left[1 - e^{-2\gamma\left(\frac{\alpha}{x}\right)^2}\right]^{\theta}\right)^{n-1} \left(1 - e^{-2\gamma\left(\frac{\alpha}{x}\right)^2}\right)^{\theta-1} .$$

5. Maximum likelihood estimation

The maximum likelihood estimates of the unknown parameters for the *TIITLGIR* distribution are determined based on complete samples. Let $X_1, \ldots, X_n$ be observed values from the *TIITLGIR* distribution with set of parameters $\varphi = (\alpha, \theta, \gamma)^T$. The total log-likelihood function for the vector of parameters φ can be expressed as

$$\ln L(\varphi) = n \ln 4\theta + n\, ln\gamma + 2n\, ln\, \alpha - 3\sum_{i=1}^{n} \ln x_i - 2\gamma \sum_{i=1}^{n} \left(\frac{\alpha}{x_i}\right)^2 + (\theta - 1)\sum_{i=1}^{n} \ln\left[1 - e^{-2\gamma\left(\frac{\alpha}{x_i}\right)^2}\right].$$

The elements of the score function $U(\varphi) = (U_\alpha, U_\theta, U_\gamma)$ are given by

$$U_\alpha = \frac{2n}{\alpha} - 4\gamma\alpha\sum_{i=1}^{n} x_i^{-\beta} + 4\gamma\alpha(\theta-1)\sum_{i=1}^{n}\frac{x_i^{-2}e^{-2\gamma\left(\frac{\alpha}{x_i}\right)^2}}{1-e^{-2\gamma\left(\frac{\alpha}{x_i}\right)^2}} ,$$

$$U_\theta = \frac{n}{\theta} + \sum_{i=1}^{n}\ln\left[1-e^{-2\gamma\left(\frac{\alpha}{x_i}\right)^2}\right] ,$$

and

$$U_\gamma = \frac{n}{\gamma} - 2\sum_{i=1}^{n}\left(\frac{\alpha}{x_i}\right)^2 + 2(\theta-1)\sum_{i=1}^{n}\frac{\left(\frac{\alpha}{x_i}\right)^2 e^{-2\gamma\left(\frac{\alpha}{x_i}\right)^2}}{1-e^{-2\gamma\left(\frac{\alpha}{x_i}\right)^2}} .$$

Then the maximum likelihood estimators of the parameters α, θ and γ are obtained by setting U_α , U_θ and U_γ to be zero and solving them. Clearly, it is difficult to solve them, therefore applying the Newton-Raphson's iteration method and using the computer package such as Maple or R or other software.

6. Application

In this section, we provide an application to a real data set to assess the flexibility of the *TIITLGIR* model. We compare the fits of the *TIITLGIR* distribution with the *GIR* distribution.

The data set (Gross and Clark, 1975) on the relief times of twenty patients receiving an analgesic is 1.1, 1.4, 1.3, 1.7, 1.9, 1.8, 1.6, 2.2, 1.7, 2.7, 4.1, 1.8, 1.5, 1.2, 1.4, 3, 1.7, 2.3, 1.6, 2.

The ML estimates along with their standard errors (SEs) of the model parameters are provided in Tables 2 and 3. In the same tables, the analytical measures including minus *double log-likelihood* (-2log L), *Kolmogorov - smirnov* (k-s), and *p-value* are presented.

Tables 3 list the MLEs of the model parameters and their corresponding standard whereas errors and the values of -2LogL, k-s and p-value.

Table 3: MLEs and their SEs (in parentheses), -2LogL, k-s and p-value for the data set.

Model	MLE and SE			-2LogL	k-s	p-value
TIITLGIR(α, θ, γ)	1.112 (333800)	3.61 (1.334)	2.204 (1323000)	31.736	0.12642	0.90655
GIR(α, γ)	1.953 (244100)	0.724 (181000)	-	42.365	0.25659	0.14358

Table 3 compares the fits of the *TIITLGIR* distribution with the *GIR* distribution. The tables 3 show that the *TIITLGIR* model has lower values for -2LogL and k-s and biggest p-value than GIR distribution. So, it could be chosen as the best model. The fitted pdf and estimated cdf plots for the fitted models are displayed in Figure 3.

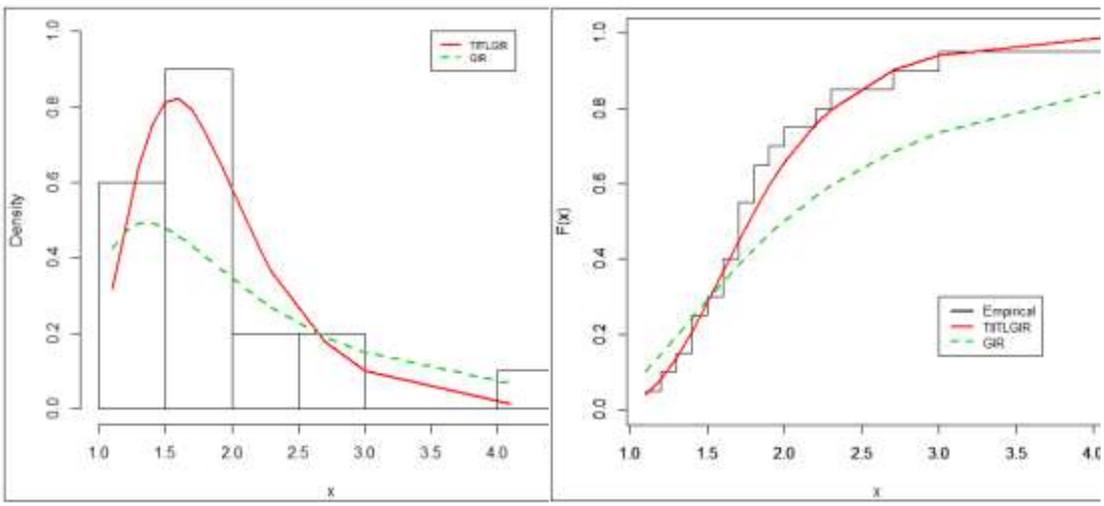

Figure 3: The empirical pdf and empirical cdf of the fitted models

7. Summary

In this paper, we propose a new three-parameter distribution named the *TIITLGIR* distribution. The pdf of *TIITLGIR* can be expressed as a linear mixture of GIR densities. We calculate explicit expressions for some of its statistical properties. We study maximum likelihood estimation. The proposed model provides better fits than some other competitive models using a real data set.

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