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Minimal Repair Warranty Cost Model and Optimal Periodic Replacement Policy for Repairable Products Based on Generalized Exponential Distribution

Iman S. Mabrouk

Departments of Mathematics, Insurance, and Applied Statistics
Helwan University, Helwan, Egypt

Belques A. Mahran

Departments of Mathematics, Insurance, and Applied Statistics
Helwan University, Helwan, Egypt

Abstract

Warranty is an essential element of marketing products. Determination of warranty costs helps manufacturers setting an appropriate warranty policy. In this paper we develop warranty cost models for new repairable products when failure times follow generalized exponential distribution under two types of minimal repair warranty, the free and the pro-rate warranty policy. The performance of the minimal repair warranty models is evaluated through an extensive simulation study which gives evidence of highly accurate simulation models. This article also derives the optimal periodic replacement policy under free repair warranty and investigates it through a numerical example.

Keywords: Generalized exponential distribution, minimal repair model, simulation model, preventive maintenance, and warranty cost analysis

1. Introduction

Warranty is a contract between a buyer and a manufacturer that becomes effective on the sale date of an item. Free and pro-rata are common types of warranty. In free warranty policy, if the
item fails before the end of warranty period it is replaced or repaired at no cost to the customer. For the pro-rata warranty, if an item fails before the end of warranty period it is replaced or repaired at a cost that depends on the age of the item at the time of failure.
Thomas and Rao [24] mentioned that the motives to establish cost models for warranty are:
(1) to determine the parameters that influence the relevant cost for producing products, and (2) to predict the amount of money that should be set aside in reserve in order to meet forthcoming expenditures resulting from warranty claims. In addition, the warranty cost per unit sale is important in the context of pricing the product, see Murthy and Djamaludin [17]. The sale price must exceed the manufacturing cost plus the warranty cost or the manufacturer incurs a loss. Generally, warranty cost per item decreases as reliability increases.

Warranty cost analysis depends on both the underlying distribution for product's lifetime and warranty terms. Gupta and Kundu [14] presented the two parameters generalized exponential distribution with density function

$$f(t) = \lambda\, \alpha(1 - e^{-\lambda t})^{\alpha-1} e^{-\lambda t}\,, \qquad t > 0،\ \lambda > 0،\ \alpha > 0, \qquad (1.1)$$

where λ is the scale parameter and α is the shape parameter.

The cumulative function and the hazard function, respectively, are

$$F(t) = (1 - e^{-\lambda t})^{\alpha}\,, \qquad t > 0،\ \lambda > 0،\ \alpha > 0 \qquad (1.2)$$

$$h(t) = \frac{\lambda\alpha\,(1-e^{-\lambda t})^{\alpha-1} e^{-\lambda t}}{1-(1-e^{-\lambda t})^{\alpha}}\,, \qquad t > 0،\ \lambda > 0،\ \alpha > 0 \qquad (1.3)$$

The generalized exponential distribution has the following advantages:

1 - The probability density distribution function varies greatly in its behavior according to the different values of the shape parameter as shown by Figure 1.1.
2- It is also clear that generalized exponential distribution is effective in the case of long right-tailed data because it has a longer tail length than other distributions, which makes it give better predictions Gupta and Kundu [15]
3- The behavior of the hazard function is increased or decreased according to the values of the shape parameter.

4- One of the main advantages of the generalized exponential distribution is that it has a nice cumulative function and reliability function which makes it a suitable distribution for use in the case of lifetime data.

Minimal repair is defined as a repair that does not affect product's failure rate. Minimal repair is generally suitable for multiple-components products in which the failed component does not affect the other components. Barlow and Hunter's study [4] was the first study to propose the concept of minimal repair. They considered a policy such that a product is replaced at regular intervals and it is minimally repaired if a failure occurs between replacement intervals. Nguyen, and Murthy [18] presented a general model for estimating warranty cost for repairable products when the manufacturer should pay all repairs costs in period T. They estimated the expected total warranty cost and prediction interval for a fixed lot size of sales. In addition, they estimated the expected number of units returned for repair and the expected warranty costs incurred in any time interval during the product life cycle assuming sales occur continuously.

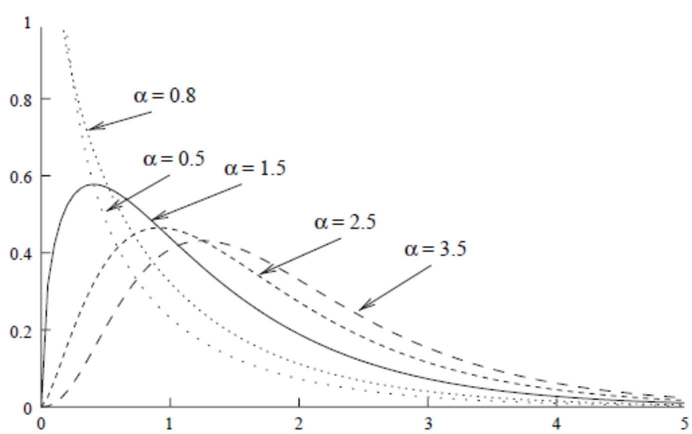

Figure 1.1: The probability density function of the generalized exponential distribution at scale parameter λ=1 and different values of the shape parameter α.

Thomas [23] developed a model for estimating the warranty reserve fund for the manufacturer where the warranty policy is pro-rata in a closed form for different cases for the product failure time such as exponential, uniform, and gamma. Jain and Maheshwari [16] proposed a hybrid warranty model for the renewing pro-rata warranty, in which the failure rate of units, cost of preventive maintenance and cost of replacement are assumed to be constant. The expected total discounted maintenance cost has been derived in an explicit form for different life time distributions, namely the exponential, Rayleigh and Weibull distributions. They also determined the optimal number and optimal period of preventive maintenance after the expiry of the warranty. Attia et. al. [3] developed a warranty cost model where the failure time for repairable products follows a log- logistic distribution. Caglar [8] reviewed the warranty cost models with an emphasis on the failure analysis of used vehicles. Expected warranty costs were calculated by taking into account age, usage. Failure intensities and characteristics are identified in order to propose a policy that highlights the trade-off between the cost and warranty length.

Ambekar and Jagtap [2] considered models that may be used to obtain estimates for the expected cost of warranty per unit sold and the expected cost per unit time over the product life cycle for products sold with free-replacement and pro rata warranties. Chen et. al. [9] considered the effect of three phases; burn-in, free replacement warranty and pro-rata warranty for repairable products under the minimal and catastrophic failure. Moreover, they developed a framework for studying the effects of various parameters, such as the burn-in time, warranty period, and distribution function on warranty costs.

For products subject to periodic replacement it is important to derive the optimal periodic replacement policy such that the long-run expected cost rate is minimized. This article also presents the optimal periodic replacement policy when failure time follows generalized exponential distribution. Moreover, it examines the effects of a free repair warranty policy on the optimal periodic replacement policy.

Replacement policy and optimal periodic replacement have received much attention in the statistical literature. Barlow and Hunter [4] have introduced a policy of replacement period. They considered a policy such that product is replaced at regular intervals and it is minimally repaired if a failure occurs between replacement intervals. Since then, the policy of replacement period has received a lot of interest under different conditions, such as the study of free-replacement warranty and pro-rata warranty studied by Blischke and Scheuer [5]. Both the buyer's and seller's points of view are considered. The basis of the analysis is comparison between warranted and unwarranted items with regard to long-run cost to the buyer and long-run profit to the seller. Boland and Proschan [7] considered expensive units in case of failure or malfunction and stated that the repair of these units with the policy of minimal repair is better than replacing them. Boland [6] introduced the replacement period policy for a complex system where the system is replaced at multiples of a certain time period while the minimal repair policy is used if there is a malfunction between replacement periods assuming that the cost of the minimal repair is a non-increasing function with the lifetime of the product. Nguyen and Murthy [19] said that the unit sold must be replaced by a new unit and it should be repaired if it has failed between replacement periods. The replacement cost is not paid by the buyer and therefore the buyer at least ensures that the product is working throughout the warranty period.
Ozekici [20] presented the optimal duration model for replacement of a multi-component system. Examples of these systems are jet engines and computers, which consist of hundreds of key components that require maintenance. The effect of the

dependence between the components on the optimum duration of replacement was studied. Sheu [22] introduced a policy of replacement period for a multi-unit system with a specific multivariate distribution. Under this policy, the system is replaced at multiples of a certain time period while the minimal repair policy is used if there is a failure occurred between the replacement periods assuming that the cost of the repair is a function of the age of the component and the number of times of repair.

Yeh et al. [25] examined the effects of a free-repair warranty on the periodic replacement policy for a repairable product. They constructed cost models for both a warranted and a non-warranted product. The corresponding optimal periodic replacement policies were derived such that the long-run expected cost rate is minimized. El-Dossouky [12] obtained the optimal time for replacement in the case of repairable products which have a free repair warranty when the failure times follow log-logarithmic distribution. In Chien [10], the free repair impacts have been presented over the optimal period of replacement with intermittent time. The failure distribution is assumed to be a binomial distribution. Chien [11] derived optimal periodic replacement policy for a GPP (Generalized Polya Process) repairable product under the free-repair warranty. Cost models from the user's perspective were developed for both a GPP repairable product with free-repair warranty and without warranty, and the corresponding optimal replacement periods are derived such that the long-run expected cost rate is minimized.

This paper considers developing warranty cost models for new repairable products when the failure time follows generalized exponential distribution under two types of minimal repair warranty. An extensive simulation study is performed to assess the performance of the minimal repair warranty models. In addition, this article also derives the optimal periodic replacement policy for failure times follow generalized exponential distribution under free repair warranty. A numerical example is developed to investigates the effects of a free repair warranty policy on the optimal periodic replacement policy.

The rest of the paper is organized as follows. In Section 2, we develop warranty cost model for two types of warranty when the repair policy is the minimal repair policy. In Section 3, a simulation study is conducted to assess the performance of warranty models presented in Section 2. Section 4 provides derivation of optimal periodic replacement time under free repair warranty.

2. Warranty cost model under the minimal repair policy

Under Minimal Repair (as bad as old), the product is warranted for a warranty period w. If the product fails before the end of warranty length it is minimally repaired to bring it to an operational condition comparable to other products having the same age (Park and Yee ([21]).

The assumptions of the minimal repair model are:

1-Successive failures are mutually independent random events.
2-Only minimal repairs are performed.
3-Repair times are negligible compared to the product life.
4-Costs of successive repairs are independent random variables with a constant average.
In Subsection 2.1 we present warranty cost model under the minimal free repair policy while in Subsection 2.2 we present warranty cost model under the minimal pro-rata repair policy

2.1- Warranty cost model under the minimal free repair policy

In this policy, when an item fails before the end of warranty period, the product is repaired and restored to the same failure rate at the time of failure at no cost to the customer, (Elsayed [13]).
The expected warranty cost for minimal free repair during the warranty period is

$$E(\boldsymbol{c_{w\_f}}) = \sum_{n=1}^{\infty} \int_0^w A\, e^{-it} f_n(t) dt \ , \tag{2.1}$$

where

$$f_n(t) = \text{h(t) poin (n-1; H (t))}, \\ = h(t)\ \frac{e^{-H(t)}\ H(t)^{n-1}}{\Gamma n} \tag{2.2}$$

and
p.m.f Probability mass function
Poin (n-1; H (t)) is a non- homogenous poisson process, p.m.f with mean H(t)
n Number of failures
A Average cost per repair
$f_n(t)$ Probability density function (p.d.f) of failure n
c_w The cost of repair during the warranty period (w)
h(t) Hazard rate function of the product at time t
H(t) Cumulative hazard function
i Nominal interest rate
In case of the generalized exponential distribution, we have

$$f_{n(t)=} \frac{\lambda\alpha}{\Gamma n}\ (1 - e^{-\lambda t})^{\alpha-1}\ e^{-\lambda t}\ (-\ln(1 - \left(1 - e^{-\lambda t}\right)^{\alpha}))^{n-1} \tag{2.3}$$

$$E(c_{w\_f}) = \sum_{n=1}^{\infty} \frac{A\lambda\alpha}{\Gamma n} \int_0^w e^{-it}\left(1 - e^{-\lambda t}\right)^{\alpha-1}\ e^{-\lambda t}\ (-\ln(1 - \left(1 - e^{-\lambda t}\right)^{\alpha}))^{n-1}\, dt \tag{2.4}$$

where Γ is the gamma function.

2.2- Warranty cost model under the minimal pro-rata repair policy

In this policy, if an item fails before the end of warranty period, it is repaired at cost that depends on the age of the item at the time of failure.
The expected warranty cost for minimal pro-rata repair during the warranty period is

$$E(c_{w\_pro}) = \sum_{n=1}^{\infty} \int_0^w A\, e^{-it} \left(1 - \frac{t}{w}\right) f_n(t)\, dt \tag{2.5}$$

$$E(c_{w\_pro}) = \sum_{n=1}^{\infty} \frac{A\lambda\alpha}{\Gamma n} \int_0^w e^{-it} \left(1 - \frac{t}{w}\right) [\left(1 - e^{-\lambda t}\right)^{\alpha-1} e^{-\lambda t} (-\ln(1 - \left(1 - e^{-\lambda t}\right)^{\alpha}))^{n-1}]dt \tag{2.6}$$

3. Simulation study

In warranty analysis, the essence of a simulation model is the generation of random variables representing item lifetimes having specified probability distributions. These are used in the simulation model to estimate warranty costs in situation that cannot adequately be investigated practically. Another use of the simulation model would be to investigate the behavior of very long-lived items or any others for which it is difficult to collect actual sufficient data. In this Section we conduct simulation study for previous models presented in Section 2 when the failure time follows generalized exponential distribution.
The inputs to the model are the value of the generalized exponential distribution parameters (α, λ), the lengths of the warranty period, w, and average cost per repair, A. The output of the simulation model is the mean expected warranty cost of the n simulated costs generated for a given set of inputs. The model is validated by comparing its output against the true expected costs generated by a mathematical model. To measure the performance of the simulation we used the Absolute Percentage Error (APE) in comparison between the simulated and theoretical results. The symbolic software Mathematica is utilized to implement the simulation model.

Now we introduced the steps employed to build the simulation model.

1- Generate random numbers following the generalized exponential distribution.
2- Compare between the warranty period for product and the generated numbers. Accordingly, if these generated random numbers are less than the warranty period then the product is subject to failure during the warranty period. Therefore, the product must be repaired. As the policy is the minimum repair policy, the product does not have the same failure rates which it has when it started its lifetime.
3- Generate new random numbers for failed items where reliability function after minimal repair (Ascher [1]) is obtained by

$$R(\tau، t + \tau) = \frac{pr\{survival\ from\ 0\ to\ t + \tau\}}{pr\{survival\ from\ 0\ to\ \tau\}}$$

4- Add the original random numbers to the new random numbers after repair.
5- If a random number after adding is within warranty period it means that the item need repair again therefore generate new random numbers using previous formula and so on.
6- All units that have failed during the warranty period are identified and multiplied by the average cost of repairing one unit. The result is the free repair costs during the warranty period.
7- The cost of free repair per unit during the warranty period is obtained by dividing of total repair costs on the total number of units.
8- The previous steps repeated R times (1000 time in our case) to acquire the average cost of warranty per unit.

The simulation program was then run to calculate estimates of these expected costs at different parameters settings. Namely, the parameter values utilized in the simulation are $\alpha = 0.5, 1.5\ and\ 2$, $\lambda = 0.6, 0.9\ and\ 1$. Other settings are warranty period, $w = 0.5, 1$, sample size $n = 20, 50, and\ 90$. The average cost is fixed at 20 and number of replication, R, is 1000. As a means of validating the simulation model, the mathematical models given in equations (2.4) and (2.6) were used to calculate true expected costs for selected sets of parameter values. The difference is measured by the Absolute Percentage Error (APE) which is calculated as follows:

$$\text{Absolute Percentage Error (APE)} = \frac{True\ value - Simulated\ valuted}{True\ value} \times 100$$

Simulation results for free minimal repair policy is given Table 3.1 and for pro-rate minimal repair policy is given it Table 3.2. For free repair policy, it can be noticed that most of APE values is around zero and one. In addition, the highest value of APE is 3.7% for $n = 20$, while for $n = 90$ it is 2.7%. That is, APE decreases when sample size increase as expected. For pro-rata repair policy, the highest value of APE is 3.2% for $n = 20$, while for $n = 90$ it is 2.1%. That APE decrease when sample size increase as expected.

For relationships between the parameters and the expected warranty cost the following can be noticed:

1- Increasing shape parameter (α) leads to a decrease in expected minimal warranty cost.
2- Increasing scale parameter (λ) leads to an increase in expected minimal warranty cost.
3- Increasing of warranty period (w) leads to an increase in expected minimal warranty cost.

Table 3.1: Simulation results for warranty cost model under the minimal free repair policy

α	λ	*w*	True Expected Cost	Simulation model n=20	APE	Simulation model n=50	APE	Simulation model n=90	APE
0.5	0.6	0.5	14.098	14.172	0.525	14.041	0.404	14.072	0.184
		1	21.852	22.204	1.611	22.079	1.039	21.986	0.613
1.5	0.6	0.5	2.789	2.863	2.653	2.862	2.617	2.829	1.434
		1	7.017	7.281	3.762	7.198	2.579	7.151	1.910
2	0.6	0.5	1.368	1.408	2.924	1.398	2.193	1.393	1.827
		1	4.411	4.518	2.426	4.495	1.904	4.472	1.383
0.5	0.9	0.5	18.250	18.537	1.573	18.379	0.707	18.245	0.027
		1	28.848	28.592	0.887	28.868	0.069	28.834	0.049
1.5	0.9	0.5	4.850	4.951	2.082	4.944	1.938	4.888	0.784
		1	11.876	12.151	2.316	12.127	2.114	12.125	2.097
2	0.9	0.5	2.770	2.857	3.141	2.818	1.733	2.794	0.866
		1	8.418	8.663	2.910	8.649	2.744	8.646	2.708
0.5	1	0.5	19.550	19.683	0.680	19.612	0.317	19.528	0.113
		1	31.076	31.028	0.154	31.053	0.074	31.054	0.071
1.5	1	0.5	5.587	5.684	1.736	5.664	1.378	5.655	1.217
		1	13.577	13.463	0.840	13.946	2.718	13.847	1.989
2	1	0.5	3.311	3.35	1.178	3.341	0.906	3.327	0.483
		1	9.893	10.185	2.952	10.102	2.113	10.1	2.092

4. Optimal periodic replacement policy for repairable Products under free repair warranty

To construct the warranty cost model needed to find optimal periodic replacement policy for repairable products under free repair warranty the following notation will be used:

w: Warranty period
x: Lifetime for a product
$h(u)$: Hazard rate of the product at time u
C_{d}: Downtime cost for each failure of a product
C_p: Purchasing cost for a product
C_{MR}: Minimal repair cost for each failure of a product
$C_1(\mathrm{t})$: Cycle cost when the period for replacement is $t \geq w > 0$
$C_2(\mathrm{t})$: Cycle cost when the period for replacement is $t < w > 0$
$C_{\mathrm{Ri}}(\mathrm{t})$: Expected cost rate which is $E(C_i(t))/t$ for $i = 1,2$
t: Time at which the product is replaced
t_w^*: Optimal period for replacement when the warranty period is w

Table 3.2: Simulation results for warranty cost model under the minimal pro-rata repair policy

α	**λ**	**W**	**True Expected Cost**	**Simulation model n=20**	**APE**	**Simulation model n=50**	**APE**	**Simulation model n=90**	**APE**
0.5	0.6	0.5	8.863	8.762	1.140	8.823	0.451	8.857	0.068
		1	13.482	13.545	0.467	13.534	0.386	13.497	0.111
1.5	0.6	0.5	1.163	1.1982	3.027	1.198	3.009	1.178	1.290
		1	3.003	3.100	3.230	3.004	0.033	3.051	1.598
2	0.6	0.5	0.486	0.497	2.263	0.493	1.440	0.480	1.234
		1	1.641	1.694	3.230	1.682	2.498	1.664	1.402
0.5	0.9	0.5	11.332	11.368	0.318	11.345	0.115	11.334	0.018
		1	17.515	17.572	0.325	17.540	0.143	17.513	0.011
1.5	0.9	0.5	2.047	2.078	1.514	2.058	0.537	2.057	0.489
		1	5.167	5.320	2.961	5.291	2.400	5.280	2.187
2	0.9	0.5	1.007	1.020	1.291	1.016	0.894	1.007	0
		1	3.231	3.299	2.105	3.276	1.393	3.274	1.331
0.5	1	0.5	12.095	12.029	0.546	12.056	0.322	12.072	0.190
		1	18.781	18.954	0.921	18.825	0.234	18.798	0.091
1.5	1	0.5	2.366	2.402	1.522	2.372	0.254	2.371	0.211
		1	5.933	6.082	2.511	6.014	1.365	6.014	1.365
2	1	0.5	1.211	1.242	2.560	1.233	1.817	1.201	0.826
		1	3.831	3.957	3.289	3.927	2.506	3.907	1.984

Under periodic replacement policy, a repairable product is replaced at multiples of a certain period. When the product fails between these periods, a minimal repair is performed with a downtime cost $C_d > 0$ and a repair cost $C_{MR} > 0$. Thus the total cost expected between any two successive replacements is:

$$(C_d + C_{MR}) \int_0^t h(u)\, du \tag{4.1}$$

In order to obtain the optimum time for replacement, the warranty will be considered free. This means that if the product fails before the warranty period expires, the minimal repair policy will be performed without the buyer having any burdens for the repair process, but the buyer will bear the downtime cost.
Products sold with free repair have renewable warranty since the product is given a warranty period starting over with each replacement and any damage caused by the product is covered within the warranty period. However, the repair cost incurred within the warranty period is covered by the warranty. Therefore, for a warranted product, the cost model is established for two cases: $t > w$ and $t < w$. These two cases will be presented in Subsections 4.1 and 4.2. In subsection 4.3, we present derivation of optimal periodic replacement and subsection 4.4 presents a numerical example.

4.1: The first case $t \geq w$

When the replacement period t is greater than or equal to the warranty period w, there are three possible situations. The first situation occurs when the product fails within the warranty period $x \leq w \leq t$. In this case, a downtime cost is incurred. In the second situation, the failure occurs after the warranty period and before and before the preventive replacement $(w < x < t)$. Thus, an additional minimal repair cost C_{MR} will exist. In the last situation, the product reaches the age t $(x = t)$ and the preventive replacement is carried out with cost C_p.
For replacement period $t \geq w$, the expected total cost in the renewal cycle is (Yeh et al. [25])

$$E(C_1(t)) = C_d \int_0^w h(u)\, du + (C_d + C_{MR}) \int_w^t h(u)\, du + C_p, \tag{4.2}$$

and the expected cost rate in this case is calculated as follows:

$$C_{R1}(t) = \frac{E(C_1(t))}{t} \tag{4.3}$$

4.2: The second case $t < w$

When the replacement period is less than the warranty period, all minimal repair costs will be covered by the product warranty. In this case, a downtime cost and preventive replacement cost will only occur. The expected total cost will be

$$E(C_2(t)) = C_d \int_0^t h(u)\, du + C_p, \tag{4.4}$$

and the expected cost rate in this case is

$$C_{R2}(\mathrm{t})=\frac{\mathrm{E}(C_2(\mathrm{t}))}{\mathrm{t}} \qquad (4.5)$$

4.3 Derivation of optimal periodic replacement time under free repair warranty

To derive the optimal periodic replacement time under free repair warranty, one must minimize the expected cost rate. The previous two cases have been investigated.

Case 1: t ≥ w

First, we find the expected total cost:

$$E(C_1(\mathrm{t}))=C_d\int_0^{\mathrm{w}}\mathrm{h(u)\,du}+(C_d+C_{MR})\int_{\mathrm{w}}^{\mathrm{t}}\mathrm{h(u)\,du}+C_p$$

$$=C_d\int_0^{\mathrm{w}}\frac{\lambda\alpha\,(1-e^{-\lambda u})^{\alpha-1}e^{-\lambda u}}{1-(1-e^{-\lambda u})^{\alpha}}du+(C_d+C_{MR})\int_{\mathrm{w}}^{\mathrm{t}}\frac{\lambda\alpha\,(1-e^{-\lambda u})^{\alpha-1}e^{-\lambda u}}{1-(1-e^{-\lambda u})^{\alpha}}\mathrm{du}+C_p$$

$$=C_{MR}\ln\left(1-\left(1-e^{-\lambda w}\right)^{\alpha}\right)-(C_d+C_{MR})\ln\left(1-\left(1-e^{-\lambda t}\right)^{\alpha}\right)+C_p$$

After that we find the expected cost rate as follows:

$$C_{R1}(\mathrm{t})=\frac{\mathrm{E}(C_1(\mathrm{t}))}{\mathrm{t}}=\frac{C_{MR}\ln\left(1-\left(1-e^{-\lambda w}\right)^{\alpha}\right)-(C_d+C_{MR})\ln\left(1-\left(1-e^{-\lambda t}\right)^{\alpha}\right)+C_p}{\mathrm{t}}.$$

The first derivative of the expected cost rate is

$$\frac{dC_{R1}(\mathrm{t})}{\mathrm{dt}}=\frac{\mathrm{t}\left(\alpha\lambda(C_d+C_{MR})\left(1-e^{-\lambda t}\right)^{\alpha-1}e^{-\lambda t}\right)-\mathrm{E}(C_1(\mathrm{t}))}{\mathrm{t}^2\left(1-\left(1-e^{-\lambda t}\right)^{\alpha}\right)}. \qquad (4.6)$$

Equating equation (4.6) with zero gives the optimal periodic replacement period.

Case 2: t < w

We apply the same steps in case 1. Hence, we find that the first derivative of the expected cost rate is

$$\frac{dC_{R2}(\mathrm{t})}{\mathrm{dt}}=\frac{\mathrm{t}\left(\alpha\lambda C_d\left(1-e^{-\lambda t}\right)^{\alpha-1}e^{-\lambda t}\right)-\mathrm{E}(C_2(\mathrm{t}))}{\mathrm{t}^2\left(1-\left(1-e^{-\lambda t}\right)^{\alpha}\right)} \qquad (4.7)$$

4.4 Numerical Example

In this part, we introduce a numerical example to clarify the effect of the product warranty on the optimal period of replacement when the life time of the product follows the generalized exponential distribution. The generalized exponential distribution has an incremental hazard function if the shape parameter $\alpha > 1$ while the value of the scale parameter does not affect the results. Therefore, we fix the value of the parameter λ at 2, the warranty period w at 1 and the cost of the purchase C_p at 200. Results are presented in Table 4.1.

Based on the results of Table 4.1 we note that:

1 - The value of t_{w}^* decreases when the value of α is increased, which means that the product with a very high failure function must undergo the replacement process frequently.
2- The value of t_{w}^* decreases when the downtime cost c_{d} increases.
3- The value of t_{w}^* decreases when the cost of repair c_{MR} increases.
Table 4.1: The results of the optimal duration of substitution in the case of different values for α ، c_d ،c_{mR}

α	c_d	c_{mR}	t_{w}^*	$C_R(t_{\mathrm{w}}^*)$
1.5	500	50	2.04	530.73
	700	50	1.25	685.87
	800	50	1.07	753.45
2	500	50	1.20	452.53
	700	50	1	557.08
	800	50	0.948	607.78
2.5	500	50	1.09	390.33
	700	50	o.974	467.50
	800	50	0.880	503.45
1.5	500	100	1.23	547.17
	700	100	1	688.82
	800	100	1	758.65
2	500	100	1	455.06
	700	100	1	557.08
	800	100	0.948	607.78
2.5	500	100	1	391.13
	700	100	0.974	467.50
	800	100	0.880	503.46
1.5	500	150	1.02	549.15
	700	150	1	688.82
	800	150	0.978	758.64
2	500	150	1	455.06
	700	150	1	557.084
	800	150	0.948	607.78
2.5	500	150	1	391.13
	700	150	1.974	467.52
	800	150	0.880	503.46

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