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Statistical Analysis for an Imprecise Flood Dataset Using the Generalized Inverse Lindley Distribution

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Abstract

In general, data measurements of continuous data such as volume, time, and length are always imprecise real numbers (Viertl (2011)). More specifically, quantities related to environment such as amounts of chemicals released to the environment, flood levels and Snowfall precipitation are always imprecise data. In this paper, we are interested in conducting statistical analysis for a flood data set taking in consideration that the data is imprecise and using the generalized inverse Lindley distribution. The analysis includes both maximum likelihood and Bayesian estimation. The maximum likelihood estimates are found by utilizing two algorithms; the Newton-Raphson and expectation maximization algorithms. The three proposed methods for estimating the parameters of the generalized Lindley distribution are applied then to the flood dataset where approximately equal estimates are obtained.

Keywords: Statistical Analysis, Generalized Inverse Lindley, Maximum likelihood estimation, Basyian estimation, Imprecise data

1 Introduction

One of the most common data assumptions in statistics is the accuracy and precision of the data. However, several quantities and in specific environmental quantities are not precise. In this case, fuzzy techniques should be employed besides statistical methods to draw statistical analysis about the dataset of interest. Here, our interest is to statistically analyze a flood dataset which was of interest in many previous studies. For example, the dataset under consideration analyzed by

Dumonceaux and Antle (1973), Cheng and Amin (1981, 1983), Ahmed (1994), Upadhyay and Peshwani(2003), Shao (2004), Maswadah (2010), Provost and Mabrouk (2010), Madi and Raqab (2013), Singh et al. (2013a), Sharma et al. (2015), Ouédraogo et al .(2017).

The generalized inverse Lindley distribution was fitted to this flood dataset by Sharma et al. (2015). We follow the same approach but we incorporate fuzzy techniques in the statistical analysis since the dataset is not precise. Let Y be a random variable which follows Lindley distribution proposed by Lindley (1958). The probability density function of Y is given by

$$f(y;\, \lambda) = \frac{\lambda^2}{\lambda+1}[1+x]\, e^{-\,\lambda y}, \quad \lambda > 0, x > 0. \tag{1.1}$$

Then the random variable $X = Y^{\frac{-1}{\alpha}}$ follows the generalized inverse Lindley distribution with probability density function and cumulative density function, respectively, given by

$$f(x;\, \alpha, \lambda) = \frac{\alpha\, \lambda^2}{\lambda+1}[1+x^{-\alpha}]x^{-\alpha-1}\, e^{-\,\lambda/x^{\alpha}}, \quad \alpha > 0, \lambda > 0, y > 0, \tag{1.2}$$

$$F(y;\, \alpha, \lambda) = \left[\frac{1+\lambda+\lambda x^{-\alpha}}{1+\lambda}\right] e^{-\,\lambda\,/x^{\alpha}}, \tag{1.3}$$

where λ is the scale parameter and α is the shape parameter.

The generalized inverse Lindley distribution introduced by Sharma et. al. in 2015 where they presented some statistical properties of the proposed distribution. Methods of maximum likelihood, least squares and maximum product spacing were given for estimating the parameters of the distribution. Other methods for estimating the parameters of the distribution were presented by Qoshja and Hoxha (2016). Asgharzadeh et al. (2016) used the generalized Lindley distribution as a non-composite distribution for fitting the Danish fire insurance. The distribution performs very well compared to almost all commonly known heavy tailed distribution and some of the presented composite distributions.

The word fuzzy means indistinct, unclear, or distorted. Aonther meaning is not clearly expressed or thought out. In fuzzy logic, computations depends on “the degree of truth” rather than the usual true or false which utilized in Boolean logic. In Boolean logic, only two values are possible: true or false. True is equivalent to 100% membership degree while false means zero membership degree. In fuzzy logic, the membership function takes any value from zero to 100% not only zero or 100% as in the Boolean logic.

Fuzzy logic can be viewed as a generalization of Boolean logic. Recently, several articles worked on generalizing classical statistical methods such that it can handle fuzzy or imprecise data. For example, Viertl (2006) generalized the classical statistical inference methods for univariate fuzzy data. Zarei et al. (2012) employed vague set theory for the Bayesian estimation of failure rate and mean time to failure. Wu (2014) considered Bayesian estimation under fuzzy environments for lifetime data. For fuzzy data, statistical inference was conducted for lifetime distributions by Pak (2017), Pak and Mahmoudi (2018).

The objective of this paper is to provide statistical inference for the flood data set considering the dataset is not precise and utilizing the generalized inverse Lindley distribution. First, the Maximum likelihood and the Bayesian estimation are employed to derive the estimates of the parameters for fuzzy data. For the maximum likelihood estimation, two algorithms are used: The Newton Raphson and the expectation maximization algorithm. Second, the flood dataset was fitted to the generalized Lindley distribution using the three proposed methods. The remainder of this paper is organized as follows. Section 2 presents the Maximum Likelihood estimation for the parameters of the generalized inverse Lindley distribution when the data is imprecise using two algorithms while Section 3 gives the Bayesian estimation. Methods of estimation presented in Section 2 and 3 applied to the flood dataset in Section 4.

2 Maximum Likelihood Estimation

Let $X_1, X_2, \dots, X_n$ be a random sample of size n drawn from the generalized inverse Lindley distribution with probability density function given by (1.2). Assume that these variables are independent and identically distributed. Also, assume that the information about the measurements of these experimental units is perceived imprecisely. The partial information about $\boldsymbol{x}$ is presumed to be available in the form of fuzzy observation $\tilde{\boldsymbol{x}} = \tilde{x}_1, \tilde{x}_2, \dots, \tilde{x}_n$ with the Borel measurable membership function $\mu_{\tilde{\boldsymbol{x}}}(\mathbf{x})$. Zadeh's definition of the probability of a fuzzy event (Zadeh (1968)) can by employed to define the likelihood function as

$$L(\alpha, \lambda; \tilde{x}_1, \tilde{x}_2, \dots, \tilde{x}_n) = \prod_{i=1}^{n} \int \frac{\alpha\, \lambda^2}{\lambda + 1} (1 + x^{-\alpha}) x^{-\alpha-1}\, e^{-\lambda/x^{\alpha}} \mu_{\tilde{x}_i}(x)\, dx$$

$$= \frac{\alpha^n \lambda^{2n}}{(1+\lambda)^n} \prod_{i=1}^{n} \int [(1 + x^{-\alpha}) x^{-\alpha-1}]\, e^{-\lambda/x^{\alpha}} \mu_{\tilde{x}_i}(x)\, dx \quad (2.1)$$

The corresponding log-likelihood function $L^*(\alpha, \lambda) = \log(L(\alpha, \lambda; \tilde{x}_1, \tilde{x}_2, \dots, \tilde{x}_n))$ is

$$L^*(\alpha,\lambda) = n\log(\alpha) + 2\,n\log(\lambda) - n\log(\lambda+1) + \sum_{i=1}^{n} log \int (1+x^{-\alpha})x^{-\alpha-1}\, e^{-\frac{\lambda}{x^{\alpha}}}\mu_{\tilde{x}_i}(\mathrm{x})dx \tag{2.2}$$

The maximum likelihood estimate of the parameter can be achieved by maximizing the observed-data log-likelihood (2.2). Thus equating the derivative of the log-likelihood with respect to zero, we have

$$\frac{\partial}{\partial\alpha}L^*(\alpha,\lambda) = \frac{n}{\alpha} + \sum_{i=1}^{n} \frac{\int \left[\frac{\lambda + x^{\alpha}(-2 - x^{\alpha} + \lambda)}{x^{3\alpha+1}}\right]\log(x)\, e^{-\frac{\lambda}{x^{\alpha}}}\mu_{\tilde{x}_i}(\mathrm{x})dx}{\int (1+x^{-\alpha})x^{-\alpha-1}\, e^{-\frac{\lambda}{x^{\alpha}}}\mu_{\tilde{x}_i}(\mathrm{x})dx} = 0 \tag{2.3}$$

$$\frac{\partial}{\partial\lambda}L^*(\alpha,\lambda) = \frac{2n}{\lambda} - \frac{n}{1+\lambda} - \sum_{i=1}^{n} \frac{\int \left[\frac{1+x^{\alpha}}{x^{3\alpha+1}}\right]\, e^{-\frac{\lambda}{x^{\alpha}}}\mu_{\tilde{x}_i}(\mathrm{x})dx}{\int (1+x^{-\alpha})x^{-\alpha-1}\, e^{-\frac{\lambda}{x^{\alpha}}}\mu_{\tilde{x}_i}(\mathrm{x})dx} = 0. \tag{2.4}$$

Since there are no closed form of the solutions to the likelihood equations Equation (2.3) and (2.4), an iterative numerical algorithm should be used to obtain the maximum likelihood estimates. In the following, we describe the Newton-Raphson method and the expectation maximization algorithm to determine the maximum likelihood estimates of α and λ.

2.1 Newton-Raphson Algorithm

In the Newton-Raphson algorithm, the solution of the likelihood equation is a result of an interative procedure. Let $\boldsymbol{\gamma} = (\alpha,\lambda)^{T}$ be the parameter vector. In (h+1)th iteration, the updated parameter can be attained as

$$\boldsymbol{\gamma}^{(h+1)} = \boldsymbol{\gamma}^{(h)} - \left[\frac{\partial^2 L^*(\boldsymbol{\gamma};\widetilde{\boldsymbol{x}})}{\partial\boldsymbol{\gamma}\partial\boldsymbol{\gamma}^T}\middle|\boldsymbol{\gamma} = \boldsymbol{\gamma}^{(h)}\right]^{-1}\left[\frac{\partial L^*(\boldsymbol{\gamma};\widetilde{\boldsymbol{x}})}{\partial\boldsymbol{\gamma}}\middle|\boldsymbol{\gamma} = \boldsymbol{\gamma}^{(h)}\right] \tag{2.5}$$

where

$$\frac{\partial L^*(\boldsymbol{\gamma};\widetilde{\boldsymbol{x}})}{\partial\boldsymbol{\gamma}} = \begin{pmatrix} \frac{\partial L^*(\alpha,\lambda;\widetilde{\boldsymbol{x}})}{\partial\alpha} \\ \frac{\partial L^*(\alpha,\lambda;\widetilde{\boldsymbol{x}})}{\partial\lambda} \end{pmatrix},$$

and

$$\frac{\partial^2 L^*(\boldsymbol{\gamma};\widetilde{\boldsymbol{x}})}{\partial\boldsymbol{\gamma}\partial\boldsymbol{\gamma}^T}=\begin{pmatrix}\frac{\partial^2 L^*(\alpha,\lambda;\widetilde{\boldsymbol{x}})}{\partial\alpha^2} & \frac{\partial^2 L^*(\alpha,\lambda;\widetilde{\boldsymbol{x}})}{\partial\alpha\partial\lambda}\\ \frac{\partial^2 L^*(\alpha,\lambda;\widetilde{\boldsymbol{x}})}{\partial\alpha\partial\lambda} & \frac{\partial^2 L^*(\alpha,\lambda;\widetilde{\boldsymbol{x}})}{\partial\lambda^2}\end{pmatrix}$$

The second-order derivatives of the log-likelihood with respect to the parameter required for proceeding with the Newton-Raphson algorithm can be attained as follows:

$$\frac{\partial^2 L^*(\alpha,\lambda;\widetilde{\boldsymbol{x}})}{\partial\alpha^2}=-\frac{n}{\alpha^2}$$
$$+\sum_{i=1}^{n}\left(\frac{\int x^{-1-4\alpha}(x^{3\alpha}+x^{2\alpha}(4-3\lambda)+x^{\alpha}(-5+\lambda)\lambda+\lambda^2)\mathrm{Log}[x]^2 e^{-\frac{\lambda}{x^\alpha}}\mu_{\tilde{x}_i}(\mathrm{x})dx}{\int(1+x^{-\alpha})x^{-\alpha-1}\,e^{-\frac{\lambda}{x^\alpha}}\mu_{\tilde{x}_i}(\mathrm{x})dx}\right.$$
$$\left.-\left[\frac{\int\left[\frac{\lambda+x^{\alpha}(-2-x^{\alpha}+\lambda)}{x^{3\alpha+1}}\right]\log(x)\,e^{-\frac{\lambda}{x^\alpha}}\mu_{\tilde{x}_i}(\mathrm{x})dx}{\int(1+x^{-\alpha})x^{-\alpha-1}\,e^{-\frac{\lambda}{x^\alpha}}\mu_{\tilde{x}_i}(\mathrm{x})dx}\right]^2\right)$$

$$\frac{\partial^2 L^*(\alpha,\lambda;\widetilde{\boldsymbol{x}})}{\partial\lambda^2}=-\frac{2n}{\lambda^2}+\frac{n}{(1+\lambda)^2}-\sum_{i=1}^{n}\left(\frac{-\int x^{-1-4\alpha}\,(1+x^{\alpha})\,e^{-\frac{\lambda}{x^\alpha}}\mu_{\tilde{x}_i}(\mathrm{x})dx}{\int(1+x^{-\alpha})x^{-\alpha-1}\,e^{-\frac{\lambda}{x^\alpha}}\mu_{\tilde{x}_i}(\mathrm{x})dx}\right.$$
$$\left.+\left[\frac{\int\left[\frac{1+x^{\alpha}}{x^{3\alpha+1}}\right]\,e^{-\frac{\lambda}{x^\alpha}}\mu_{\tilde{x}_i}(\mathrm{x})dx}{\int(1+x^{-\alpha})x^{-\alpha-1}\,e^{-\frac{\lambda}{x^\alpha}}\mu_{\tilde{x}_i}(\mathrm{x})dx}\right]^2\right)$$

$$\frac{\partial^2 L^*(\alpha,\lambda;\widetilde{\boldsymbol{x}})}{\partial\alpha\partial\lambda}=\sum_{i=1}^{n}\frac{\int x^{-1-4\alpha}(x^{\alpha}(-3-2x^{\alpha}+\lambda)+\lambda)\,\log(x)\,e^{-\frac{\lambda}{x^\alpha}}\mu_{\tilde{x}_i}(\mathrm{x})dx}{\int(1+x^{-\alpha})x^{-\alpha-1}\,e^{-\frac{\lambda}{x^\alpha}}\mu_{\tilde{x}_i}(\mathrm{x})dx}$$
$$+\sum_{i=1}^{n}\left(\frac{\int\left[\frac{\lambda+x^{\alpha}(-2-x^{\alpha}+\lambda)}{x^{3\alpha+1}}\right]\,\log(x)\,e^{-\frac{\lambda}{x^\alpha}}\mu_{\tilde{x}_i}(\mathrm{x})dx}{\int(1+x^{-\alpha})x^{-\alpha-1}\,e^{-\frac{\lambda}{x^\alpha}}\mu_{\tilde{x}_i}(\mathrm{x})dx}\right.$$
$$\left.\times\frac{\int x^{-1-3\alpha}(1+x^{\alpha})e^{-x^{-\alpha}\lambda}\mu_{\tilde{x}_i}(\mathrm{x})dx}{\int(1+x^{-\alpha})x^{-\alpha-1}\,e^{-\frac{\lambda}{x^\alpha}}\mu_{\tilde{x}_i}(\mathrm{x})dx}\right)$$

The iteration process should continue until convergence is achieved, i.e., until $\left\|\boldsymbol{\gamma}^{(h+1)} - \boldsymbol{\gamma}^{(h)}\right\| < \varepsilon$, for some prefixed $\varepsilon > 0$. In this paper, the maximum likelihood estimate of (α, λ)obtained via Newton-Raphson algorithm is referred to as $(\hat{\alpha}_{NR}, \hat{\lambda}_{NR})$.

In the following we discuss the expectation maximization algorithm since it is viable alternative to the Newton- Raphson algorithm.

2.2 EM Algorithm

The expectation maximization algorithm is a widely applicable approach to iterative computation of maximum likelihood estimates in case of incomplete-data problems. The observed fuzzy data $\widetilde{\boldsymbol{x}}$ can be seen as incomplete specification of a complete vector $\boldsymbol{x}$. Hence, the expectation maximization algorithm is applicable to acquire the maximum likelihood estimates of the unknown parameters. We make use of the fuzzy expectation maximization algorithm (see Denoeux (2011)) to determine the maximum likelihood estimates of α and λ

The complete-data likelihood function for a complete known realization $\boldsymbol{x} = (x_1, x_2, \dots, x_n)$ of $\boldsymbol{X}$ is

$$L_0(\alpha, \lambda; \boldsymbol{x}) = \frac{\alpha^n \lambda^{2n}}{(1+\lambda)^n} e^{-\lambda \sum_{i=1}^{n} 1/x_i^{\alpha}} \prod_{i=1}^{n} [(1 + x_i^{-\alpha}) x_i^{-\alpha-1}] \qquad (2.6)$$

From equation (2.6), the log-likelihood function for the complete data vector $\boldsymbol{x}$ is

$$\log L_0\,(\alpha, \lambda; \boldsymbol{x}) = n \log(\alpha) + 2n \log(\lambda) - n \log(1 + \lambda) + \sum_{i=1}^{n} \log(1 + x_i^{-\alpha}) \\ -(\alpha + 1) \sum_{i=1}^{n} \log(x_i) - \lambda \sum_{i=1}^{n} x_i^{-\alpha} \qquad (2.7)$$

After taking the derivatives with respective to α and λ, respectively, on (2.7), we have the following equations:

$$\frac{n}{\alpha} = \sum_{i=1}^{n} \log(x_i) + \sum_{i=1}^{n} \frac{x_i^{-\alpha} \log(x_i)}{1 + x_i^{-\alpha}} - \lambda \sum_{i=1}^{n} x_i^{-\alpha} \log(x_i) \qquad (2.8)$$

$$\frac{2n}{\lambda} = \frac{n}{1 + \lambda} + \sum_{i=1}^{n} x_i^{-\alpha} \qquad (2.9)$$

Now, the expectation maximization algorithm is given by the follow steps:

1. Choose starting values of α and λ, say $\alpha^{(0)}$ and $\lambda^{(0)}$ and set $h = 0$

2. In the (h+1)th iteration, two steps should be performed:

- The E-Step: This step requires the following conditional expectations:

$$E_{1i} = E_{\alpha^{(h)},\lambda^{(h)}}(\log(x)|\tilde{x}_i) = \frac{\int \log(x)(1+x^{-\alpha})x^{-\alpha-1}e^{-\frac{\lambda}{x^\alpha}}\mu_{\tilde{x}_i}(\mathrm{x})dx}{\int(1+x^{-\alpha})x^{-\alpha-1}\, e^{-\frac{\lambda}{x^\alpha}}\mu_{\tilde{x}_i}(\mathrm{x})dx}$$

$$E_{2i} = E_{\alpha^{(h)},\lambda^{(h)}}\left(\frac{{x_i}^{-\alpha}\log(x)}{1+x^{-\alpha}}\middle|\tilde{x}_i\right) = \frac{\int x^{-\alpha}\log(x)x^{-\alpha-1}\, e^{-\frac{\lambda}{x^\alpha}}\mu_{\tilde{x}_i}(\mathrm{x})dx}{\int(1+x^{-\alpha})x^{-\alpha-1}\, e^{-\frac{\lambda}{x^\alpha}}\mu_{\tilde{x}_i}(\mathrm{x})dx}$$

$$E_{3i} = E_{\alpha^{(h)},\lambda^{(h)}}({x_i}^{-\alpha}\log(x)|\tilde{x}_i) = \frac{\int x^{-\alpha}\log(x)(1+x^{-\alpha})x^{-\alpha-1}\, e^{-\frac{\lambda}{x^\alpha}}\mu_{\tilde{x}_i}(\mathrm{x})dx}{\int(1+x^{-\alpha})x^{-\alpha-1}\, e^{-\frac{\lambda}{x^\alpha}}\mu_{\tilde{x}_i}(\mathrm{x})dx}$$

$$E_{4i} = E_{\alpha^{(h)},\lambda^{(h)}}({x_i}^{-\alpha}|\tilde{x}_i) = \frac{\int x^{-\alpha}(1+x^{-\alpha})x^{-\alpha-1}\, e^{-\frac{\lambda}{x^\alpha}}\mu_{\tilde{x}_i}(\mathrm{x})dx}{\int(1+x^{-\alpha})x^{-\alpha-1}\, e^{-\frac{\lambda}{x^\alpha}}\mu_{\tilde{x}_i}(\mathrm{x})dx}$$

Hence, equations (2.8) and (2.9) are replaced by

$$\frac{n}{\alpha} = \sum_{i=1}^{n} E_{1i} + \sum_{i=1}^{n} E_{2i} - \lambda \sum_{i=1}^{n} E_{3i} \tag{2.10}$$

$$\frac{2n}{\lambda} = \frac{n}{1+\lambda} + \sum_{i=1}^{n} E_{4i} \tag{2.11}$$

- The M-Step: This step requires to solve the equations (2.10) and (2.11) and obtain the next values, $\alpha^{(h+1)}$ and $\lambda^{(h+1)}$, of α and λ, respectively as follows:

$$\lambda^{(h+1)} = \frac{-\sum_{i=1}^{n} E_{4i} + n + \sqrt{\sum_{i=1}^{n} {E_{4i}}^2 + 6\,n\sum_{i=1}^{n} E_{4i} + n^2}}{2\sum_{i=1}^{n} E_{4i}}$$

$$\alpha^{(h+1)} = \frac{n}{\sum_{i=1}^{n} E_{1i} + \sum_{i=1}^{n} E_{2i} - \lambda\sum_{i=1}^{n} E_{3i}}$$

3. We check the convergence where if it occurs then $\alpha^{(h+1)}$ and $\lambda^{(h+1)}$ are the maximum likelihood estimates of α and λ via the expectation maximization, referred to as $(\hat{\alpha}_{em}, \hat{\lambda}_{em})$; otherwise, we set $h = h + 1$ and go to step 2.

3 Bayesian Estimation

This Section describes the Bayes estimate of the unknown parameters α and λ. In the Bayesian estimation setting the unknown parameter is presumed to act as random variable with distribution usually known as prior probability distribution. In this section, we assume the conjugate prior for α to be Gamma(a, b) density of the form:

$$\pi_1(\alpha) \propto \alpha^{a-1}e^{-\alpha b}, \qquad \alpha > 0, \tag{3.1}$$

and for the parameter λ to be Gamma(c, d) density of the form:

$$\pi_2(\lambda) \propto \lambda^{c-1}e^{-\lambda d}, \qquad \lambda > 0, \tag{3.2}$$

where $a > 0$, $b > 0$, $c > 0$ and $d > 0$. Using these priors, the joint posterior density function of α and λ given the data can be obtained as

$$\pi(\alpha, \lambda|\tilde{\boldsymbol{x}}) = \frac{\pi_1(\alpha)\, \pi_2(\lambda)\ell(\alpha, \lambda; \tilde{\boldsymbol{x}})}{\int_0^\infty \int_0^\infty \pi_1(\alpha)\, \pi_2(\lambda)\ell(\alpha, \lambda; \tilde{\boldsymbol{x}})\, d\alpha d\lambda} \tag{3.3}$$

where

$$\ell(\alpha, \lambda; \tilde{\boldsymbol{x}}) = \frac{\alpha^{n+a-1}\lambda^{2n+c-1}}{(1+\lambda)^n} e^{-\alpha b-\lambda d} \prod_{i=1}^n \int (1 + x^{-\alpha})x^{-\alpha-1}\, e^{-\frac{\lambda}{x^\alpha}} \mu_{\tilde{x}_i}(\mathrm{x})dx$$

is the likelihood function for the fuzzy sample $\tilde{\boldsymbol{x}}$. Under the squared error loss function, the Bayes estimate of any function of α and λ, say $g(\alpha, \lambda)$, is given by

$$E(g(\alpha, \lambda)|\tilde{\boldsymbol{x}}) = \frac{\int_0^\infty \int_0^\infty g(\alpha, \lambda)\pi_1(\alpha)\, \pi_2(\lambda)\ell(\alpha, \lambda; \tilde{\boldsymbol{x}})\, d\alpha d\lambda}{\int_0^\infty \int_0^\infty \pi_1(\alpha)\, \pi_2(\lambda)\ell(\alpha, \lambda; \tilde{\boldsymbol{x}})\, d\alpha d\lambda}$$

$$= \frac{\int_0^\infty \int_0^\infty g(\alpha, \lambda)e^{Q(\alpha,\lambda)} d\alpha d\lambda}{\int_0^\infty \int_0^\infty e^{Q(\alpha,\lambda)}\, d\alpha d\lambda} \tag{3.4}$$

where $Q(\alpha, \lambda) = \ln[\pi_1(\alpha)\pi_2(\lambda)] + \ln \ell(\alpha, \lambda; \tilde{\boldsymbol{x}}) \equiv \rho(\alpha, \lambda) + L(\alpha, \lambda)$. Equation (3.4) cannot be obtained analytically. Therefore, we adopt Tierney and Kadane's approximation for computing the Bayes estimates.

The expression in (3.4) can be expressed as

$$E(g(\alpha, \lambda)|\tilde{\boldsymbol{x}}) == \frac{\int_0^\infty \int_0^\infty e^{nH^*(\alpha,\lambda)} d\alpha d\lambda}{\int_0^\infty \int_0^\infty e^{nH^*(\alpha,\lambda)}\, d\alpha d\lambda} \tag{3.5}$$

where

$$H(\alpha,\lambda)=\frac{1}{n}\ Q(\alpha,\lambda),$$

and

$$H^{*}(\alpha,\lambda)=\frac{1}{n}[\ln g(\alpha,\lambda)+Q(\alpha,\lambda)]$$

An approximation of (3.5) can be achieved by Tierney and Kadane (1986) using Laplace's method as follows:

$$\hat{g}_{BT}(\theta)=\left[\frac{\det\Sigma^{*}}{\det\Sigma}\right]^{1/2}\ e^{n\,[H^{*}(\bar{\alpha}^{*},\bar{\lambda}^{*})-\,H(\bar{\alpha},\bar{\lambda})]}, \tag{3.6}$$

where $(\bar{\alpha}^{*},\bar{\lambda}^{*})$ and $(\bar{\alpha},\bar{\lambda})$ maximize $H^{*}(\alpha,\lambda)$ and $H(\alpha,\lambda)$, respectively. Σ and Σ^{*} are the negative of the inverse Hessians of $H(\alpha,\lambda)$ and $H^{*}(\alpha,\lambda)$ at $(\bar{\alpha},\bar{\lambda})$ and $(\bar{\alpha}^{*},\bar{\lambda}^{*})$, respectively.

Applying this approximation to obtain the Bayes estimate of the parameter α and λ, we have

$$\begin{aligned}H(\alpha,\lambda)=\frac{1}{n}\{&k+(n+a-1)\log(\alpha)+(2n+c-1)\log(\lambda)\\&-n\log(1+\lambda)-\alpha b-\lambda d+\sum_{i=1}^{n}log\int\frac{(1+x^{-\alpha})}{x^{\alpha+1}}\ e^{-\frac{\lambda}{x^{\alpha}}}\mu_{\tilde{x}_i}(\mathrm{x})dx\}\end{aligned}$$

where k is a constant. $(\bar{\alpha},\bar{\lambda})$ can be obtained by solving the following two equations

$$\begin{aligned}\frac{\partial}{\partial\alpha}H(\alpha,\lambda)=\frac{1}{n}\Bigg\{&\frac{(n+a-1)}{\alpha}-b\\&+\sum_{i=1}^{n}\frac{\int\left[\frac{\lambda+x^{\alpha}(-2-x^{\alpha}+\lambda)}{x^{3\alpha+1}}\right]\log(x)\ e^{-\frac{\lambda}{x^{\alpha}}}\mu_{\tilde{x}_i}(\mathrm{x})dx}{\int(1+x^{-\alpha})x^{-\alpha-1}\ e^{-\frac{\lambda}{x^{\alpha}}}\mu_{\tilde{x}_i}(\mathrm{x})dx}\Bigg\}\end{aligned}$$

$$\frac{\partial}{\partial\lambda}H(\alpha,\lambda)=\frac{1}{n}\left\{\frac{(2n+c-1)}{\lambda}-\frac{n}{1+\lambda}-d\right.$$
$$\left.-\sum_{i=1}^{n}\frac{\int\left[\frac{1+x^{\alpha}}{x^{3\alpha+1}}\right]e^{-\frac{\lambda}{x^{\alpha}}}\mu_{\tilde{x}_i}(\mathrm{x})dx}{\int(1+x^{-\alpha})x^{-\alpha-1}\,e^{-\frac{\lambda}{x^{\alpha}}}\mu_{\tilde{x}_i}(\mathrm{x})dx}\right\}$$

The determinant of the negative of the inverse Hessian $H(\alpha,\lambda)$ at $(\bar{\alpha},\bar{\lambda})$ can be found using the second derivative of $H(\alpha,\lambda)$. The determinant of Σ is given by

$$\det\Sigma=(H_{11}H_{22}-H_{12}^{2})^{-1} \qquad (3.8)$$

where

$$H_{11}=\frac{1}{n}\left\{-\frac{(n+a-1)}{\bar{\alpha}^{2}}\right.$$
$$+\sum_{i=1}^{n}\left(\frac{\int x^{-1-4\bar{\alpha}}\left(x^{3\bar{\alpha}}+x^{2\bar{\alpha}}\left(4-3\bar{\lambda}\right)+x^{\bar{\alpha}}\left(-5+\bar{\lambda}\right)\bar{\lambda}+\bar{\lambda}^{2}\right)\mathrm{Log}[x]^{2}e^{-\frac{\bar{\lambda}}{x^{\bar{\alpha}}}}\mu_{\tilde{x}_i}(\mathrm{x})dx}{\int(1+x^{-\bar{\alpha}})x^{-\bar{\alpha}-1}\,e^{-\frac{\bar{\lambda}}{x^{\bar{\alpha}}}}\mu_{\tilde{x}_i}(\mathrm{x})dx}\right.$$
$$\left.\left.-\left[\frac{\int\left[\frac{\bar{\lambda}+x^{\bar{\alpha}}(-2-x^{\bar{\alpha}}+\bar{\lambda})}{x^{3\alpha+1}}\right]\log(x)\,e^{-\frac{\bar{\lambda}}{x^{\bar{\alpha}}}}\mu_{\tilde{x}_i}(\mathrm{x})dx}{\int(1+x^{-\bar{\alpha}})x^{-\bar{\alpha}-1}\,e^{-\frac{\bar{\lambda}}{x^{\bar{\alpha}}}}\mu_{\tilde{x}_i}(\mathrm{x})dx}\right]^{2}\right)\right\}$$

$$H_{22}=\frac{1}{n}\left\{-\frac{(2n+c-1)}{\bar{\lambda}^{2}}+\frac{n}{\left(1+\bar{\lambda}\right)^{2}}\right.$$
$$\left.-\sum_{i=1}^{n}\left(\frac{-\int(x^{-1-4\bar{\alpha}}+x^{-1-3\bar{\alpha}})\,e^{-\frac{\bar{\lambda}}{x^{\bar{\alpha}}}}\mu_{\tilde{x}_i}(\mathrm{x})dx}{\int(1+x^{-\bar{\alpha}})x^{-\bar{\alpha}-1}\,e^{-\frac{\bar{\lambda}}{x^{\bar{\alpha}}}}\mu_{\tilde{x}_i}(\mathrm{x})dx}+\left[\frac{\int\left[\frac{1+x^{\alpha}}{x^{3\alpha+1}}\right]e^{-\frac{\bar{\lambda}}{x^{\bar{\alpha}}}}\mu_{\tilde{x}_i}(\mathrm{x})dx}{\int(1+x^{-\alpha})x^{-\bar{\alpha}-1}\,e^{-\frac{\bar{\lambda}}{x^{\bar{\alpha}}}}\mu_{\tilde{x}_i}(\mathrm{x})dx}\right]^{2}\right)\right\}$$

$$H_{12}=\frac{1}{n}\left\{\sum_{i=1}^{n}\frac{\int x^{-1-4\bar{\alpha}}\left(x^{\alpha}\left(-3-2x^{\bar{\alpha}}+\bar{\lambda}\right)+\bar{\lambda}\right)\log(x)\,e^{-\frac{\bar{\lambda}}{x^{\bar{\alpha}}}}\mu_{\tilde{x}_i}(\mathrm{x})dx}{\int(1+x^{-\bar{\alpha}})x^{-\bar{\alpha}-1}\,e^{-\frac{\bar{\lambda}}{x^{\bar{\alpha}}}}\mu_{\tilde{x}_i}(\mathrm{x})dx}\right.$$
$$+\sum_{i=1}^{n}\left(\frac{\int\left[\frac{\bar{\lambda}+x^{\bar{\alpha}}(-2-x^{\bar{\alpha}}+\bar{\lambda})}{x^{3\bar{\alpha}+1}}\right]\log(x)\,e^{-\frac{\bar{\lambda}}{x^{\bar{\bar{\alpha}}}}}\mu_{\tilde{x}_i}(\mathrm{x})dx}{\int(1+x^{-\bar{\alpha}})x^{-\bar{\alpha}-1}\,e^{-\frac{\bar{\lambda}}{x^{\bar{\alpha}}}}\mu_{\tilde{x}_i}(\mathrm{x})dx}\right.$$

$$\times \frac{\int x^{-1-3\overline{\alpha}}(1+x^{\overline{\alpha}})e^{-x^{-\overline{\alpha}}\overline{\lambda}}\mu_{\tilde{x}_i}(\mathrm{x})dx}{\int(1+x^{-\overline{\alpha}})x^{-\overline{\alpha}-1}\, e^{-\frac{\overline{\lambda}}{x^{\overline{\alpha}}}}\mu_{\tilde{x}_i}(\mathrm{x})dx}\Bigg)\Bigg\}$$

Following the same augments with $g(\alpha, \lambda) = \alpha$ and λ, respectively, in $H^*(\alpha, \lambda)$, $\hat{\alpha}_{BT}, \hat{\lambda}_{BT}$ in equation (3.8) can be obtained directly.

4 Application

In this Section, we apply the proposed methods in Section 2 and 3 on a flood dataset which is presented in Table 1. The dataset represents maximum flood levels, measured in millions cubic feet per second, for the Susquehanna River at Harrisburg, Pennsylvania over 20 four-year periods (see Dumonceaux and Antle (1973)). This data set was fitted to the generalized inverse Lindley distribution by Sharma et al. (2015). Here we make use of the same distribution but we take in consideration that the data measurements are imprecise. We presume that the imprecision in the dataset can be represented by the following fuzzy information system:

$$\mu_{\tilde{x}_1}(x) = \begin{cases} 1 & x \le 0.20, \\ \dfrac{0.30 - x}{0.10} & .20 \le x \le 0.30, \\ 0 & otherwise, \end{cases}$$

$$\mu_{\tilde{x}_2}(x) = \begin{cases} \frac{x-0.20}{0.10} & .20 \le x \le 0.30, \\ \frac{0.35-x}{0.05} & .30 \le x \le 0.35, \\ 0 & otherwise, \end{cases}$$

$$\mu_{\tilde{x}_3}(x) = \begin{cases} \dfrac{x - 0.30}{0.05} & .30 \le x \le 0.35, \\ \dfrac{0.40 - x}{0.05} & .35 \le x \le 0.40, \\ 0 & otherwise, \end{cases}$$

$$\mu_{\tilde{x}_4}(x) = \begin{cases} \dfrac{x - 0.35}{0.05} & .35 \le x \le 0.40, \\ \dfrac{0.45 - x}{0.05} & .40 \le x \le 0.45, \\ 0 & otherwise, \end{cases}$$

$$\mu_{\tilde{x}_5}(x) = \begin{cases} \dfrac{x - 0.40}{0.05} & .40 \le x \le 0.45 \\ \dfrac{.50 - x}{0.05} & 0.45 \le x \le .50 \\ 0 & otherwise \end{cases}$$

$$\mu_{\tilde{x}_6}(x) = \begin{cases} \dfrac{x-0.45}{0.05} & 0.45 \le x \le .50 \\ \dfrac{0.60-x}{0.10} & .50 \le x \le 0.60 \\ 0 & otherwise \end{cases}$$

$$\mu_{\tilde{x}_7}(x) = \begin{cases} \dfrac{x-0.50}{0.10} & .50 \le x \le 0.60 \\ \dfrac{0.65-x}{0.05} & 0.60 \le x \le 0.65 \\ 0 & otherwise \end{cases}$$

$$\mu_{\tilde{x}_8}(x) = \begin{cases} \dfrac{x-0.60}{0.05} & 0.60 \le x \le 0.65 \\ 1 & x \ge 0.65 \\ 0 & otherwise \end{cases}$$

We employ the Newton-Raphson and the expectation maximization algorithms for computing the maximum likelihood estimates. For computing the Bayes estimates, we considered (a,b,c,d)=(0,0,0,0), that is the priors are non-informative. Table 2 presents the estimates obtained from the three methods. From Table 2, it can be seen that the estimates in the three methods are very close to each other.

Table 1: Maximum Flood levels dataset

0.654	0.613	0.402	0.379	0.269	0.740	0.416	0.338	0.315	0.449
0.297	0.423	0.379	0.3235	0.418	0.412	0.494	0.392	0.484	0.265

Table 2: Parameter estimates for the flood dataset sing the proposed methods

Parameter **Method**	**Estimate of α**	**Estimate of λ**
Newton-Raphson	2.98	.103
Expectation Maximization	3.074	0.093
Bayes	2.916	0.098

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