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Statistical Inference for the Parameter of the Inverse Lindley Distribution Based on Imprecise Data with Simulation Study

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Abstract

Uncertainty refers to situations which involve imperfect or unknown information. The most common type of uncertainty encountered in Statistics is randomness. By randomness we mean uncertainty about the result of a random experiment although all its all outcomes are known ahead of time. Nonetheless other sources of uncertainty also are common in real life. One of the most famous sources of uncertainty is imprecision. Imprecision can be recognized due to machine errors or recollecting data. In this case, fuzzy techniques besides statistical methods should be employed to deal with such setting. The focus of this paper is estimate the parameter of the inverse Lindley distribution in presence of imprecise data. Both the classical maximum likelihood estimation and the Baysian estimation have been utilized to estimate the parameter on the inverse Lindley distribution along with fuzzy techniques. In addition, the confidence interval has been derived in the case of the maximum likelihood estimation. An extensive simulation study has been conducted for the two proposed methods of estimation to evaluate their performance in estimating the parameter. The simulation study showed that the parameter can be estimated with high degree of accuracy using the two methods of estimation. Also, the simulation carried on the confidence interval coverage which was approximately found to be equal to its nominal level used in the study specifically when the sample size is large.

Keywords: Inverse Lindley distribution, Maximum likelihood estimation, Basyian estimation, Imprecise data, Simulation Model

1 Introduction

A mixture of exponential(θ) and gamma(2,θ) distributions with mixing proportion [θ/(θ+1)], called Lindley distribution, was proposed by Lindley (1958). This distribution is capable of modeling lifetime data especially in applications modeling stress-strength reliability. The probability density function and cumulative distribution function of the Lindley distribution are, respectively, given by

$$f(y;\ \theta) = \frac{\theta^2}{\theta+1}(1+y)\, e^{-\theta\, y}, \quad \theta > 0, y > 0, \tag{1.1}$$
$$F(y;\ \theta) = 1 - \left(1 + \frac{\theta y}{\theta+1}\right) e^{-\theta\, y}, \quad \theta > 0, y > 0, \tag{1.2}$$

Ghitany et el (2008) showed that mathematical properties of the Lindley distribution are more flexible than those of the exponential distribution and presented an application that showed that the Lindley distribution models the dataset better than the exponential distribution.

The probability distribution of the random variable $= 1/y$, called the inverse Lindley distribution, was presented by Sharma et al. (2015). They proved that the proposed distribution is capable of modeling survival data with upside-down bathtub shaped hazard rate models. Also, they presented an extensive study of the mathematical properties of the proposed distribution. The inverse Lindley distribution is a mixture of the inverse of inverse exponential distribution and the inverse gamma distribution at specific parameter settings.

The probability density function and cumulative distribution function of the inverse Lindley distribution are, respectively, given by

$$f(x;\ \theta) = \frac{\theta^2}{\theta+1}\left(\frac{1+x}{x^3}\right) e^{-\theta\,/x}, \quad \theta > 0, x > 0, \tag{1.3}$$
$$F(x;\ \theta) = \left(1 + \frac{\theta}{x(\theta+1)}\right) e^{-\theta/\,x}, \quad \theta > 0, x > 0, \tag{1.4}$$

Sharma et al. (2015) estimated the stress–strength reliability for the inverse Lindley distribution using both classical and Bayesian method. Basu et al. (2018) presented point and interval estimates under both the classical and Bayesian paradigm for inverse Lindley distribution for Type-I hybrid censored data.
All these inference techniques for estimating the parameter of Lindley distribution are based on precisely (crisp) data. In real world situations, data may not be measured and recorded precisely due to human errors, machine errors or some unexpected situations. The problem addressed in this paper, is different from imprecision arising from random inspection times. The imprecision of interest here is the result of reporting results of a random experiment imprecisely due to its limited perception or recollection of the precise numerical values.

Usually, Statistics deals with randomness using probabilities. A random experiment can be defined an experiment whose all possible outcomes are known in advance however which outcome will occur is not known before running the experiment. Examples are: chance that a coin lands head, chance that the Conservative Party wins 300 seat or more in the next election? (see French (1995)).

Randomness is one source of uncertainty. Other possible source of uncertainty is imprecision. Imprecision means that the observer reported to the statistician the results of a random experiment imprecisely due to machine errors or lack of knowledge. In this case the data may be called imprecise/fuzzy data.

The word fuzzy means unclear, distorted, or indistinct. Also it may mean not clearly thought out or expressed. Fuzzy logic is a computational approach which rely on "*degrees of truth*" rather than the usual "true or false" or what we call the Boolean logic. In Boolean logic there are only two possible results which is true (belong to with certainty degree equal to 100%) or false (not belong to with certainty degree equal to 100% or belong to with certainty or membership equal to 0%). In fuzzy logic, we have *degrees of truth or memberships* which take values from zero to one (one completely belong to and zero completely does not belong to). For example, we may have degree of membership 30% or 70% not only zero or 1(100%).

It is obvious that there are relationship between fuzzy set and crisp set or the fuzzy logic and the Boolean logic. Fuzzy logic can be seen as a generalization of the Boolean logic. In recent years many papers worked on generalizing the classical statistical methods to be capable of dealing with fuzzy data. The Bayesian estimation under fuzzy environments for lifetime data was considered by Wu (2014). Gil et al. (2006) conducted a backward analysis on the interpretation, impact, and modeling of fuzzy random variable. For univariate fuzzy data, a generalization of the classical statistical inference methods was proposed by Viertl (2006). Using vague set theory the Bayesian estimation of failure rate and mean time to failure was considered by Zarei et al. (2012) for complete and censored data. A series of studies were conducted to attain statistical inference for lifetime distributions when the data is fuzzy (see Pak et al. (2013), Pak (2017), Pak and Mahmoudi (2018)).

The aim of this paper is to provide the Maximum likelihood and the Bayesian estimation for the inverse Lindley distribution parameters when the data is imprecise using statistical methods along with fuzzy techniques. The rest of this paper is organized as follows. Section 2 presents the Maximum Likelihood estimation and confidence interval for the inverse Lindley distribution when the data is imprecise while Section 3 gives the Bayesian estimation. In Section 4, we provide a simulation study for the two methods of estimation besides the confidence interval for the first method of estimation. The performance of the simulation is measured by the mean squared errors (MSE) and the average bias values (AB) of the estimates.

2 Maximum Likelihood estimation and confidence interval

Suppose that $X_1, X_2, \dots, X_n$ is a random sample of size n drawn from inverse Lindley distribution with probability density function given by (1.3). It is assumed that these variables are independent and identically distributed. Assume that the available information about the measurements of these experimental units can not be exactly perceived, that is the data is imprecise. Presume that this partial information about $\boldsymbol{x}$ is available in the form of fuzzy observation $\tilde{\boldsymbol{x}} = \tilde{x}_1, \tilde{x}_2, \dots, \tilde{x}_n$ with the Borel measurable membership function $\mu_{\tilde{\boldsymbol{x}}}(\mathbf{x})$. Using Zadeh's definition of the probability of a fuzzy event (Zadeh (1968)), the likelihood function of θ is given by

$$L(\theta; \tilde{x}_1, \tilde{x}_2, \dots, \tilde{x}_n) = \prod_{i=1}^{n} \int \frac{\theta^2}{\theta+1} \left(\frac{1+x}{x^3}\right) e^{-\theta/x} \mu_{\tilde{x}_i}(x)\, dx \tag{2.1}$$

$$= \frac{\theta^{2n}}{(1+\theta)^n} \prod_{i=1}^{n} \int \left(\frac{1+x}{x^3}\right) e^{-\theta/x} \mu_{\tilde{x}_i}(x)\, dx$$

The corresponding log-likelihood function $L^*(\theta) = \log(L(\theta; \tilde{x}_1, \tilde{x}_2, \dots, \tilde{x}_n))$ is

$$L^*(\theta) = 2\, n \log(\theta) - n \log(\theta + 1) + \sum_{i=1}^{n} log \int \left(\frac{1+x}{x^3}\right) e^{-\theta/x} \mu_{\tilde{x}_i}(\mathrm{x}) dx \tag{2.2}$$

The maximum likelihood estimate of the parameter can be achieved by maximizing the observed-data log-likelihood (2.2). Thus, equating the derivative of the log-likelihood with respect to zero, we have

$$\frac{\partial}{\partial\theta} L^*(\theta) = \frac{2n}{\theta} - \frac{n}{1+\theta} - \sum_{i=1}^{n} \frac{\int \frac{1+x}{x^4} e^{-\theta/x} \mu_{\tilde{x}_i}(\mathrm{x}) dx}{\int \frac{1+x}{x^3} e^{-\theta/x} \mu_{\tilde{x}_i}(\mathrm{x}) dx} = 0. \tag{2.3}$$

Equation (2.3) is not easy to solve, however it can be seen that it has a unique solution (Pak et al. (2014)) where an iterative numerical search method can be used to obtain the maximum likelihood inference. A well-known method is the Newton-Raphson algorithm which can be implemented easily. By using this method, we can also compute the asymptotic variance of the MLE and construct its asymptotic confidence interval. In the following, the Newton- Raphson method is described.

In each iteration of the Newton- Raphson algorithm, we calculate the new estimate $\hat{\theta}$ as

$$\hat{\theta} = \theta_0 + \varepsilon,$$

where θ_0 is the previous estimate and ε is the correction. The iteration method depends on Taylor series expansion of equation (2.3) in the neighborhood of the previous estimate. Using Taylor's theorem and neglecting power of ε above the first order we obtain the following equation which should be solved for ε:

$$\varepsilon \frac{\partial^2}{\partial\theta^2} L^*(\theta_0) = -\frac{\partial}{\partial\theta} L^*(\theta_0)$$

where

$$\frac{\partial^2}{\partial\theta^2} L^*(\theta) = -\frac{2n}{\theta^2} + \frac{n}{(1+\theta)^2} - \sum_{i=1}^{n} \left[\frac{\int \frac{1+x}{x^5} e^{-\theta/x} \mu_{\tilde{x}_i}(\mathrm{x}) dx}{\int \frac{1+x}{x^3} e^{-\theta/x} \mu_{\tilde{x}_i}(\mathrm{x}) dx} - \left[\frac{\int \frac{1+x}{x^4} e^{-\theta/x} \mu_{\tilde{x}_i}(\mathrm{x}) dx}{\int \frac{1+x}{x^3} e^{-\theta/x} \mu_{\tilde{x}_i}(\mathrm{x}) dx} \right]^2 \right]$$

In this paper, the maximum likelihood estimate of θ obtained via Newton-Raphson algorithm is referred to as $\hat{\theta}_{NR}$. Since the maximum likelihood estimate of θ is now obtainable, the asymptotic normality of the maximum likelihood estimate can be used to construct the approximate confidence interval for the parameter θ. The asymptotic distribution of the maximum likelihood estimate of θ is given by (Miller 1981):

$$\left(\hat{\theta}_{NR} - \theta\right) \to N\left(0, Var(\theta)\right).$$

The variance $Var(\theta)$ can be approximated as

$$Var\left(\hat{\theta}_{NR}\right) = \frac{1}{I(\hat{\theta}_{NR})},$$

where $I(\hat{\theta}_{NR})$ denotes the observed Fisher information which can be attained by

$$I\left(\hat{\theta}_{NR}\right) = -\frac{\partial^2}{\partial\theta^2} L^*(\theta)|_{\theta=\hat{\theta}_{NR}}.$$

Hence, the approximated $100(1-\alpha)\%$ confidence interval for the parameter θ can be computed as

$$[\hat{\theta}_L, \hat{\theta}_U] = \hat{\theta}_{NR} \pm z_{\frac{\alpha}{2}}\sqrt{Var(\hat{\theta}_{NR})},$$

where $z_{\frac{\alpha}{2}}$ is an upper percentile of the standard normal variable.

3 Bayesian estimation

This Section describes the Bayes estimate of the unknown parameter θ. In the Bayesian estimation the unknown parameter is assumed to act as random variable with distribution usually known as prior probability distribution. Here, conjugate prior for θ is assumed to be Gamma(a, b) density of the form:

$$\pi(\theta) \propto \theta^{a-1}e^{-\theta b}, \qquad \theta > 0, \tag{3.1}$$

where $a > 0$ and $b > 0$. By combining (2.1) with (3.1), the joint density function of the data and θ becomes

$$\pi(data, \theta) \propto \frac{\theta^{2n+a-1}}{(1+\theta)^n} e^{-\theta b} \prod_{i=1}^{n} \int \left(\frac{1+x}{x^3}\right) e^{-\frac{\theta}{x}} \mu_{\tilde{x}_i}(x)\, dx, \tag{3.2}$$

Consequently, the posterior density function of θ given the data can be obtained as

$$\pi(\theta|data) = \frac{\pi(data, \theta)}{\int_0^\infty \pi(data, \theta)\, d\theta}. \tag{3.3}$$

Under squared error loss function, the Bayes estimate of any function of θ, say $h(\theta)$, is the posterior mean which is obtained by

$$\int_0^\infty \pi(\theta|data)\, h(\theta) d\theta \tag{3.4}$$

The last two equations do not simplified to nice closed forms due to the complex form of the likelihood function. However, the Tierney and Kadane's approximation can be employed for computing the Bayes estimate of θ. The first step to take on the Tierney and Kadane's approximation is to rewrite the expression in (3.4) as

$$\int_0^\infty \pi(\theta|data)\, h(\theta) d\theta = \frac{\int_0^\infty e^{n\, F^*(\theta)} d\theta}{\int_0^\infty e^{n\, F(\theta)}\, d\theta}, \tag{3.5}$$

where

$$F(\theta) = \frac{1}{n} \ln \pi(data, \theta),$$

and

$$F^{*}(\theta) = F(\theta) + \frac{1}{n} \ln h(\theta)$$

An approximation of (3.5) is attained by Tierney and Kadane (1986) using Laplace's method as follows:

$$\hat{h}_{BT}(\theta) = \left[\frac{\emptyset^{*}}{\emptyset}\right]^{1/2} e^{n\,[F^{*}(\bar{\theta}^{*}) - F(\bar{\theta})]}, \tag{3.6}$$

where $\emptyset^{*}$ and $\emptyset$ are minus the inverse of the second derivatives of $F^{*}(\theta)$ and $F(\theta)$ respectively, and $\bar{\theta}^{*}$and $\bar{\theta}$ maximize $F^{*}(\theta)$ and $F(\theta)$ respectively.

Applying this approximation to acquire the Bayes estimate of the parameter θ and setting $h(\theta) = \theta$, we have

$$\begin{aligned} F(\theta) = \frac{1}{n}\Bigg\{&(2n + a - 1)\log(\theta) - n\log(1+\theta) - \theta b \\ &+ \sum_{i=1}^{n} log \int \left(\frac{1+x}{x^{3}}\right) e^{-\theta/x} \mu_{\tilde{x}_i}(\mathrm{x})dx\Bigg\}, \end{aligned} \tag{3.7}$$

$$\begin{aligned} F^{*}(\theta) = \frac{1}{n}\Bigg\{&(2n + a)\log(\theta) - n\log(1+\theta) - \theta b \\ &+ \sum_{i=1}^{n} log \int \left(\frac{1+x}{x^{3}}\right) e^{-\theta/x} \mu_{\tilde{x}_i}(\mathrm{x})dx\Bigg\}, \end{aligned} \tag{3.8}$$

The Bayes estimate of θ under squared error loss accordingly can be obtain be obtained by substituting from (3.7) and (3.8) in (3.6) with the form

$$\begin{aligned} &\hat{\theta}_{BT}(\theta) \\ &= \left[\frac{\emptyset^{*}}{\emptyset}\right]^{1/2} \left(\frac{\bar{\theta}^{*2n+a}}{\bar{\theta}^{2n+a-1}}\right)\left(\frac{1+\bar{\theta}}{1+\bar{\theta}^{*}}\right)^{n} e^{b\,[\bar{\theta} - \bar{\theta}^{*}]} \prod_{i=1}^{n} \frac{\int \left(\frac{1+x}{x^{3}}\right) e^{-\frac{\bar{\theta}^{*}}{x}}\, \mu_{\tilde{x}_i}(\mathrm{x})dx}{\int \left(\frac{1+x}{x^{3}}\right) e^{-\frac{\bar{\theta}}{x}}\, \mu_{\tilde{x}_i}(\mathrm{x})dx} \end{aligned} \tag{3.9}$$

4 Simulation Study

This section provides some simulation results with the purpose of comparing the performances of the various methods introduced in the previous sections. The performances measured in terms of their average biases and mean squared errors. Besides, the average lengths of the confidence and their coverage percentages are computed in the simulation study.

Different choices of sample sizes (n) are considered while the parameter θ was fixed at 1. The fuzzy random sample generated from the distribution given in (1.3) by utilizing the algorithm presented in Pak et al (2014). First, we generate random variables that follow the inverse Lindley distribution when $\theta = 1$ then each realization of the generated samples was fuzzified by using the following fuzzy information system $\{\tilde{x}_1, \tilde{x}_2, \dots, \tilde{x}_8\}$ which corresponds to the following membership function

$$\mu_{\tilde{x}_1}(x) = \begin{cases} 1 & x \le 0.05, \\ \dfrac{0.25 - x}{0.20} & .05 \le x \le 0.25, \\ 0 & otherwise, \end{cases}$$

$$\mu_{\tilde{x}_2}(x) = \begin{cases} \frac{x-0.05}{0.20} & .05 \le x \le 0.25, \\ \frac{0.5-x}{0.25} & .25 \le x \le 0.5, \\ 0 & otherwise, \end{cases}$$

$$\mu_{\tilde{x}_3}(x) = \begin{cases} \dfrac{x - 0.25}{0.25} & .25 \le x \le 0.5, \\ \dfrac{0.75 - x}{0.25} & .5 \le x \le 0.75, \\ 0 & otherwise, \end{cases}$$

$$\mu_{\tilde{x}_4}(x) = \begin{cases} \dfrac{x - 0.5}{0.25} & .5 \le x \le 0.75, \\ \dfrac{1 - x}{0.25} & .75 \le x \le 1, \\ 0 & otherwise, \end{cases}$$

$$\mu_{\tilde{x}_5}(x) = \begin{cases} \dfrac{x - 0.75}{0.25} & .75 \le x \le 1 \\ \dfrac{1.5 - x}{0.5} & 1 \le x \le 1.5 \\ 0 & otherwise \end{cases}$$

$$\mu_{\tilde{x}_6}(x) = \begin{cases} \dfrac{x - 1}{0.5} & 1 \le x \le 1.5 \\ \dfrac{2 - x}{0.5} & 1.5 \le x \le 2 \\ 0 & otherwise \end{cases}$$

$$\mu_{\tilde{x}_7}(x) = \begin{cases} \dfrac{x-1.5}{0.5} & 1.5 \le x \le 2 \\ 3-x & 2 \le x \le 3 \\ 0 & otherwise \end{cases}$$

$$\mu_{\tilde{x}_8}(x) = \begin{cases} x-2 & 2 \le x \le 3 \\ 1 & x \ge 3 \\ 0 & otherwise \end{cases}$$

In the simulation model of the maximum likelihood estimate for the parameter θ, the true value of θ was used as the initial guess value for θ. Moreover, the approximate 95% confidence interval was computed and the performance was measured by closeness of the coverage probability to the nominal value (95%). In the simulation model of the Bayes estimate for the parameter θ, it is assumed that θ has gamma prior where two cases are considered the informative gamma prior ($a = b = 2$) and the non-informative gamma prior ($a = b = 0$). The process was replicated 1000 time where the mean squared errors (MSE) and the average bias values (AB) of the estimates were reported in Tables 4.1–4.2.

Table 4.1: AB and MSE of the maximum likelihood estimates of θ, coverage probabilities and expected width of the confidence interval

n	Maximum likelihood estimate		Confidence Interval	
	AB	MSE	Coverage	Length
15	-0.0654	0.0757	0.934	0.7737
25	-0.0358	0.0246	0.941	0.5780
50	-0.0247	0.0113	0.951	0.3993
100	-0.0201	0.0076	0.959	0.1842

Table 4.2: AB and MSE of the Bayes estimates of θ

n	Non-informative ($a = b = 0$)		Informative $a = b = 2$	
	AB	MSE	AB	MSE
15	-0.0726	0.07708	-0.0610	0.0605
25	-0.0397	0.0253	-0.0364	0.0223
50	-0.0267	.01163	-0.0259	0.0110
100	-0.0217	0.0078	-0.0207	0.0075

Simulation results show that the performances of all estimators are good enough even when the sample size is sample. However, as expected, all the performance measures get better when the sample size increases. The performance of the Bayes estimates with informative prior is slightly better than that with non-informative prior. The performance of the confidence interval is very good specifically when the sample size n is large. It can be seen that the coverage probability is almost 95% for all sample sizes. For the width of the confidence intervals, it is clear that it gets narrow down when the sample size becomes bigger.

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