International Journal of Contemporary Mathematical Sciences
Vol. 14, 2019, no. 1, 1 - 7
HIKARI Ltd, www.m-hikari.com
https://doi.org/10.12988/ijcms.2019.81234

Construction of Orthogonal Arrays of Strength 3 by Difference Schemes

Pang Shanqi

College of Mathematics and Information Science
Henan Normal University
Xinxiang, 453007, China

Lin Xiao

College of Mathematics and Information Science
Henan Normal University
Xinxiang, 453007, China

Abstract

Difference schemes are a simple but powerful tool for the construction of orthogonal arrays. In this study, we construct some new OAs of strength 3 by using difference scheme of strength 3, orthogonal partition and the Kronecker product.

Mathematics Subject Classification: 05B10,05B15

Keywords: Orthogonal array, Difference scheme, Orthogonal partition, Kronecker product

1 Introduction

Besides being used for the construction of various other combinatorial configurations, orthogonal arrays (OAs) have, since their introduction by Rao [13] in 1947, gained prominence among statisticians for their properties as fractional factorial designs. They are widely applied to the design of experiments (DOE), statistics[6, 9, 15], coding theory[1, 6], computer science[9], and quantum information[4], as well as drug screening. In addition to their orthogonality, OAs of high strength have the projection properties of high order and uniformity, which are employed in theoretical studies and computer experiments.

Many statisticians and mathematicians [8] are devoting efforts to the construction of OAs. In addition, an increasing number of authors are focusing on designs with orthogonality, projection properties, and uniformity. The main methods in the literature include Latin squares, Hadamard matrices, difference schemes [7, 10], orthogonal decompositions of projection matrices [10, 16], and finite projective geometries [14]. Hedayat et al. [7] established the relations between difference schemes and OAs of strength t, and also proposed a method to construct symmetric OAs. Du et al.[5] also proposed the recursive construction of symmetrical OAs of strength m using orthogonal partitions.

In this study, we first use difference schemes and orthogonal partition to construct OAs of strength 3. In Section 2 we introduce concepts of OA, difference scheme and orthogonal partition. In Section 3 we give some lemmas, and construct some new OAs with strength 3 by using difference scheme, orthogonal partition and the Kronecker product.

2 Preliminary Notes

In order to present our results, we make some preparations. Let Z_s^k denote the k dimensional space over a ring $Z_s = \{0, 1, \cdots, s-1\}$. Let A^T be the transpose of matrix A and $(s) = (0, 1, \cdots, s-1)^T$. The Kronecker product $B \otimes C$ and Kronecker sum $\oplus$ are respectively defined as: $B \otimes C = (b_{ij}C)_{su\times tv}$ and $B \oplus C = (b_{ij} \oplus C)_{su\times tv}$ if $B = (b_{ij})_{s\times t}$ and $C = (c_{kl})_{u\times v}$ are based on Z_s.

Definition 2.1 *([11]) An $n \times m$ matrix A, having k_i columns with p_i (levels), $i = 1, \ldots, v$, $m = \sum_{i=1}^{v} k_i$, $p_i \neq p_j$, for $i \neq j$, is called an orthogonal array $OA(n, m, p_1^{k_1} p_2^{k_2} \cdots p_v^{k_v}, t)$ of strength t and size n, if each $n \times t$ submatrix of A contains all possible $1 \times t$ row vectors with the same frequency. An OA with $p_1 = p_2 = \cdots = p_v = p$ is called symmetric; otherwise, it is called asymmetric.*

Definition 2.2 *Let S be an Abelian group of order s and $G = S^k$. Suppose $E = \{(\underbrace{g, g, \cdots, g}_{k}) : g \in S\}$ is a subgroup of G and $E^{(i_1,i_2,\ldots,i_t)} = E|_{S^t}$, where $(i_1, i_2, ..., i_t) \subseteq \{1, 2, ..., k\}$. An $r \times k$ array D on (G, E) is a difference scheme of strength t if for every $r \times t$ subarray each coset of $E^{(i_1,i_2,\ldots,i_t)}$ is represented equally often when the rows of the subarray are viewed as elements of G^t. We denote such an array by $D_t(r, k, s)$ on S.*

Definition 2.3 *Let A be the orthogonal array $OA(n,m,p_1^{k_1}\ p_2^{k_2}\cdots p_v^{k_v},t)$ and $\{A_1,A_2,\ldots,A_u\}$ be a set of orthogonal arrays $OA(\frac{n}{u},m,\ p_1^{k_1}p_2^{k_2}\cdots p_v^{k_v},t_1)$. If $\bigcup_{i=1}^{u} A_i = A$ and $A_i\bigcap A_j=\emptyset$ for $i\neq j$, then $\{A_1,A_2,\ldots,A_u\}$ is said to be an orthogonal partition of strength t_1 of the OA A. In particular, when $t_1=0$, $\{A_1,A_2,\ldots,A_u\}$ is still an orthogonal partition of A of strength 0, A_i can be seen as an $OA(\frac{n}{u},m,\ p_1^{k_1}p_2^{k_2}\cdots p_v^{k_v},0)$ for $1\leq i\leq u$.*

For simplicity in the following descriptions, we introduce some notations: $(A_{j[l_1,l_2,\cdots,l_k]},n)=\begin{pmatrix} A_{jl_1}\otimes 1_n\\ A_{jl_2}\otimes 1_n\\ \cdots\\ A_{jl_k}\otimes 1_n\end{pmatrix}$, and $(m,A_{j[l_1,\cdots,l_k]})=\begin{pmatrix} 1_m\otimes A_{jl_1}\\ 1_m\otimes A_{jl_2}\\ \cdots\\ 1_m\otimes A_{jl_k}\end{pmatrix}$, where $l_1,\cdots,l_k,j,m$ and n are integers.

3 Main Results

First, we need the following auxiliary results.

Lemma 3.1 *[12] Let $\{A_1,A_2,\ldots,A_u\}$ be an orthogonal partition of strength t_1 of an $OA(n,k,\ p,t')$, and $\{B_1,B_2,\ldots,B_v\}$ be an orthogonal partition of strength t_2 of an $OA(m,s,p,t'')$. Let $h=l.c.m.\{u,v\}$. Then, the matrix $M=(1_{\frac{h}{u}}\otimes(A_{[1,2,\ldots,u]},\ \frac{m}{v}),\ 1_{\frac{h}{v}}\otimes(\frac{n}{u},\ B_{[1,2,\ldots,v]}))$ is an $OA(\frac{mnh}{uv},k+s,p,t)$, where $t=t_1+t_2+1\leq min\{t',t''\}$.*

Lemma 3.2 *([3]) Let D be a difference scheme $D_3(r,k,s)$ on S. If there is a column vector $\alpha=(a_1,a_2,\ldots,a_N)^T$ such that any element $x\in S$ appears $\frac{N}{s}$ times in $\{a_1,a_2,\ldots,a_N\}$, then $\alpha\oplus D$ is an orthogonal array $OA(Nr,k,s,3)$.*

Corollary 3.3 *Let D be a difference scheme $D_3(r,k,s)$ on $S=\{0,1,\ldots,s-1\}$, then $D\oplus(s)$ is also an orthogonal array $OA(rs,k,s,3)$.*

Proof. It follows from the properties of Kronecker product and permutation matrix.

Lemma 3.4 *([2]) If H is a Hadamard matrix of order k for $k\geq 4$, then $A=\begin{pmatrix} H_k\\ -H_k\end{pmatrix}$ is an orthogonal array $OA(2k,k,2,3)$. That is to say, if element 0 is replaced by element -1 of H_k and $-H_k$ to obtain matrix H'_k and $-H'_k$, respectively, then $B=\begin{pmatrix} H'_k\\ -H'_k\end{pmatrix}$ is an $OA(2k,k,2,3)$ with elements from $S=\{0,1\}$ and H'_k is also a difference scheme $D_3(k,k,2)$ on S.*

Lemma 3.5 *([3]) For a prime or prime power s, there is a difference scheme $D_3(s^2, s, s)$ on Z_s.*

Lemma 3.6 *([3]) A difference scheme $D_3(s^2, 4, s)$ exists on Z_s for any $s \geq 2$.*

In the following, we give some results of constructing orthogonal arrays.

Theorem 3.7 *There is an orthogonal partition with strength 1 of the orthogonal array obtained by Lemma 3.2.*

Proof. Assume that $L = D_3(r, k, s) \oplus (s)$ is obtained by Corollary 3.8, and D_i is the i-th row of difference scheme $D_3(r, k, s)$. Let $A_i = D_i \oplus (s)$, then it is clear that $\{A_1, A_2, \ldots, A_r\}$ is an orthogonal partition of L of strength 1.

Theorem 3.8 *If s is a power of two, then new asymmetric orthogonal arrays $OA(2s^3, s^2 + s, s^s 2^{s^2}, 3)$ can be constructed.*

Proof. By Lemmas 3.4 and 3.5, we can obtain difference schemes $D_3(s^2, s^2, 2)$ and $D_3(s^2, s, s)$ on Z_s where s is a power of two. From Corollary 3.3, we can get two orthogonal arrays $A = OA(2s^2, s^2, 2^{s^2}, 3)$ and $B = OA(s^3, s, s^s, 3)$. From Theorem 3.7, we know that there are orthogonal partitions $\{A_1, A_2, \ldots, A_{s^2}\}$ and $\{B_1, B_2, \ldots, B_{s^2}\}$ of strength 1 of A and B, respectively.

From Lemma 3.1, let $h = u = v = s^2$, $t_1 = t_2 = 1$, so we can construct $OA(2s^3, s^2 + s, s^s 2^{s^2}, 3) = ((A_{[1,2,\ldots,s^2]}, s), (2, B_{[1,2,\ldots,s^2]}))$.

Theorem 3.9 *If s is a power of two, then there are new asymmetric orthogonal arrays $OA(2s^3, s^2 + 4, s^4 2^{s^2}, 3)$.*

Proof. By Lemmas 3.4 and 3.6, we can get difference schemes $D_3(s^2, s^2, 2)$ and $D_3(s^2, 4, s)$ on Z_s where s is a power of two. From Corollary 3.3, we can obtain orthogonal arrays $A = OA(2s^2, s^2, 2^{s^2}, 3)$ and $B = OA(s^3, 4, s^4, 3)$. From Theorem 3.7, we know that $\{A_1, A_2, \ldots, A_{s^2}\}$ and $\{B_1, B_2, \ldots, B_{s^2}\}$ are orthogonal partitions of strength 1 of A and B, respectively.

From Lemma 3.1, let $h = u = v = s^2$, $t_1 = t_2 = 1$, so we can obtain $OA(2s^3, s^2 + 4, s^4 2^{s^2}, 3) = ((A_{[1,2,\ldots,s^2]}, s), (2, B_{[1,2,\ldots,s^2]}))$.

Theorem 3.10 *If s is s a prime power, then there are new asymmetric orthogonal arrays $OA(8s^3, 4s^2 + s, s^s 2^{4s^2}, 3)$.*

Proof. By Lemmas 3.4 and 3.5, we can obtain difference schemes $D_3(4s^2, 4s^2, 2)$ and $D_3(s^2, s, s)$ on Z_s where s is a prime or prime power. We can easily get $D_3(4s^2, s, s)$ from $D_3(s^2, s, s)$. From Corollary 3.3, there are two orthogonal arrays $A = OA(8s^2, 4s^2, 2^{4s^2}, 3)$ and $B = OA(4s^3, s, s^s, 3)$. From Theorem 3.7, we know that there are orthogonal partitions $\{A_1, A_2, \ldots, A_{4s^2}\}$ and $\{B_1, B_2, \ldots, B_{4s^2}\}$ of strength 1 of A and B, respectively.

From Lemma 3.1, let $h = u = v = 4s^2$, $t_1 = t_2 = 1$, so we can construct $OA(8s^3, 4s^2 + s, s^s 2^{4s^2}, 3) = ((A_{[1,2,\ldots,4s^2]}, s), (2, B_{[1,2,\ldots,4s^2]}))$.

Theorem 3.11 *If $s \geq 2$, then there are new asymmetric orthogonal arrays $OA(8s^3, 4s^2 + 4, s^4 2^{4s^2}, 3)$.*

Proof. By Lemmas 3.4 and 3.6, we can obtain difference schemes $D_3(4s^2, 4s^2, 2)$ and $D_3(s^2, 4, s)$ on Z_s where $s \geq 2$. We can easily get $D_3(4s^2, 4, s)$ from $D_3(s^2, 4, s)$. From Corollary 3.3, there are orthogonal arrays $A = OA(8s^2, 4s^2, 2^{4s^2}, 3)$ and $B = OA(4s^3, 4, s^4, 3)$. From Theorem 3.7, we know that $\{A_1, A_2, \ldots, A_{4s^2}\}$ and $\{B_1, B_2, \ldots, B_{4s^2}\}$ are orthogonal partitions of strength 1 of A and B, respectively.

From Lemma 3.1, let $h = u = v = 4s^2$, $t_1 = t_2 = 1$, so we can obtain $OA(8s^3, 4s^2 + 4, s^4 2^{4s^2}, 3) = ((A_{[1,2,\ldots,4s^2]}, s), (2, B_{[1,2,\ldots,4s^2]}))$.

Example 3.12 *Construction of new tight orthogonal arrays $OA(128, 22, 4^6 2^{16}, 3)$.*

$$\text{Let } D^T = \begin{pmatrix} 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&1&2&3&0&1&2&3&0&1&2&3&0&1&2&3 \\ 0&2&3&1&2&0&1&3&3&1&0&2&1&3&2&0 \\ 0&0&0&0&1&1&1&1&2&2&2&2&3&3&3&3 \\ 0&1&2&3&2&3&0&1&3&2&1&0&1&0&3&2 \\ 0&2&3&1&1&3&2&0&2&0&1&3&3&1&0&2 \end{pmatrix}.$$

It is clear that D is a difference scheme $D_3(16, 6, 4)$ on Z_4. By Corollary 3.8, we can obtain orthogonal array $A = OA(64, 6, 4, 3)$. Then $\{A_1, A_2, \ldots, A_{16}\}$ is an orthogonal partition of strength 1 of A by Theorem 3.7.

Suppose $B = OA(32, 16, 2, 3) = D_3(16, 16, 2) \oplus (2)$. Then $\{B_1, B_2, \ldots, B_{16}\}$ is an orthogonal partition of strength 1 of B.

It follows from Lemma 3.1 that $M = ((A_{[1,2,\ldots,16]}, 2),\ (4, B_{[1,2,\ldots,16]}))$ is an $OA(128, 22, 4^6 2^{16}, 3)$.

From the above ways, we can get the following table.

Table 1. Construction of orthogonal arrays

OA by difference scheme	OA by difference scheme	OAs	Ref
$OA(72, 36, 2, 3)$: $D_3(36, 36, 2) \oplus (2)$	$OA(216, 4, 6, 3)$: $D_3(36, 4, 6) \oplus (6)$	$OA(432, 40, 6^4 2^{36}, 3)$	Theorem 3.9
$OA(72, 36, 2, 3)$: $D_3(36, 36, 2) \oplus (2)$	$OA(108, 4, 3, 3)$: $D_3(36, 4, 3) \oplus (3)$	$OA(216, 40, 3^4 2^{36}, 3)$	Theorem 3.11
$OA(128, 64, 2, 3)$: $D_3(64, 64, 2) \oplus (2)$	$OA(512, 8, 8, 3)$: $D_3(64, 8, 8) \oplus (8)$	$OA(1024, 72, 8^8 2^{64}, 3)$	Theorem 3.8
$OA(128, 64, 2, 3)$: $D_3(64, 64, 2) \oplus (2)$	$OA(256, 6, 4, 3)$: $D_3(64, 6, 4) \oplus (4)$	$OA(512, 70, 4^6 2^{64}, 3)$	Lemma 3.4, Example 3.12
$OA(288, 144, 2, 3)$: $D_3(144, 144, 2) \oplus (2)$	$OA(12^3, 4, 12, 3)$: $D_3(144, 4, 12) \oplus (12)$	$OA(2 \times 12^3, 148, 12^4 2^{144}, 3)$	Theorem 3.9
$OA(288, 144, 2, 3)$: $D_3(144, 144, 2) \oplus (2)$	$OA(576, 6, 4, 3)$: $D_3(144, 6, 4) \oplus (4)$	$OA(1152, 150, 4^6 2^{144}, 3)$	Lemma 3.4, Example 3.12

Acknowledgements. This research is supported by the National Natural Science Foundation of China (grant no. 11571094).

References

[1] J. Bierbrauer, *Introduction to Coding Theory*, Chapman and Hall/CRC, Boca Raton, FL, 2005.

[2] A. E. Brouwer, A. M. Cohen and M. V. M. Nguyen, Orthogonal arrays of strength 3 and small run size, *J. Statist. Plann. Inference*, **136** (2006), 3268-3280. https://doi.org/10.1016/j.jspi.2004.12.012

[3] G. Chen and J. Lei, Constructions of mixed orthogonal arrays of strength three (in Chinese), *Sci. Sin. Math.*, **47** (2017), 545-564. https://doi.org/10.1360/012016-23

[4] G. Dardo and Z. Karol, Genuinely multipartite entangled states and orthogonal arrays, *Phys. Rev. A*, **90** (2014), no. 2, 022316. https://doi.org/10.1103/physreva.90.022316

[5] J. Du, S. Wang, Q. Wen and S. Pang, Recursive construction of symmetrical orthogonal arrays of strength m, *Acta Math. Appl. Sinica*, **35** (2012), no. 2, 232-244, (in Chinese).

[6] A. S. Hedayat, N. J. A. Sloane and J. Stufken, *Orthogonal Arrays: Theory and Applications*, Springer-Verlag, New York, 1999. https://doi.org/10.1007/978-1-4612-1478-6

[7] A.S. Hedayat, J. Stufken and G. Su, On difference schemes and orthogonal arrays of strength t, *J. Statist. Plann. Inference*, **56** (1996), no. 2, 307-324. https://doi.org/10.1016/s0378-3758(96)00026-2

[8] Y. He and B. Tang, A characterization of strong orthogonal arrays of strength three, *Ann. Statist.*, **42** (2014), no. 4, 1347-1360. https://doi.org/10.1214/14-aos1225

[9] L. Ji and J. Yin, Constructions of new orthogonal arrays and covering arrays of strength three, *J. Combin. Theory, Ser. A*, **117** (2010), no. 3, 236-247. https://doi.org/10.1016/j.jcta.2009.06.002

[10] S. Pang, *Construction Methods of Orthogonal Arrays and Their Applications*, Diss., (in Chinese), Xidian Univ. Xian, China, 2004.

[11] S. Pang, Y. Zhang and S. Liu, Further results on the orthogonal arrays obtained by generalized Hadamard product, *Statist. Probab. Lett.*, **68** (2004), no. 1, 17-25. https://doi.org/10.1016/j.spl.2004.03.001

[12] S. Pang, X. Lin and J. Wang, Construction of Asymmetric Orthogonal Arrays of Strength t from Orthogonal Partition of Small Orthogonal Arrays, *IEICE Transactions on Fundamentals of Electronics, Communications and Computer Sciences*, **E101** (2018), 1267-1272. https://doi.org/10.1587/transfun.e101.a.1267

[13] C. R. Rao, Factorial experiments derivable from combinatorial arrangements of arrays, *J. Roy. Statist. Soc.*, **9** (1947), 128-139. https://doi.org/10.2307/2983576

[14] C. Y. Suen and A. Dey, Construction of asymmetric orthogonal arrays through finite geometries, *J. Statist. Plann. Inference*, **115** (2003), no. 2, 623-635. https://doi.org/10.1016/s0378-3758(02)00165-9

[15] Y. Zhang, Orthogonal arrays obtained by repeating-column difference matrices, *Discrete Math.*, **307** (2007), no. 2, 246-261. https://doi.org/10.1016/j.disc.2006.06.029

[16] Y. Zhang, Y. Lu and S. Pang, Orthogonal arrays obtained by orthogonal decomposition of projection matrices, *Statist. Sinica*, **9** (1999), 595-604.

Received: December 18, 2018; Published: January 15, 2019