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The k-th Largest Numbers of Maximum Independent Sets in Quasi-Forest Graphs

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Abstract

Let $G = (V, E)$ be a simple undirected graph. A subset I of the vertex set $V(G)$ is *independent* if there is no edge of G between any two vertices of I. A *maximum independent set* is an independent set of maximum size. A graph G with vertex set $V(G)$ is called a *quasi-forest graph*, if there exists a vertex $x \in V(G)$ such that $G - x$ is a forest. In this paper we complete the determination of the k-th $(3 \leq k \leq \lfloor n/2 \rfloor)$ largest numbers of maximum independent sets among all quasi-forest graphs of order $n \geq 6$ and characterize the extremal graphs.

Mathematics Subject Classification: 05C51

Keywords: maximum independent set, tree, quasi-tree graph, forest, quasi-forest graph, extremal graph

1 Introduction

Let $G = (V, E)$ be a simple undirected graph. The vertex set of a graph G is referred to as $V(G)$, its edge set as $E(G)$. A subset $I \subseteq V(G)$ is an *independent set* of G if no two vertices of I are adjacent in G. An independent set I' of G is called *maximum* if G has no independent set I with $|I'| < |I|$. The set of all maximum independent sets of a graph G is denoted by $\mathrm{XI}(G)$ and its cardinality by $xi(G)$.

For notation and terminology in graphs we follow [1] in general. A graph is *connected* when there is a path between every pair of vertices. A *triangle-free graph* is a graph in which no three vertices form a triangle of edges. An acyclic graph, one not containing any cycles, is called a *forest*. A connected forest is called a *tree*. The problem of determining the largest number of maximum independent sets of a graph was studied for various classes of graphs, including general graphs, trees, forests, graphs with at most one cycle, graphs with at most r cycle, connected graphs and triangle-free graphs, see [3, 7]. Lin [5] investigated the second largest and the third largest cardinality of $xi(G)$ among all trees and forests. Recently, Lin and Jou [6] investigated the k-th largest cardinality of $xi(G)$ among all forests of order n.

A connected graph (respectively, graph) G with vertex set $V(G)$ is called a *quasi-tree graph* (respectively, *quasi-forest graph*), if there exists a vertex $x \in V(G)$ such that $G - x$ is a tree (respectively, forest). The problem of determining the largest and the second largest numbers of maximum independent sets among all quasi-tree graphs and quasi-forest graphs was solved by Lin [4].

The purpose of this paper is to determine the k-th $(3 \leq k \leq \lfloor n/2 \rfloor)$ largest numbers of maximum independent sets among all quasi-forest graphs of order $n \geq 6$. Extremal graphs achieving these values are also given.

2 Preliminary

In this section, we describe some notations and preliminary results. For a graph $G = (V, E)$, the cardinality of $V(G)$ is called the *order*, and it is denoted by $|G|$. A maximal connected subgraph of G is called a *component* of G. A component of odd (respectively, even) order is called an *odd* (respectively, *even*) *component*. For a set $A \subseteq V(G)$, the *deletion* of A from G is the graph $G - A$ obtained from G by removing all vertices in A and their incident edges.

Two graphs G_1 and G_2 are *disjoint* if $V(G_1) \cap V(G_2) = \emptyset$. The *union* of two disjoint graphs G_1 and G_2 is the graph $G_1 \cup G_2$ with vertex set $V(G_1 \cup G_2) = V(G_1) \cup V(G_2)$ and edge set $E(G_1 \cup G_2) = E(G_1) \cup E(G_2)$. Let nG be the short notation for the union of n copies of disjoint graphs isomorphic to G.

Denote by P_n a *path* with n vertices and C_n a *cycle* with n vertices. Throughout this paper, for simplicity, let $r = \sqrt{2}$.

Lemma 2.1. ([2], [3]) *If G is the union of two disjoint graphs G_1 and G_2, then $xi(G) = xi(G_1) \cdot xi(G_2)$.*

The result of the largest number of maximum independent sets among all trees is described in Theorem 2.2.

Theorem 2.2. ([2], [3]) *If T is a tree of order $n \geq 2$, then*

$$xi(T) \leq t_1(n) = \begin{cases} r^{n-2}+1, & \text{if } n \text{ is even,} \\ r^{n-3}, & \text{if } n \text{ is odd.} \end{cases}$$

Furthermore, $xi(T) = t_1(n)$ if and only if $T = T_1(n)$, where

$$T_1(n) = \begin{cases} T_{1e}(n), & \text{if } n \text{ is even,} \\ T_{1o}(n), & \text{if } n \text{ is odd.} \end{cases}$$

The graphs $T_{1e}(n)$ and $T_{1o}(n)$ are shown in Figure 1.

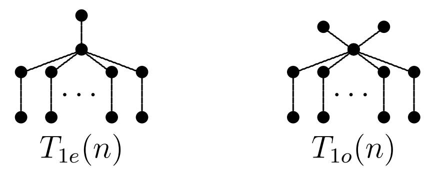


Figure 1: The graphs $T_{1e}(n)$ and $T_{1o}(n)$

Define the graph $F_i(n)$ of order $n \geq 2$, $i = 1, 2, \ldots, \lfloor n/2 \rfloor$ as follows.

$$F_i(n) = \begin{cases} T_{1e}(2i) \cup \frac{n-2i}{2} P_2, & \text{if } n \text{ is even,} \\ T_{1e}(2i) \cup P_1 \cup \frac{n-2i-1}{2} P_2, & \text{if } n \text{ is odd.} \end{cases}$$

Let $f_i(n) = xi(F_i(n))$. For simple calculation, we have that

$$f_i(n) = \begin{cases} r^{n-2} + r^{n-2i}, & \text{if } n \text{ is even,} \\ r^{n-3} + r^{n-2i-1}, & \text{if } n \text{ is odd.} \end{cases}$$

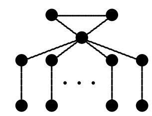
The result of the k-th ($1 \leq k \leq \lfloor n/2 \rfloor$) largest numbers of maximum independent sets among all forests is described in Theorem 2.3.

Theorem 2.3. ([2], [3], [6]) *For integers k, $n \geq 2$ and $1 \leq k \leq \lfloor n/2 \rfloor$. If F is a forest of order n having $F \neq F_i(n)$, for $i = 1, 2 \ldots, k-1$, then $xi(F) \leq f_k(n)$. Furthermore, $xi(F) = f_k(n)$ if and only if $F = F_k(n)$ or $2T_{1e}(4) \cup F_1(n-8)$ with $k = 4$.*

The result of the largest number of maximum independent sets among all quasi-tree graphs is described in Theorem 2.4.

Theorem 2.4. ([4]) *If Q is a quasi-tree graph of order $n \geq 2$, then*

$$xi(Q) \leq qt_1(n) = \begin{cases} r^{n-2}+1, & \text{if } n \text{ is even,} \\ r^{n-1}+1, & \text{if } n \text{ is odd.} \end{cases}$$



$QT_{1o}(n)$

Figure 2: The graph $QT_{1o}(n)$

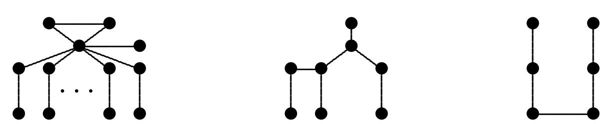
Furthermore, $xi(Q) = qt_1(n)$ *if and only if* $Q = QT_1(n)$*, where*

$$QT_1(n) = \begin{cases} T_{1e}(n), & \text{if } n \text{ is even,} \\ QT_{1o}(n) \text{ or } C_5, & \text{if } n \text{ is odd.} \end{cases}$$

The graph $QT_{1o}(n)$ *is shown in Figure* 2.

The result of the second largest number of maximum independent sets among all quasi-tree graphs of even order is described in Theorem 2.5.

Theorem 2.5. ([4]) *If* Q *is a quasi-tree graph of even order* $n \geq 6$ *having* $Q \neq QT_1(n)$*, then* $xi(Q) \leq r^{n-2}$ *with the equality holding if and only if* $Q = QT_{2e}(n)$ *or* T_8 *or* P_6*, where* $QT_{2e}(n)$*,* T_8 *and* P_6 *are shown in Figure* 3.



$QT_{2e}(n)$ T_8 P_6

Figure 3: The graphs $QT_{2e}(n)$, T_8 and P_6

The results of the largest and the second largest numbers of maximum independent sets among all quasi-forest graphs are described in Theorems 2.6 and 2.7, respectively.

Theorem 2.6. ([4]) *If* Q *is a quasi-forest graph of order* $n \geq 2$*, then*

$$xi(Q) \leq qf_1(n) = \begin{cases} r^n, & \text{if } n \text{ is even,} \\ 3r^{n-3}, & \text{if } n \text{ is odd.} \end{cases}$$

Furthermore, $xi(Q) = qf_1(n)$ *if and only if* $Q = QF_1(n)$*, where*

$$QF_1(n) = \begin{cases} F_1(n), & \text{if } n \text{ is even,} \\ C_3 \cup F_1(n-3), & \text{if } n \text{ is odd.} \end{cases}$$

Theorem 2.7. ([4]) *If Q is a quasi-forest of order $n \geq 4$ having $Q \neq QF_1(n)$, then*

$$xi(Q) \leq qf_2(n) = \begin{cases} 3r^{n-4}, & \text{if } n \text{ is even,} \\ 5r^{n-5}, & \text{if } n \text{ is odd.} \end{cases}$$

Furthermore, $xi(Q) = qf_2(n)$ if and only if $Q = QF_2(n)$, where

$$QF_2(n) = \begin{cases} F_2(n) \text{ or } C_3 \cup F_1(n-3), & \text{if } n \text{ is even,} \\ QT_{1o}(5) \cup F_1(n-5) \text{ or } C_5 \cup F_1(n-5), & \text{if } n \text{ is odd.} \end{cases}$$

3 Main results

In this section, we determine the k-th $(3 \leq k \leq \lfloor n/2 \rfloor)$ largest values of $xi(G)$ among all quasi-forest graphs of order $n \geq 6$. Moreover, the extremal graphs achieving these values are also determined.

Define the graphs $QF_i(n)$ of order $n \geq 6$, $i = 3, 4, \ldots, \lfloor n/2 \rfloor$ as follows.

$$QF_i(n) = \begin{cases} QT_{1o}(2i-1) \cup F_1(n-2i+1), & \text{if } n \text{ is even,} \\ QT_{1o}(2i+1) \cup F_1(n-2i-1), & \text{if } n \text{ is odd.} \end{cases}$$

Let $qf_i(n) = xi(QF_i(n))$. For simple calculation, we have that

$$qf_i(n) = \begin{cases} r^{n-2} + r^{n-2i}, & \text{if } n \text{ is even,} \\ r^{n-1} + r^{n-2i-1}, & \text{if } n \text{ is odd.} \end{cases}$$

Lemma 3.1. *If Q is a quasi-forest graph of odd order $n \geq 7$ with $Q \neq QF_i(n)$ for $i = 1, 2 \ldots, k-1$ and $3 \leq k \leq (n-1)/2$, then $xi(Q) \leq qf_k(n)$. Furthermore, the equality holds if and only if $Q = QF_k(n)$ or $C_3 \cup F_2(n-3)$ for $k = 3$.*

Proof. Since $f_1(n) < qf_k(n)$ for n is odd, by Theorem 2.3, we assume that Q is not a forest. Then there exists a component H containing at least one cycle, where $|H| = m$. We consider the following two cases.

Case 1: m is even. Since H contains at least one cycle, it follows that $H \neq QT_1(m)$. By Lemma 2.1, Theorems 2.3 and 2.5, we have that

$$\begin{aligned} xi(Q) &= xi(H \cup (Q - V(H))) \\ &= xi(H) \cdot xi(Q - V(H)) \\ &\leq r^{m-2} \cdot r^{n-m-1} \\ &= r^{n-3} \\ &< r^{n-1} + r^{n-2k-1} \\ &= qf_k(n). \end{aligned}$$

Case 2: m is odd. Since H contains at least one cycle, it follows that $m \geq 3$. For the case that $Q-V(H) \neq F_1(n-m)$, by Lemma 2.1, Theorems 2.3 and 2.4, we have that

$$\begin{aligned}
xi(Q) &= xi(H \cup (Q - V(H))) \\
&= xi(H) \cdot xi(Q - V(H)) \\
&\leq qt_1(m) \cdot f_2(n-m) \\
&= (r^{m-1} + 1) \cdot 3r^{n-m-4} \\
&= 3r^{n-5} + 3r^{n-m-4} \\
&\leq 3r^{n-5} + 3r^{n-7} \\
&= 9r^{n-7} \\
&= qf_3(n).
\end{aligned}$$

Furthermore, the equalities holding imply that $m = k = 3$, $H = C_3$ and $Q - V(H) = F_2(n-3)$, that is, $Q = C_3 \cup F_2(n-3)$.

On the other hand, we assume that $Q - V(H) = F_1(n-m)$. Since $Q \neq QF_i(n)$ for $i = 1, 2, \ldots, k-1$, by Lemma 2.1, Theorems 2.3 and 2.4, we have that

$$\begin{aligned}
xi(Q) &= xi(H \cup (Q - V(H))) \\
&= xi(H) \cdot xi(Q - (V(H))) \\
&\leq \begin{cases} (qt_1(m) - 1) \cdot f_1(n-m), & \text{if } m \leq 2k-1, \\ qt_1(m) \cdot f_1(n-m), & \text{if } m \geq 2k+1, \end{cases} \\
&= \begin{cases} r^{m-1} \cdot r^{n-m}, & \text{if } m \leq 2k-1, \\ (r^{m-1} + 1) \cdot r^{n-m}, & \text{if } m \geq 2k+1, \end{cases} \\
&= \begin{cases} r^{n-1}, & \text{if } m \leq 2k-1, \\ r^{n-1} + r^{n-m}, & \text{if } m \geq 2k+1, \end{cases} \\
&\leq r^{n-1} + r^{n-2k-1} \\
&= qf_k(n).
\end{aligned}$$

Furthermore, the equalities holding imply that $m = 2k+1$, $H = QT_{1o}(2k+1)$ and $Q - V(H) = F_1(n-2k-1)$. In conclusion, $Q = QF_k(n) = QT_{1o}(2k+1) \cup F_1(n-2k-1)$. □

Lemma 3.2. *If Q is a quasi-forest graph of even order $n \geq 6$ with $Q \neq QF_i(n)$ for $i = 1, 2 \ldots, k-1$ and $3 \leq k \leq n/2$, then $xi(Q) \leq qf_k(n)$. Furthermore, the equality holds if and only if $Q = F_k(n)$ or $QF_k(n)$ or $C_5 \cup F_1(n-5)$ for $k = 3$ or $2T_{1e}(4) \cup F_1(n-8)$, $C_3 \cup F_2(n-3)$ for $k = 4$.*

Proof. Since $f_k(n) = qf_k(n)$ for n is even, by Theorem 2.3, we assume that Q is not a forest. We consider the following two cases.

Case 1: All components of Q are even. Let H' be an even component containing at least one cycle, where $|H'| = m'$. Note that $H' \neq QT_1(m')$. By Lemma 2.1, Theorems 2.3 and 2.5, we have that

$$\begin{aligned} xi(Q) &= xi(H' \cup (Q - V(H'))) \\ &= xi(H') \cdot xi(Q - V(H')) \\ &\leq r^{m'-2} \cdot r^{n-m'} \\ &= r^{n-2} \\ &< r^{n-2} + r^{n-2k} \\ &= qf_k(n). \end{aligned}$$

Case 2: There is an odd component H'' of Q, where H'' is a tree of order m''. Suppose that $m'' \geq 3$, by Lemma 2.1, Theorems 2.2 and 2.6, then

$$\begin{aligned} xi(Q) &= xi(H'' \cup (Q - V(H''))) \\ &= xi(H'') \cdot xi(Q - (V(H''))) \\ &\leq r^{m''-3} \cdot 3r^{n-m''-3} \\ &= 3r^{n-6} \\ &< qf_k(n). \end{aligned}$$

For the case that $m'' = 1$, by Theorem 2.7 and Lemma 3.1, we have that $Q - V(H) = QT_{1o}(2k-1) \cup F_1(n-2k)$ or $C_5 \cup F_1(n-6)$ for $k = 3$ or $C_3 \cup F_2(n-4)$ for $k = 4$. In conclusion, $Q = QF_k(n) = QT_{1o}(2k-1) \cup F_1(n-2k+1)$ or $C_5 \cup F_1(n-5)$ for $k = 3$ or $C_3 \cup F_2(n-3)$ for $k = 4$. □

The result for the k-th ($3 \leq k \leq \lfloor n/2 \rfloor$) largest numbers of maximal independent sets among all quasi-forest graphs, now follow from Lemmas 3.1 and 3.2, and it is summarized in the following theorem.

Theorem 3.3. *For integers $k, n \geq 6$ and $3 \leq k \leq \lfloor n/2 \rfloor$. If Q is a quasi-forest graph of order n with $Q \neq QF_i(n)$, for $i = 1, 2 \ldots, k-1$, then $xi(Q) \leq qf_k(n)$. Furthermore, the equality holds if and only if*

(1) $Q = QF_k(n)$,

(2) $Q = C_3 \cup F_2(n-3)$ *for n is odd and $k = 3$,*

(3) $Q = F_k(n)$ *for n is even,*

(4) $Q = C_5 \cup F_1(n-5)$ *for n is even and $k = 3$,*

(5) $Q = 2T_{1e}(4) \cup F_1(n-8)$ *for n is even and $k = 4$,*

(6) $Q = C_3 \cup F_2(n-3)$ *for n is even and $k = 4$.*

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