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Boundary Values of Harmonic Functions and Modulus of Continuity (Part I)

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Abstract

In this paper, we will present a few theorems that will lead us to Theorem 1.3, the first part of which is about a necessary and sufficient condition for a harmonic function u on the open unit disk to be uniquely represented as the Poisson integral of a continuous function f on the unit circle in terms of modulus of continuity.

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1 Introduction

Let U denote the open unit disk and T denote the unit circle on the complex plane $\mathbb{C}$. Let $\mathbf{R}$ denote the set of all real numbers. We will first state a set of definitions about modulus of functions followed by another set of definitions on measure-limit.

Definition 1.1 *(a) The r-th order modulus of continuity of a function $f \in L^p(T), 1 \leq p < +\infty$, is defined by*

$$\omega_r(t; f)_p = \sup_{0\leq h\leq t} \|\Delta_h^r f\|_p,$$

where $(\Delta_h f)(e^{i\theta}) = f(e^{i(\theta+h)}) - f(e^{i\theta})$ *and for* $r = 1, 2, 3, \ldots$ *and* $t \geq 0$, *let*

$$(\Delta_h^{r+1} f)(e^{i\theta}) = (\Delta_h(\Delta_h^r f))(e^{i\theta}) = (\Delta_h^r f)(e^{i(\theta+h)}) - (\Delta_h^r f)(e^{i\theta}).$$

When $p = +\infty$, *the* L^p*-norm is replaced by the supremum norm in* $C(T)$, *i.e.* $\omega_r(t; f)_\infty = \sup_{0 \leq h \leq t} \|\Delta_h^r f\|_{C(T)}$, *where* $t \geq 0$ *and* $f \in C(T)$.

(b) For a harmonic function $u : U \to \mathbb{C}$, *we define the* β*-th order harmonic modulus of continuity of* u *to be* $\omega^*_\beta(t; u)_p = \limsup_{r \to 1^-} \omega_\beta(t; u_r)_p$ *for every* $p \in [1, +\infty]$, *where* u_r *is the restriction of* u *to* $\partial B(0, r)$.

(c) For a bounded function g *on* T, *the local modulus of continuity of* g *is defined for every* $t \in [0, 2\pi]$ *as* $\omega(t; \cdot; g) : T \to \mathbb{R}$ *given by*

$$\omega(t; e^{i\theta}; g) = \sup_{\alpha_1, \alpha_2 \in [\theta - t/2, \theta + t/2]} |g(e^{i\alpha_1}) - g(e^{i\alpha_2})|.$$

Moreover, the averaged modulus of continuity of g *is defined to be*

$$\tau(t; g) = \int_T \omega(t; z; g) d\sigma(z) = \|\omega(t; \cdot; g)\|_1$$

for every $t \in [0, 2\pi]$, *where* σ *is the normalized Lebesgue measure on* T.

(d) For $u \in Har(U)$, *we define the harmonic averaged modulus of continuity of* u *to be* $\tau^*(t; u) = \limsup_{r \to 1^-} \tau(t; u^r)$ *for every* $t \in [0, 2\pi]$.

Definition 1.2 *For* $g : T \to \mathbb{C}$ *measurable and* $z_0 \in T$,

(a) a complex number $L \in \mathbb{C}$ *is said to be a measure-limit of* g *at* z_0 *if for any* $\varepsilon > 0$, *there is an open arc* $I \subseteq T$ *centered at* z_0 *such that* $\sigma(I \cap \{z \in T : |g(z) - L| \geq \varepsilon\}) = 0$,

(b) g *is said to have measure-limit everywhere on* T *if for every* $z \in T$, *there exists* $L \in \mathbb{C}$ *such that* L *is a measure-limit of* g *at* z.

In terms of modulus of continuity, we will give necessary and sufficient conditions for a harmonic function $u : U \to \mathbf{R}$ to be represented by the Poisson integral of a continuous function or a Riemann integrable function on T. Below we would like to present part (1) of the following theorem and in a subsequent paper, we will present part (2). Here is the main result:

Theorem 1.3 *For* $u \in Har(U)$,

(1) $\lim_{t \to 0+} \omega^*_1(t; u)_\infty = 0$ *iff there is a unique* $f \in C(T)$ *such that* $u = P[f]$,

(2) $\lim_{t \to 0+} \tau^*(t; u) = 0$ *iff there is a unique* $f \in R(T)$ *such that* $u = P[f]$,

where $P[f]$ *denote the Poisson integral of* f *and* $R(T)$ *is the class of all Riemann integrable functions on* T.

For remarks, we first observe that if $L_1, L_2 \in \mathbb{C}$, are measurable-limits of g at $z_0 \in T$, then $L_1 = L_2$. The reason is as follows. Suppose $L_1 \neq L_2$. Let $\varepsilon_1 = |L_1 - L_2|/2 > 0$ and there are two open arcs $I_1, I_2 \subseteq T$ centered at $z_0 \in T$ such that $\sigma(I_i \cap \{z \in T : |g(z) - L_i| \geq \varepsilon_1\} = 0$ for $i = 1, 2$. Then

$$\sigma(I_1 \cap I_2 \cap (\{z \in T : |g(z) - L_1| \geq \varepsilon_1\} \cup \{z \in T : |g(z) - L_2| \geq \varepsilon_1\})) = 0.$$

So $|g(z_1) - L_1| < \varepsilon_1$ and $|g(z_1) - L_2| < \varepsilon_1$ for some $z_1 \in I_1 \cap I_2$. Then

$$|L_1 - L_2| \leq |g(z_1) - L_1| + |L_2 - g(z_1)| < 2\varepsilon_1 = |L_1 - L_2|,$$

which is a contradiction. Next we will prove

Theorem 1.4 *For measurable $f^* : T \to \mathbb{C}$, f^* has measure-limit everywhere on T if and only if there exists a $g \in C(T)$ such that $f^* = g$ a.e. $[\sigma]$ on T. (Note that there exists a $f^* : T \to \mathbb{C}$ measurable function such that there does not exist any $g \in C(T)$ with $f^* = g$ almost everywhere on T, for example, the characteristic function χ_A where $A = \{e^{i\theta} : 0 \leq \theta \leq \pi\}$.)*

Proof. Suppose that $f^* : T \to \mathbb{C}$ is measurable and $f^* = g$ a.e. $[\sigma]$ on T for some $g \in C(T)$. Let $e^{i\theta} \in T$ and $\varepsilon_0 > 0$. Then there exists an open arc $I \subseteq T$ centered at $e^{i\theta}$ such that for all $z \in I$, $|g(e^{i\theta}) - g(z)| < \varepsilon_0$. Since $f^* = g$ a.e. $[\sigma]$ on T, for almost all $z \in I$, $|f^*(z) - g(e^{i\theta})| < \varepsilon_0$. So

$$\sigma(I \cap \{z \in T : |f^*(z) - g(e^{i\theta})| \geq \varepsilon_0\}) = 0.$$

Therefore, for every $e^{i\theta} \in T$, $g(e^{i\theta})$ is the measure-limit of f^* at $e^{i\theta}$.

Conversely, suppose $f^* : T \to \mathbb{C}$ is measurable and f^* has measure-limit everywhere on T. By remark above, f^* has a unique measure-limit at each point $z \in T$. Hence we may define $L : T \to \mathbb{C}$ by $L(z)$ to be the measure-limit of f^* at z for every $z \in T$.

We claim $f^* \in L^\infty(T)$. To see that take $\varepsilon = 1$. For all $z \in T$, there is an open arc $I_z \subseteq T$ centered at z such that $\sigma(I_z \cap \{\alpha \in T : |f^*(\alpha) - L(z)| \geq \varepsilon\}) = 0$. Since $\{I_z : z \in T\}$ is an open cover of T, by the compactness of T, there exists a finite subset $G \subseteq T$ such that $T = \cup\{I_z : z \in G\}$. Hence, for almost all $\alpha \in T$, $|f^*(\alpha)| < M$, where $M = \max\{1 + |L(w)| : w \in G\}$.

Next we claim that for every $\varepsilon > 0$ and $e^{i\theta} \in T$, there exists an open arc $I \subseteq T$ centered at $e^{i\theta} \in T$ such that for every open arc $I' \subseteq I$ (the center of I' may not be $e^{i\theta}$), we have

$$\Big|\frac{\int_{I'} f^*(w) d\sigma(w)}{\sigma(I')} - L(e^{i\theta})\Big| < \varepsilon.$$

By the previous claim $f^* \in L^\infty(T)$, so $\int_{I'} f^*(w)\, d\sigma(w)$ exists. Let $e^{i\theta} \in T$ and $\varepsilon_0 > 0$. Then there exists an open arc $I \subseteq T$ centered at $e^{i\theta}$ such that

$$\sigma(I \cap \{z \in T : |f^*(z) - L(e^{i\theta})| \geq \varepsilon_0/2\}) = 0.$$

For open arc $I' \subseteq I$, let $A = I' \cap \{z \in T : |f^*(z) - L(e^{i\theta})| < \varepsilon_0/2\}$, $B = I' \cap \{z \in T : |f^*(z) - L(e^{i\theta})| \geq \varepsilon_0/2\}$. Then $\sigma(A) = \sigma(I')$ and $\sigma(B) = 0$. For every open arc $I' \subseteq I$,

$$\begin{aligned}
&\Big|\frac{\int_{I'} f^*(w)d\sigma(w)}{\sigma(I')} - L(e^{i\theta})\Big| \\
= \ & \Big|\frac{\int_{I'} f^*(w)d\sigma(w) - L(e^{i\theta})d\sigma(w)}{\sigma(I')}\Big| \leq \frac{\int_{I'} |f^*(w) - L(e^{i\theta})|d\sigma(w)}{\sigma(I')} \\
= \ & \frac{\int_A |f^*(w) - L(e^{i\theta})|d\sigma(w)}{\sigma(I')} + \frac{\int_B |f^*(w) - L(e^{i\theta})|d\sigma(w)}{\sigma(I')} \\
\leq \ & \frac{\varepsilon_0}{2} \cdot \frac{\sigma(A)}{\sigma(I')} + 0 < \varepsilon_0.
\end{aligned}$$

As a consequence, for every $e^{i\theta} \in T$, the limit of $(\int_I f^*(w)d\sigma(w))/\sigma(I)$ goes to $L(e^{i\theta})$ as the open arcs I centered at $e^{i\theta}$ shrink to $e^{i\theta}$.

Next we will show $L : T \to \mathbb{C}$ is continuous on T. Let $e^{i\theta} \in T$ and $\varepsilon_0 > 0$. By the second claim, we see that there exists an open arc $I \subseteq T$ centered at $e^{i\theta}$ such that for every open arc $I' \subseteq I$ (the center of I' may not be at $e^{i\theta}$),

$$\Big|\frac{\int_{I'} f^*(w)d\sigma(w)}{\sigma(I')} - L(e^{i\theta})\Big| < \frac{\varepsilon_0}{2}.$$

Let $w_0 \in I$. By the second claim, there exists an open arc $J \subseteq T$ centered at w_0 such that for every open arc $J' \subseteq J$ centered at w_0,

$$\Big|\frac{\int_{J'} f^*(w)d\sigma(w)}{\sigma(J')} - L(w_0)\Big| < \frac{\varepsilon_0}{2}.$$

Then there exists an open arc $K \subseteq I \cap J$ centered at w_0. Hence,

$$\begin{aligned}
|L(w_0) - L(e^{i\theta})| &\leq \Big|L(e^{i\theta}) - \frac{\int_K f^*(w)d\sigma(w)}{\sigma(K)}\Big| + \Big|\frac{\int_K f^*(w)d\sigma(w)}{\sigma(K)} - L(w_0)\Big| \\
&< \frac{\varepsilon_0}{2} + \frac{\varepsilon_0}{2} = \varepsilon_0.
\end{aligned}$$

Finally, L is continuous on T. Since $f^* \in L^\infty(T)$, the limit of $\int_I f^*(w)d\sigma(w))/\sigma(I)$ goes to $f^*(e^{i\theta})$ for every Lebesgue point $e^{i\theta}$ of f^*, where the limit is taken as the open arcs I centered at $e^{i\theta}$ shrink to $e^{i\theta}$. By the third claim, $L = f^*$ a.e. $[\sigma]$ on T and we are done.

Next we recall the following lemma (which can be found from [1], [2], [3]).

Lemma 1.5 *If $f \in L^p(T)$ with $p \in [1, +\infty)$ or $f \in C(T)$ with $p = +\infty$, then for all positive real t, all positive integers β and n, we have $\omega_\beta(nt; f)_p \leq n^\beta \omega_\beta(t; f)_p$.*

Theorem 1.6 *Let $u \in Har(U)$, $p \in [1, +\infty]$ and β be a positive integer. Then*

(a) for every $t \in [0, 2\pi]$ and $0 \leq r_1 < r_2 < 1$, $\omega_\beta(t; u_{r_1})_p \leq \omega_\beta(t; u_{r_2})_p$.

(b) for every $t \in (0, 2\pi]$, $\sup_{r\in(0,1)} \|u_r - u(0)\|_p \leq \left(\frac{4\pi}{t}\right)^\beta \omega^*_\beta(t; u)_p$.

Proof. (a) Let $u \in Har(U)$ and β be a positive integer. In case $p \in [1, +\infty)$, for every $h \in [0, +\infty)$ and $z \in U$, let $(T_h u)(z) = u(ze^{ih})$, $(Iu)(z) = u(z)$, then

$$((T_h - I)^\beta u) = (\Delta_h^\beta u)(z) = \sum_{j=0}^{\beta} \left(\begin{array}{c} \beta \\ j \end{array}\right) (-1)^{\beta-j} u(ze^{ijh}).$$

Hence $\Delta_h^\beta u \in Har(U)$. Then $|\Delta_h^\beta u|$ is a subharmonic function on U. Since x^p is an increasing convex function on $[0, +\infty)$, by Jensen's inequality, $|\Delta_h^\beta u|^p$ is subharmonic on U. By a theorem stated in [4], pp. 337, $\|\Delta_h^\beta u_{r_1}\|_p \leq \|\Delta_h^\beta u_{r_2}\|_p$ for every $h \in [0, +\infty), 0 \leq r_1 < r_2 < 1$. So $\omega_\beta(t; u_{r_1})_p \leq \omega_\beta(t; u_{r_2})_p$ for all $t \in [0, +\infty)$ and $0 \leq r_1 < r_2 < 1$.

For $p = +\infty$, since $\Delta_h^\beta u$ is harmonic in U, for all $0 \leq r_1 < r_2 < 1$, $\theta \in \mathbb{R}$,

$$(\Delta_h^\beta u)(r_1 e^{i\theta}) = \frac{1}{2\pi} \int_{t=-\pi}^{t=\pi} \frac{r_2^2 - r_1^2}{r_2^2 - 2r_2 r_1 \cos(\theta - t) + r_1^2} (\Delta_h^\beta u)(r_2 e^{it}) dt.$$

Hence, $\|\Delta_h^\beta u_{r_1}\|_{C(T)} \leq \|\Delta_h^\beta u_{r_2}\|_{C(T)}$. So $\omega_\beta(t; u_{r_1})_\infty \leq \omega_\beta(t; u_{r_2})_\infty$ for all $t \in [0, +\infty)$ and $0 \leq r_1 < r_2 < 1$.

(b) Let $u \in Har(U), p \in [1, +\infty], r \in (0, 1), \theta$ be real and β be a positive integer. We have

$$\begin{aligned} & \frac{1}{2\pi} \int_{h=0}^{h=2\pi} (\Delta_h^\beta u_r)(e^{i\theta}) dh \\ = \;& \frac{1}{2\pi} \int_{h=0}^{h=2\pi} \sum_{j=0}^{\beta} \left(\begin{array}{c} \beta \\ j \end{array}\right) (-1)^{\beta-j} u(re^{i\theta} e^{ijh}) dh \end{aligned}$$

$$\begin{aligned}
&= (-1)^\beta u(re^{i\theta}) + \sum_{j=1}^{\beta} \begin{pmatrix} \beta \\ j \end{pmatrix} (-1)^{\beta-j} \frac{1}{2\pi}\frac{1}{j}\int_{w=0}^{w=2j\pi} u(re^{i\theta}e^{iw})dw \\
&= (-1)^\beta u(re^{i\theta}) + \sum_{j=1}^{\beta} \begin{pmatrix} \beta \\ j \end{pmatrix} (-1)^{\beta-j} u(0) \\
&= (-1)^\beta u(re^{i\theta}) - (-1)^\beta u(0) = (-1)^\beta (u(re^{i\theta}) - u(0)).
\end{aligned}$$

In case $p \in [1, +\infty)$, for every $r \in (0, 1)$,

$$\begin{aligned}
\|u_r - u(0)\|_p &= \Big(\frac{1}{2\pi}\int_{\theta=0}^{\theta=2\pi} |u_r(e^{i\theta}) - u(0)|^p d\theta\Big)^{1/p} \\
&= \Big(\frac{1}{2\pi}\int_{\theta=0}^{\theta=2\pi} \Big|\frac{1}{2}\int_{h=0}^{h=2\pi} (\Delta_h^\beta u_r)(e^{i\theta})dh\Big|^p d\theta\Big)^{1/p} \\
&\le \Big(\frac{1}{2\pi}\int_{\theta=0}^{\theta=2\pi} \frac{1}{2}\int_{h=0}^{h=2\pi} \Big|(\Delta_h^\beta u_r)(e^{i\theta})\Big|^p dh\ d\theta\Big)^{1/p} \quad \textit{by Jensen} \\
&= \Big(\frac{1}{2\pi}\int_{h=0}^{h=2\pi} \frac{1}{2}\int_{\theta=0}^{\theta=2\pi} \Big|(\Delta_h^\beta u_r)(e^{i\theta})\Big|^p d\theta\ dh\Big)^{1/p} \quad \textit{by Fubini} \\
&= \Big(\frac{1}{2\pi}\int_{h=0}^{h=2\pi} \|\Delta_h^\beta u_r\|_p^p\ dh\Big)^{1/p}.
\end{aligned}$$

Let $t \in (0, 2\pi]$. Then $\frac{2\pi}{n+1} < t \le \frac{2\pi}{n}$ for some positive integer n. By lemma 1.5,

$$\begin{aligned}
\omega_\beta(2\pi; u_r)_p &= \omega_\beta\Big((n+1)\frac{2\pi}{n+1}; u_r\Big)_p \le (n+1)^\beta \omega_\beta\Big(\frac{2\pi}{n+1}; u_r\Big)_p \\
&\le (n+1)^\beta \omega_\beta(t; u_r)_p = \Big(\frac{n+1}{n}\Big)^\beta n^\beta \omega_\beta(t; u_r)_p \\
&\le \Big(\frac{n+1}{n}\Big)^\beta \Big(\frac{2\pi}{t}\Big)^\beta \omega_\beta(t; u_r)_p \le \Big(\frac{4\pi}{t}\Big)^\beta \omega_\beta(t; u_r)_p
\end{aligned}$$

So for every $r \in (0, 1)$,

$$\|u_r - u(0)\|_p \le \Big(\frac{1}{2\pi}\int_{h=0}^{h=2\pi} \|\Delta_h^\beta u_r\|_p^p\, dh\Big)^{1/p} \le (\sup_{0\le h\le 2\pi} \|\Delta_h^\beta u_r\|_p^p)^{1/p} \le (4\pi)^\beta \frac{\omega_\beta(t; u_r)_p}{t^\beta}.$$

Then for every $t \in (0, 2\pi]$,

$$\begin{aligned}
\sup_{r\in(0,1)} \|u_r - u(0)\|_p &\le \sup_{r\in(0,1)} \Big(\frac{4\pi}{t}\Big)^\beta \omega_\beta(t; u_r)_p = \Big(\frac{4\pi}{t}\Big)^\beta \sup_{r\in(0,1)} \omega_\beta(t; u_r)_p \\
&= \Big(\frac{4\pi}{t}\Big)^\beta \limsup_{r\to 1^-} \omega_\beta(t; u_r)_p = \Big(\frac{4\pi}{t}\Big)^\beta \omega_\beta^*(t; u)_p.
\end{aligned}$$

In case $p = +\infty$, for every $r \in (0, 1)$ and real θ,

$$
\begin{aligned}
|u_r(e^{i\theta}) - u(0)| &= \Big|\frac{1}{2\pi}\int_{h=0}^{h=2\pi}(\Delta_h^\beta u_r)(e^{i\theta})\ dh\Big| \le \frac{1}{2\pi}\int_{h=0}^{h=2\pi}\Big|(\Delta_h^\beta u_r)(e^{i\theta})\Big|dh \\
&\le \sup_{0\le h\le 2\pi}\|\Delta_h^\beta u_r\|_\infty = \omega_\beta(2\pi; u_r)_\infty.
\end{aligned}
$$

So for $r \in (0,1)$, $\|u_r - u(0)\|_\infty \le \omega_\beta(2\pi; u_r)_\infty$. Let $t \in (0, 2\pi]$. Similar as above, we have $\omega_\beta(2\pi; u_r)_\infty \le (4\pi/t)^\beta \omega_\beta(t; u_r)_\infty$. Then for every $t \in (0, 2\pi]$,

$$
\begin{aligned}
\sup_{r\in(0,1)}\|u_r - u(0)\|_\infty &\le \Big(\frac{4\pi}{t}\Big)^\beta \sup_{r\in(0,1)}\omega_\beta(t; u_r)_\infty = \Big(\frac{4\pi}{t}\Big)^\beta \limsup_{r\to 1^-}\omega_\beta(t; u_r)_\infty \\
&= \Big(\frac{4\pi}{t}\Big)^\beta \omega_\beta^*(t; u)_\infty.
\end{aligned}
$$

Theorem 1.7 *(a) For $u \in Har(U), p \in [1, +\infty]$ and positive integer β, if $\omega_\beta^*(t; u)_p = o(t^\beta)$ as $t \to 0^+$, then u is a constant function on U.*

(b) For positive integer β, $f^* \in L^p(T)$ if $p \in [1, +\infty)$ or $f^* \in C(T)$ if $p = +\infty$, we have if $\omega_\beta^*(t; u)_p = o(t^\beta)$ as $t \to 0^+$, then $f^* : T \to \mathbb{C}$ is a constant function.

Proof. Part (a) of the theorem follows immediately from Theorem 1.6(b).

For part (b), we claim for all $t \in (0, 2\pi]$, $\omega_\beta^*(t; u)_p = \omega_\beta(t; f^*)_p$, where $u = P[f^*]$ is the Poisson integral of f^*. It is not difficult to show that for $p \in [1, +\infty)$, $\lim_{r\to 1^-}\|u_r - f^*\|_p = 0$, where u_r is the restriction of $u = P[f^*]$ to $\partial B(0, r)$. Moreover, if $f^* \in C(T)$, then by the uniform continuity of $\tilde{u}$ ($\tilde{u}$ is the continuous extension of u to $\bar{U}$ with $\tilde{u}|_T = f^*$), we get $\lim_{r\to 1^-}\|u_r - f^*\|_\infty = 0$. For every $h \in [0, +\infty)$, we have for all $z \in U$ and $\omega \in T$,

$$
(\Delta_h^\beta u)(z) = \sum_{j=0}^{\beta}\begin{pmatrix}\beta\\ j\end{pmatrix}(-1)^{\beta-j}u(ze^{ijh}),\ (\Delta_h^\beta f^*)(w) = \sum_{j=0}^{\beta}\begin{pmatrix}\beta\\ j\end{pmatrix}(-1)^{\beta-j}f^*(we^{ijh}).
$$

So for all $r \in (0,1)$ and all $e^{i\theta} \in T$,

$$
(\Delta_h^\beta u_r)(e^{i\theta}) - (\Delta_h^\beta f^*)(e^{i\theta}) = \sum_{j=0}^{\beta}\begin{pmatrix}\beta\\ j\end{pmatrix}(-1)^{\beta-j}[u_r(e^{i\theta}e^{ijh}) - f^*(e^{i\theta}e^{ijh})].
$$

Then for all $r \in (0,1)$,

$$
\|\Delta_h^\beta u_r - \Delta_h^\beta f^*\|_p \le \sum_{j=0}^{\beta}\begin{pmatrix}\beta\\ j\end{pmatrix}\|u_r - f^*\|_p = 2^\beta\|u_r - f^*\|_p.
$$

Therefore, $\lim_{r\to 1^-}\|\Delta_h^\beta u_r\|_p = \|\Delta_h^\beta f^*\|_p$. By the proof of Theorem 1.6(a), we know for all $h \in [0,+\infty)$ and $0 \le r_1 < r_2 < 1$, $\|\Delta_h^\beta u_{r_1}\|_p \le \|\Delta_h^\beta u_{r_2}\|_p$. So $\sup_{r\in(0,1)}\|\Delta_h^\beta u_r\|_p = \lim_{r\to 1^-}\|\Delta_h^\beta u_r\|_p = \|\Delta_h^\beta f^*\|_p$. Then

$$\begin{aligned}\omega_\beta^*(t;u)_p &= \limsup_{r\to 1^-}\omega_\beta(t;u_r)_p = \limsup_{r\to 1^-}\sup_{0\le h\le t}\|\Delta_h^\beta u_r\|_p = \sup_{r\in(0,1)}\sup_{0\le h\le t}\|\Delta_h^\beta u_r\|_p \\ &= \sup_{0\le h\le t}\sup_{r\in(0,1)}\|\Delta_h^\beta u_r\|_p = \sup_{0\le h\le t}\|\Delta_h^\beta f^*\|_p = \omega_\beta(t;f^*)_p.\end{aligned}$$

Finally, suppose that $\omega_\beta(t;f^*)_p = o(t^\beta)$ as $t\to 0^+$. Then by the previous claim $\omega_\beta^*(t;P[f^*])_p = o(t^\beta)$ as $t\to 0^+$. By part (a), $P[f^*]$ is a constant function on U. If $p\in[1,+\infty)$, then $P[f^*]$ has non-tangential limit $f^*(e^{i\theta})$ at every Lebesgue point $e^{i\theta}$ of f^*. So f^* is a constant function in $L^p(T)$ (in equivalence class sense). If $p=+\infty$, then $f^*\in C(T)$, $P[f^*]$ has a continuous extension to $\bar{U}$ with f^* as its boundary value function. Therefore, f^* is a constant function on T and we are done.

Remarks. (a) There is a version of the above theorem for the functions defined on closed interval $[a,b]$, namely if $f:[a,b]\to\mathbb{C}$ and $\omega_\beta(t;f)_p = o(t^\beta)$ as $t\to 0^+$, then f is a polynomial of degree $\beta-1$. This result can be found in [1] and [2]. The situation for functions on $[a,b]$ and T are different.

(b) Trigonometric polynomials of degree n are of the form $\sum_{j=-n}^{n} c_j e^{ij\theta}$, where $|c_n|+|c_{-n}|\ne 0$. Consider $\beta=2$ and $p\in[1,+\infty]$. Define $f:T\to\mathbb{C}$ by $f(e^{i\theta}) = c_1e^{i\theta}$, where $c_1\ne 0$. Clearly, f is a trigonometric polynomial of degree 1. For every $e^{i\theta}$ and $h\in[0,+\infty)$,

$$\begin{aligned}(\Delta_h^2 f)(e^{i\theta}) &= (\Delta_h f)(e^{i(\theta+h)}) - (\Delta_h f)(e^{i\theta}) = f(e^{i\theta+2h}) - 2f(e^{i(\theta+h)}) + f(e^{i\theta}) \\ &= c_1e^{i(\theta+2h)} - 2c_1e^{i(\theta+h)} + c_1e^{i\theta} = c_1e^{i\theta}(e^{2ih}-2e^{ih}+1).\end{aligned}$$

Then for all $p\in[1,+\infty)$,

$$\|\Delta_h^2 f\|_p = \left(\tfrac{1}{2\pi}\int_{\theta=0}^{\theta=2\pi}|c_1e^{i\theta}(e^{2ih}-2e^{ih}+1)|^p d\theta\right)^{1/p} = |c_1(e^{2ih}-2e^{ih}+1)|.$$

For $p=+\infty$, $\|\Delta_h^2 f\|_\infty = |c_1(e^{2ih}-2e^{ih}+1)|$. By l'Hopital's rule,

$$\lim_{h\to 0^+}\frac{\|\Delta_h^2 f\|_p}{h^2} = \lim_{h\to 0^+}\frac{|c_1(e^{2ih}-2e^{ih}+1)|}{h^2} = |c_1|\ne 0.$$

So $\omega_2(t;f)_p \ne o(t^2)$ as $t\to 0^+$.

Now we are ready to prove one of the main results in the paper.

Theorem 1.8 *For $u\in Har(U)$, $\lim_{t\to 0^+}\omega_1^*(t;u)_\infty = 0$ if and only if there is a unique $f^*\in C(T)$ such that $u=P[f^*]$.*

Proof. For the if-direction, let $u : U \to \mathbb{C}$ be harmonic on U such that there is a unique $f^* \in C(T)$ such that $u = P[f^*]$. In the proof of Theorem 1.7(b), we showed that for all $t \in (0, 2\pi]$, $\omega_1^*(t; u)_\infty = \omega_1(t; f^*)_\infty$. Since $f^* \in C(T)$, i.e. $\lim_{t\to 0^+} \omega_1(t; f^*)_\infty = 0$, we have $\lim_{t\to 0^+} \omega_1^*(t; u)_\infty = 0$.

For the only-if direction, let $u : U \to \mathbb{C}$ be a harmonic function on U such that $\lim_{t\to 0^+} \omega_1^*(t; u)_\infty = 0$. We claim that u is in fact a bounded harmonic function on U. The proof is as follows. Since $\lim_{t\to 0^+} \omega_1^*(t; u)_\infty = 0$, let $\varepsilon = 1$, then for all $0 < t < \delta_0, \omega_1^*(t; u)_\infty < 1$ is true for some $\delta_0 \in (0, 2\pi)$. Let $t_0 = \delta_0/2$. By Theorem 1.6(b), we have $\sup_{r\in(0,1)} \|u_r - u(0)\|_\infty \le (4\pi/t_0)\omega_1^*(t_0; u)_\infty$. So $\sup_{r\in(0,1)} \|u_r - u(0)\|_\infty \le 4\pi/t_0 < +\infty$. Therefore, u is bounded on U and there is a unique $g^* \in L^\infty(T)$ such that $u = P[g^*]$.

Next we claim that g^* has measure-limit everywhere on T. The proof goes as follows. Let $e^{i\theta} \in T$. Suppose $\{r_n\}$ is a sequence of numbers in $(0, 1)$ with limit 1. Since u is a bounded function on U, $\{u_{r_n}(e^{i\theta})\}$ is a bounded sequence in $\mathbb{C}$. Then there exists a subsequence $\{r_{n_k}\}$ of $\{r_n\}$ such that $\lim_{k\to\infty} u_{r_{n_k}}(e^{i\theta}) = L$ for some $L \in \mathbb{C}$.

We claim that L is the measure-limit of g^* at $e^{i\theta}$. Let $\varepsilon_0 > 0$. Then for all $t \in (0, \delta_0), \omega_1^*(t; u)_\infty < \varepsilon_0/2$ is true for some $\delta_0 \in (0, 2\pi)$. Now

$$\begin{aligned}\sup_{r\in(0,1)} \sup_{0\le h\le \delta_0/2} \|\Delta_h^1 u_r\|_\infty &= \sup_{r\in(0,1)} \omega_1(\delta_0/2; u_r)_\infty = \limsup_{r\to 1^-} \omega_1(\delta_0/2; u_r)_\infty \\ &= \omega_1^*(\delta_0/2; u)_\infty < \varepsilon_0/2,\end{aligned}$$

where the second equality is due to Theorem 1.6(a).

Next, we define $I = \{e^{iw} \in T : \theta - \delta_0/2 < w < \theta + \delta_0/2\}$. Then I is an open arc centered at $e^{i\theta}$. Let e^{iw_0} be a Lebesgue point of g^* and $|\theta - w_0| < \delta_0/2$. We have for all $r \in (0, 1)$,

$$|u_r(e^{i\theta}) - u_r(e^{iw_0})| \le \sup_{0\le h\le \delta_0/2} \|\Delta_h^1 u_r\|_\infty < \varepsilon_0/2.$$

Since e^{iw_0} is a Lebesgue point of g^*, u has non-tangential limit $g^*(e^{iw_0})$ at e^{iw_0}. So as $k \to \infty$, $u_{r_{n_k}}(e^{iw_0}) \to g^*(e^{iw_0})$. Moreover, as $k \to \infty$, $u_{r_{n_k}}(e^{i\theta}) \to L$ and $|u_{r_{n_k}}(e^{i\theta}) - u_{r_{n_k}}(e^{iw_0})| \to |L - g^*(e^{iw_0})|$. Therefore, if $z \in I$ and z is a Lebesgue point of g^*, then $|L - g^*(z)| \le \varepsilon_0/2 < \varepsilon_0$. So

$$\sigma(I \cap \{e^{iw} \in T : |L - g^*(e^{iw})| \ge \varepsilon_0\}) = 0.$$

Finally, by Theorem 1.4, there exists an $f^* \in C(T)$ such that $g^* = f^*$ a.e. $[\sigma]$ on T. Then $u = P[g^*] = P[f^*]$. Therefore, there is a unique $f^* \in C(T)$ such that $u = P[f^*]$, which completes the proof.

Now that Theorem 1.8 (which is part (1) of Theorem 1.3) is proved. We will proceed to the proof of part (2) of Theorem 1.3, which is much longer and we will do a separate paper to cover the details.

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