International Journal of Contemporary Mathematical Sciences
Vol. 14, 2019, no. 4, 179 - 186
HIKARI Ltd, www.m-hikari.com
https://doi.org/10.12988/ijcms.2019.9715

Boundary Values of Harmonic Functions and Modulus of Continuity (Part II)

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Abstract

We will present a proof of part (b) of our main theorem in a previous paper, which is Theorem 1.4 below. It is about a necessary and sufficient condition for a harmonic function on the open unit disk to be uniquely represented as the Poisson integral of a Riemann-integrable function on the unit circle in terms of modulus of continuity.

Mathematics Subject Classification: 30E25

Keywords: Boundary value functions of harmonic functions, modulus of continuity

1 Introduction

Let U denote the open unit disk, $\overline{U}$ the closed unit disk and T the unit circle on the complex plane $\mathbb{C}$. Let $\mathbf{R}$ denote the set of all real numbers and $\mathbf{N}$ the set of all positive integers. Let $Har(U)$ denote the set of all harmonic functions on U. The abbreviation a.e. means almost everywhere.

In [2], pp. 12-14, there is a result of K. G. Ivanov (Lemma 1.1, Lemma 1.2 and Theorem 1.3), which proved that if $f : [a, b] \to \mathbf{R}$ is bounded measurable, then $\omega(t; \cdot; f) : [a, b] \to \mathbf{R}$ is also bounded measurable with respect to Lebesgue measure σ, where $\omega(t; e^{i\theta}; f) = \sup |f(e^{i\alpha_1}) - f(e^{i\alpha_2})|$ over all $\alpha_1, \alpha_2 \in [\theta - t/2, \theta + t/2]$. As a corollary, by taking the case $f(a) = f(b)$, we get Ivanov's result is also true for functions of the form $f : T \to \mathbf{R}$. In [1], we defined $\tau(t; f) = \|\omega(t; \cdot; f)\|_{L^1(T)}$ for $t \in [0, 2\pi]$. Next we will prove some useful facts.

Theorem 1.1 *For a nonconstant $u \in Har(U)$,*

(a) if $p \in [1,+\infty]$, $\beta \in \mathbf{N}$, $t \in (0,2\pi)$ and $r \in (0,1)$, then $\omega_\beta(t;u_r)_p > 0$;

(b) if $t \in (0,2\pi)$, then $\inf_{r\in(0,1)}(\tau(t;u_r)/\omega_1(t;u_r)_\infty) > 0$.

Proof. For (a), suppose $u \in Har(U)$ is not constant on U. Assume there are $p \in [1,+\infty], \beta \in \mathbf{N}$, $t_0 \in (0,2\pi)$ and $r_1 \in (0,1)$ such that $\omega_\beta(t_0;u_{r_1})_p = 0$. Define $g : \overline{U} \to \mathbf{R}$ by $g(z) = u(r_1 z)$ for all $z \in \overline{U}$. Then g is harmonic on U and continuous on $\overline{U}$. Using the proof of Theorem 1.7(b) in [1], we showed for all $t \in (0,2\pi]$, $\omega^*_\beta(t;g)_p = \omega_\beta(t;g|_T)_p$. However, $\omega_\beta(t_0;g|_T)_p = \omega_\beta(t_0;u_{r_1})_p$. So $\omega^*_\beta(t_0;g)_p = 0$. By Theorem 1.6 (b) in [1], we see $\sup_{r\in(0,1)} \|g_r - g(0)\|_p = 0$, i.e. g is constant on $\overline{U}$. So $u|_{\overline{B(0,r_1)}}$ is constant. Applying the Cauchy-Riemann equations, u is constant on U, which is a contradiction.

For (b), suppose $u \in Har(U)$ is not constant on U. By (a), we have $\omega_1(t;u_r) > 0$ for all $t \in (0,2\pi)$, $r \in (0,1)$. Assume there is $t_0 \in (0,2\pi)$ such that $\inf_{r\in(0,1)}(\tau(t;u_r)/\omega_1(t;u_r)_\infty) = 0$. Since $u \in Har(U)$, so $u_r \in C(T)$ for every $r \in (0,1)$. By the corollary of Ivanov's theorem, $\omega(t;\cdot;u_r)$ is bounded measurable for all $t \in [0,2\pi]$ and $\tau(t;u_r) = \|\omega(t;\cdot;u_r)\|_{L^1(T)}$ is well-defined. For all $n \in \mathbf{N}$, there is $r_n \in (0,1)$ such that

$$0 \le \frac{\tau(t_0;u_{r_n})}{\omega_1(t_0;u_{r_n})_\infty} < \frac{1}{n}, \quad \textit{then} \ \lim_{n\to\infty} \frac{1}{2\pi}\int_0^{2\pi} \frac{\omega(t_0;e^{i\theta};u_{r_n})}{\omega_1(t_0;u_{r_n})_\infty} d\theta = 0.$$

Define $\psi_n(\theta) = \omega(t_0;e^{i\theta};u_{r_n})/\omega_1(t_0;u_{r_n})_\infty$ for all $\theta \in [0,2\pi]$. Then $\{\psi_n\}$ is a sequence of measurable function on $[0,2\pi]$ that converges to 0 in measure. So there exists a subsequence $\{\psi_{n_k}\}$ converges to 0 a.e. on $[0,2\pi]$ with respect to Lebesgue measure. Without loss of generality, we may assume $\lim_{k\to\infty}\psi_{n_k}(0) = 0$. Then there exists a partition P of $[0,2\pi]$ such that

(1) $P = \{0 = \theta_0 < \theta_1 < \cdots < \theta_n = 2\pi\}$, and $t_0/2 < \theta_{j+1} - \theta_j$ for $0 \le j < n$,

(2) if $[0,2\pi] \subseteq \bigcup_{i=0}^{n} [\theta_i - \frac{t_0}{2}, \theta_i + \frac{t_0}{2}]$, then $T \subseteq \bigcup_{j=0}^{n-1} \{e^{iw} : w \in [\theta_j - \frac{t_0}{2}, \theta_j + \frac{t_0}{2}]\}$,

(3) for $0 \le j < n$, $\lim_{k\to\infty} \omega(t_0;e^{i\theta_j};u_{r_{n_k}})/\omega_1(t_0;u_{r_{n_k}})_\infty = 0$ and

(4) for $0 \le i < n$, $[\theta_i - t_0/2, \theta_i + t_0/2] \cap [\theta_{i+1} - t_0/2, \theta_{i+1} + t_0/2] \neq \emptyset$.

Take $\varepsilon_0 = 1/16$. For $0 \le j < n$, there exists $k_j \in \mathbf{N}$ such that for all $k \ge k_j$, $\omega(t_0;e^{i\theta_j};u_{r_{n_k}})/\omega_1(t_0;u_{r_{n_k}})_\infty < 1/16$. Let k^* be the maximum of $k_0, k_1, \ldots, k_{n-1}$. For all $\theta \in \mathbf{R}$ and $h \in [0,t_0]$, denote S_j to mean $\{e^{iw} : w \in [\theta_j - t_0/2, \theta_j + t_0/2]\}$ for some $j \in \{0,1,\ldots,n-1\}$. Then $e^{i\theta}, e^{i(\theta+h)} \in S_j \cup S_{j+1}$ for $0 \le j < n$. Since $[\theta_j - t_0/2, \theta_j + t_0/2] \cap [\theta_{j+1} - t_0/2, \theta_{j+1} + t_0/2] \neq \emptyset$ by (4), let w_1 be in this set. Then

$$\begin{aligned} & |u_{r_{n_{k^*}}}(e^{i\theta}) - u_{r_{n_{k^*}}}(e^{i(\theta+h)})| \\ \leq\ & |u_{r_{n_{k^*}}}(e^{i\theta}) - u_{r_{n_{k^*}}}(e^{iw_1})| + |u_{r_{n_{k^*}}}(e^{iw_1}) - u_{r_{n_{k^*}}}(e^{i(\theta+h)})| \\ <\ & (1/16)\omega_1(t_0; u_{r_{n_{k^*}}})_\infty + (1/16)\omega_1(t_0; u_{r_{n_{k^*}}})_\infty = (1/8)\omega_1(t_0; u_{r_{n_{k^*}}})_\infty. \end{aligned}$$

For $e^{i\theta} \in S_j$, $|u_{r_{n_{k^*}}}(e^{i\theta}) - u_{r_{n_{k^*}}}(e^{iw_1})| \leq \omega(t_0; e^{i\theta_j}; u_{r_{n_{k^*}}}) < (1/16)\omega_1(t_0; u_{r_{n_{k^*}}})_\infty$. As $\theta \in \mathbf{R}$, $h \in [0, t_0]$ are arbitrary, we see $\omega_1(t_0; u_{r_{n_{k^*}}})_\infty \leq (1/8)\omega_1(t_0; u_{r_{n_{k^*}}})_\infty$. So $\omega_1(t_0; u_{r_{n_{k^*}}})_\infty = 0$, which is a contradiction.

Definition 1.2 *For a function $g : T \to \mathbf{R}$, we say g is measure-continuous at $z \in T$ if for every $\varepsilon > 0$, there exists an open arc I centered at z such that $\sigma(I \cap \{w \in T : |g(w) - g(z)| \geq \varepsilon\}) = 0$. Also, we say g is measure-continuous a.e. on T if the measure of the set of the points at which g fails to be measure-continuous is zero.*

Theorem 1.3 *For $f^* : T \to \mathbf{R}$, $f^* \in L^\infty(T)$ is measure-continuous a.e. on T if and only if there exists $g : T \to \mathbf{R}$ such that (1) $f^* = g$ a.e. on T, (2) g is bounded on T, (3) g is continuous a.e. on T.*

Proof. Suppose for $f^* : T \to \mathbf{R}$, there exists $g : T \to \mathbf{R}$ such that $f^* = g$ a.e. $[\sigma]$ on T, g is bounded on T and g is continuous a.e. on T.

We claim g is measurable. This is due to g is bounded and continuous a.e. on T, hence Riemann integrable on T. Then $f^* = g$ a.e. $[\sigma]$ on T implies f^* is measurable and $f^* \in L^\infty(T)$. To see f^* is measure-continuous a.e. on T, let $e^{i\theta} \in C(g) \cap E$, where $C(g)$ is the set of all points of continuity of g and $E = \{e^{i\phi} : f^*(e^{i\phi}) = g(e^{i\phi})\}$. Let $\varepsilon_0 > 0$. Then "for all $w \in I, |g(w) - g(e^{i\theta})| < \varepsilon_0$" is true for some open arc I centered at $e^{i\theta}$. So for all $w \in I \cap E$, $|f^*(w) - f^*(e^{i\theta})| < \varepsilon_0$. Then $\sigma(I \cap \{w \in T : |f^*(w) - f^*(e^{i\theta})| \geq \varepsilon_0\}) = 0$. Since $\sigma(T \backslash (C(g) \cap E)) = 0$, f^* is measure-continuous a.e. on T.

Conversely, suppose $f^* \in L^\infty(T)$ is measure-continuous a.e. on T. Let $L_{f^*} = \{e^{i\theta} : e^{i\theta} \text{ is a Lebesgue point of } f^*\}$ and $m_c(f^*) = \{e^{i\theta} : e^{i\theta} \text{ is a measure-continuous point of } f^*\}$. Then $\sigma(T \backslash (L_{f^*} \cap m_c(f^*))) = 0$. For every $e^{i\theta} \in T$ and $n \in \mathbf{N}$, let $I_n(e^{i\theta})$ be the open arc centered at $e^{i\theta}$ with arc length $2\pi/n$. Then for every $e^{i\theta} \in T$ and $n \in \mathbf{N}$, the sequence $\{L_n(e^{i\theta})\} = \left\{\int_{I_n(e^{i\theta})} f^*\, d\sigma \Big/ \sigma(I_n(e^{i\theta}))\right\}$ is bounded as $|L_n(e^{i\theta})| \leq \|f^*\|_\infty < +\infty$. So $\{L_n(e^{i\theta})\}$ has a convergent subsequence, say $\{L_{n_k}(e^{i\theta})\}$ with limit $L(e^{i\theta})$. Then $|L(e^{i\theta})| \leq \|f^*\|_\infty$.

For an open arc J in T, define $\mu(f^*, J) = (\int_J f^* d\sigma)/\sigma(J)$. We claim that for every $e^{i\theta} \in m_c(f^*)$ and $\varepsilon > 0$, there exists an open arc I centered at $e^{i\theta}$ such that for any open arc $I' \subseteq I$ (with the center of I' may not be $e^{i\theta}$), we have $|f^*(e^{i\theta}) - \mu(f^*, I')| < \varepsilon$. (The proof is as follows. Let $e^{i\theta} \in m_c(f^*)$ and $\varepsilon_0 > 0$. By definition of measure-continuous at $e^{i\theta}$, there exist an open arc I centered at $e^{i\theta}$ such that $\sigma(I \cap \{w \in T : |f^*(w) - f^*(e^{i\theta})| \geq \varepsilon_0/2\}) = 0$. Let $A = I' \cap \{w \in T : |f^*(w) - f^*(e^{i\theta})| < \varepsilon_0/2\}$ and $B = I' \cap \{w \in T : |f^*(w) - f^*(e^{i\theta})| \geq \varepsilon_0/2\}$. Then $\sigma(A) = \sigma(I')$ and $\sigma(B) = 0$.

For every open arc $I' \subseteq I$, we have

$$\begin{aligned} & |\mu(f^*, I') - f^*(e^{i\theta})| \leq \Big(\int_{I'} |f^*(w) - f^*(e^{i\theta})| \, d\sigma(w)\Big)/\sigma(I') \\ = \; & \Big(\int_A |f^*(w) - f^*(e^{i\theta})| \, d\sigma(w)\Big)/\sigma(I') + \Big(\int_B |f^*(w) - f^*(e^{i\theta})| \, d\sigma(w)\Big)/\sigma(I') \\ \leq \; & (\varepsilon_0/2)(\sigma(A)/\sigma(I')) + 0 < \varepsilon_0.) \end{aligned}$$

Next we claim for every $e^{i\theta} \in m_c(f^*) \cap L_{f^*}$, $L : T \to \mathbf{R}$ is continuous at $e^{i\theta}$. To see that, let $e^{i\theta} \in m_c(f^*) \cap L_{f^*}$. We have

$$L(e^{i\theta}) = \lim_{k\to\infty} \frac{\int_{I_{n_k}(e^{i\theta})} f^*(w) \, d\sigma(w)}{\sigma(I_{n_k})(e^{i\theta})} = \lim_{k\to\infty} \mu(f^*, I_{n_k}(e^{i\theta})) = f^*(e^{i\theta}).$$

Let $\varepsilon_0 > 0$. By the previous claim, there exists an open arc I centered at $e^{i\theta}$ such that for every open arc $I' \subseteq I$ (the center of I' may not be $e^{i\theta}$), $|\mu(f^*; I') - L(e^{i\theta})| < \varepsilon_0/2$.

Let $w_0 \in I$. Since I is open,"for all $k \geq k_1, I_{n_k}(w_0) \subseteq I$" is true for some $k_1 \in \mathbf{N}$. Moreover, "for all $k \geq k_2, |L(w_0) - \mu(f^*, I_{n_{k^*}}(w_0))| < \varepsilon_0/2$" is true for some $k_2 \in \mathbf{N}$. Take $k^* = \max\{k_1, k_2\}$. We have

$$\begin{aligned} |L(e^{i\theta}) - L(w_0)| &\leq |L(e^{i\theta}) - \mu(f^*, I_{n_{k^*}}(w_0))| + |\mu(f^*, I_{n_{k^*}}(w_0)) - L(w_0)| \\ &< \frac{\varepsilon_0}{2} + \frac{\varepsilon_0}{2} = \varepsilon_0. \end{aligned}$$

So there exists $L : T \to \mathbf{R}$ such that $L = f^*$ a.e. $[\sigma]$ on T, L is bounded on T and L is continuous a.e. on T.

In terms of modulus of continuity, we will prove our main theorem below, which gives a necessary and sufficient condition for a harmonic function on U. to be the Poisson integral of a Riemann integrable function on T.

Theorem 1.4 *For $u \in Har(U)$, $\lim_{t\to 0^+} \tau^*(t; u) = 0$ if and only if there is a unique $f^* \in R(T)$ such that $u = P[f^*]$, where $R(T)$ is the set of all Riemann integrable functions on T. (Recall by Lebesgue's theorem, for every bounded function $f^* : T \to \mathbf{R}$, f^* is Riemann integerable if and only if f^* is continuous a.e. on T.)*

Proof. Suppose $u \in Har(U)$ satisfies $\lim_{t\to 0^+} \tau^*(t;u) = 0$. We claim u is bounded on U as follows. Since $\lim_{t\to 0^+} \tau^*(t;u) = 0$, if $\varepsilon_0 = 1$, then for all $t \in (0,\delta_0), \tau^*(t;u) < \varepsilon_0$ is true for some $\delta_0 \in (0,2\pi)$. So $\limsup_{r\to 1^-} \tau(t_0;u_r) = \tau^*(t_0;u) < \varepsilon_0 = 1$, where $t_0 = \delta_0/2$.

By Theorem 1.1(b), let $k_0 = \inf_{r\in(0,1)} \tau(t_0;u_r)/\omega_1(t_0,u_r)_\infty > 0$. For every $r(0,1)$, $\omega_1(t_0;u_r)_\infty \le \tau(t_0;u_r)/k_0$ and so

$$\begin{aligned} &\limsup_{r\to 1^-} \omega_1(t_0;u_r)_\infty = \inf_{0<\delta<1} \sup_{1-\delta<r<1} \omega_1(t_0;u_r)_\infty \\ \le\ & (\inf_{0<\delta<1} \sup_{1-\delta<r<1} \tau(t_0;u_r))/k_0 = \limsup_{r\to 1^-} \tau(t_0;u_r)/k_0 < 1/k_0. \end{aligned}$$

By Theorem 1.6(b) in [1], $\sup_{r\in(0,1)} \|u_r - u(0)\|_\infty \le (4\pi/t)\omega_1^*(t_0;u)_\infty < 4\pi/(t_0k_0) < \infty$. So u is bounded on U and there is a unique $g \in L^\infty(T)$ with $u = P[g]$.

Next, we claim g is measure-continuous a.e. on T as follows. Suppose g is not measure-continuous at $z \in T$. Now for every $j \in \mathbf{N}$, define $K_j = \{z \in T :$ for all open arc I centered at z, $0 < \sigma(I \cap \{w \in T : |g(w) - g(z)| \ge 1/j\})\}$. Let $r_n = 1 - 1/n$ for $n = 1, 2, 3, \ldots$, then $\lim_{n\to\infty} r_n = 1$.

We claim for all $j \in \mathbf{N}$, $K_j \cap \{e^{i\theta} \in T : e^{i\theta}$ is a Lebesgue point of $g\} \subseteq \cap_{t\in(0,2\pi)} \{z \in T : \liminf_{n\to\infty} \omega(t;z;u_{r_n}) \ge 1/j\}$. The reason is as follows. Let $j \in \mathbf{N}$, $z_0 \in K_j \cap \{e^{i\theta} \in T : e^{i\theta}$ is a Lebesgue point of $g\}$ and $t_2 \in (0,2\pi)$. Let $I_{z_0} = \{e^{i\phi} \in T : \theta_0 - t_2/2 < \phi < \theta_0 + t_2/2\}$, where $z_0 = e^{i\theta_0}$ for some $\theta_0 \in \mathbf{R}$. Then $0 < \sigma(I_{z_0} \cap \{w \in T : |g(w) - g(z_0)| \ge 1/j\})$ and $0 < \sigma(I_{z_0} \cap \{w \in T : |g(w) - g(z_0)| \ge 1/j\}) \cap \{w \in T : w$ is a Lebesgue point of $g\})$. So there exists $w_0 \in T$ such that

(1) $w_0 \in I_{z_0} \cap \{w \in T : |g(w) - g(z_0)| \ge 1/j\}$ and

(2) w_0 is a Lebesgue point of g (that means u has non-tangential limit at w_0).

Since z_0 is also a Lebesgue point of g, $\lim_{n\to\infty} |u_{r_n}(z_0) - u_{r_n}(w_0)| = |g(z_0) - g(w_0)| \ge 1/j$ by (1). So for all $n \in \mathbf{N}$,

$$|u_{r_n}(z_0) - u_{r_n}(w_0)| \le \sup_{\alpha_1,\alpha_2\in[\theta_0 - t_2/2, \theta_0 + t_2/2]} |u_{r_n}(e^{i\alpha_1}) - u_{r_n}(e^{i\alpha_2})| = \omega(t_2; z_0; u_{r_n})$$

and

$$\begin{aligned} \frac{1}{j} &\le \lim_{n\to\infty} |u_{r_n}(z_0) - u_{r_n}(w_0)| = \liminf_{n\to\infty} |u_{r_n}(z_0) - u_{r_n}(w_0)| \\ &= \sup_{n\ge 1} \inf_{k\ge n} |u_{r_k}(z_0) - u_{r_k}(w_0)| \le \sup_{n\ge 1} \inf_{k\ge n} \omega(t_2; z_0, u_{r_k}) = \liminf_{n\to\infty} \omega(t_2; z_0; u_{r_n}). \end{aligned}$$

Next, we claim that for all $j \in \mathbf{N}$, $\sigma(K_j) = 0$. To see that let $\varepsilon_0 > 0$ and j be a positive integer. Then "for every $t \in (0,\delta_1), \tau^*(t;u) < \varepsilon_0$" is true for some $\delta_1 \in (0, 2\pi)$. Take $t_3 = \delta_1/2$. By Fatou's lemma,

$$\begin{aligned}\int_T \liminf_{n\to\infty} \omega(t_3; z; u_{r_n}) d\sigma(z) &\leq \liminf_{n\to\infty} \int_T \omega(t_3; z; u_{r_n}) d\sigma(z) \\ &= \liminf_{n\to\infty} \tau(t_3; u_{r_n}) \leq \limsup_{r\to 1^-} \tau(t_3; u_r).\end{aligned}$$

So $\int_T \liminf_{n\to\infty} \omega(t_3; z; u_{r_n}) d\sigma(z) \leq \tau^*(t_3; u) < \varepsilon_0$. Then

$$\begin{aligned}\frac{1}{j}\sigma(\{z \in T : \liminf_{n\to\infty} \omega(t_3; z; u_{r_n}) \geq \frac{1}{j}\}) &\leq \int_T \liminf_{n\to\infty} \omega(t_3; z; u_{r_n}) d\sigma(z) \\ &\leq \tau^*(t_3; u) < \varepsilon_0.\end{aligned}$$

By (a), $\sigma(K_j) = \sigma(K_j \cap \{e^{i\theta} : e^{i\theta}$ is a Lebesgue point of $g\}) \leq \sigma(\{z \in T : \liminf_{n\to\infty} \omega(t_3; z; u_{r_n}) \geq 1/j\}) < j\varepsilon_0$. Since ε_0 is arbitrary, for all $j \in \mathbf{N}$, $\sigma(K_j) = 0$, i.e. g is measure-continuous a.e. on T. By Theorem 1.3, there exists $f^* \in R(T)$ such that $f^* = g$ a.e. $[\sigma]$ on T. So there is a unique $f^* \in R(T)$ such that $u = P[f^*]$, where the uniqueness is in the equivalence class sense of $L^\infty(T)$.

Conversely, suppose $u \in Har(U)$ and there is a unique $f^* \in R(T)$ with $u = P[f^*]$. Then f^* is bounded on T. Let $M > 0$ be such that for all $z \in T, |f^*(z)| \leq M$. Since for all $z \in U, u(z) = P[f^*](z)$, so for all $z \in U, |u(z)| \leq M$. Now f^* is Riemann integrable on T. So f^* is continuous a.e. on T.

Define $\tilde{f^*} : [0, 2\pi] \to \mathbf{R}$ by $\tilde{f^*}(\theta) = f^*(e^{i\theta})$. Then $\tilde{f}$ is continuous a.e. on $[0, 2\pi]$. Let $C(\tilde{f^*}) = \{\theta \in [0, 2\pi] : \tilde{f^*}$ is continuous at $\theta\}$. Fix $\varepsilon_0 > 0$. For every $\theta \in (0, 2\pi) \cap C(\tilde{f^*})$, there exists $\delta_\theta > 0$ such that for all $\alpha \in [0, 2\pi] \cap (\theta - \delta_\theta, \theta + \delta_\theta)$, $|f^*(e^{i\alpha}) - f^*(e^{i\theta})| < \varepsilon_0/48$. Now the collection

$$\mathcal{I} = \{(\theta - \delta, \theta + \delta) : \theta \in (0, 2\pi) \cap C(\tilde{f^*}), (\theta - \delta, \theta + \delta) \subseteq (0, 2\pi), \delta \in (0, \delta_\theta)\}.$$

is a Vitali covering of the set $(0, 2\pi) \cap C(\tilde{f^*})$ with Lebesgue measure $m((0, 2\pi) \cap C(\tilde{f^*})) = 2\pi$. By the Vitali covering lemma, there is a finite pairwise-disjoint collection $\{I_1, I_2, \ldots, I_N\}$ of intervals in $\mathcal{I}$ such that $m([0, 2\pi) \cap C(\tilde{f^*})] \setminus \cup_{i=1}^N I_i) < \pi\varepsilon_0/(8M)$. So for every $j = 1, 2, \ldots, N$, $I_j = (\theta_j - \delta_j, \theta_j + \delta_j) \subseteq (0, 2\pi)$ for some $\theta_j \in (0, 2\pi) \cap C(\tilde{(f)}^*)$, $\delta_j \in (0, \delta - \theta_j)$. Then there exists a positive number Δ such that $(2\Delta)N < \pi\varepsilon_0/(8M)$ and $\delta < \min\{\delta_1, \delta_2, \ldots, \delta_N, \pi\}$. For $j = 1, 2, \ldots, N$, let $I'_j = [\theta_j - \delta_j + \Delta, \theta_j + \delta_j - \Delta] \subseteq I_j$. Now $\delta_j - \Delta > 0$. So $m([0, 2\pi] \setminus \cup_{i=1}^N I'_i) < (\pi\varepsilon_0)/(8M) + (\pi\varepsilon_0)/(8M) = (\pi\varepsilon_0)/(4M)$.

Recall the Poisson kernel $P_r(\theta) = (1 - r^2)/(1 - 2r\cos\theta + r^2)$ for all $\theta \in \mathbf{R}$, $r \in [0, 1)$. Since for all $\delta \in (0, \pi]$, $\lim_{r\to 1^-} P_r(\theta) = 0$ uniformly in $\theta \in [\delta, 2\pi - \delta]$, so $\lim_{r\to 1^-} P_r(\theta) = 0$ uniformly in $\theta \in [\Delta, 2\pi - \Delta]$. Then "for every $r \in [r^*, 1)$

and $\theta \in [\Delta, 2\pi - \Delta]$, $|P_r(\theta)| < (\varepsilon_0/(2M))(1/48)$" is true for some $r^* \in (0,1)$. Let $j = 1, 2, \ldots, N$, $r \in [r^*, 1), \theta \in [\theta_j - \delta_j + \Delta, \theta_j + \delta_j - \Delta] = I'_j$. We have

$$\begin{aligned}
& |P[f^*](re^{i\theta}) - f^*(e^{i\theta_j})| \\
= \ & \Big|\frac{1}{2\pi}\int_0^{2\pi} P_r(\theta - t) f^*(e^{it})dt - \frac{1}{2\pi}\int_0^{2\pi} P_r(\theta - t) f^*(e^{i\theta_j})dt\Big| \\
= \ & \Big|\frac{1}{2\pi}\int_0^{2\pi} P_r(\theta - t)(f^*(e^{it}) - f^*(e^{i\theta_j}))dt\Big| \\
\leq \ & \frac{1}{2\pi}\Big(\int_0^{\theta_j-\delta_j} + \int_{\theta_j-\delta_j}^{\theta_j+\delta_j} + \int_{\theta_j+\delta_j}^{2\pi}\Big) P_r(\theta - t)|f^*(e^{it}) - f^*(e^{i\theta_j})| \ dt \\
\leq \ & \Big(\frac{1}{2\pi}\int_0^{\theta_j-\delta_j} P_r(\theta - t)dt\Big)(2M) + \Big(\frac{1}{2\pi}\int_{\theta_j-\delta_j}^{\theta_j+\delta_j} P_r(\theta - t)dt\Big)\frac{\varepsilon_0}{48} \\
& + \Big(\frac{1}{2\pi}\int_{\theta_j+\delta_j}^{2\pi} P_r(\theta - t)dt\Big)(2M) \\
\leq \ & \frac{1}{2\pi}\cdot\frac{\varepsilon_0}{2M}\cdot\frac{1}{48}(\theta_j - \delta_j)2M + \Big(\frac{1}{2\pi}\int_0^{2\pi} P_r(\theta - t)dt\Big)\frac{\varepsilon_0}{48} \\
& + \frac{1}{2\pi}\cdot\frac{\varepsilon_0}{2M}\cdot\frac{1}{48}(2\pi - \theta_j - \delta_j)\cdot 2M \\
< \ & \frac{\varepsilon_0}{48} + \frac{\varepsilon_0}{48} + \frac{\varepsilon_0}{48} = \frac{\varepsilon_0}{16}
\end{aligned}$$

due to $0 < \theta_j - \delta_j < 2\pi$ and $0 < 2\pi - \theta_j - \delta_j < 2\pi$. So for $j = 1, 2, \ldots, N$, $\theta \in I'_j, r_1, r_2 \in [r^*, 1)$, we have

$$\begin{aligned}
|P[f^*](r_1e^{i\theta}) - P[f^*](r_2e^{i\theta})| &< |P[f^*](r_1e^{i\theta}) - f^*(e^{i\theta_j})| + |f^*(e^{i\theta_j}) - P[f^*](r_2e^{i\theta})| \\
&< \varepsilon_0/16 + \varepsilon_0/16 = \varepsilon_0/8.
\end{aligned}$$

Since u_{r^*} as a restriction of u to $\partial B(0, r^*)$ is continuous on T, so for all $t \in [0, 2\pi], z \in T$, we have $\omega(t; z; u_{r^*})_\infty \leq \omega_1(t; u_{r^*})_\infty$. For every $t \in [0, 2\pi]$, we have $\tau(t; u_{r^*}) = \|\omega(t; \cdot; u_{r^*})\|_1$ $(\leq \omega_1(t; u_{r^*})_\infty)$ and $\lim_{t\to 0^+} \tau(t; u_{r^*}) = 0$ by the uniform continuity of u_{r^*}. So"for all $t \in (0, t_1], \tau(t; u_{r^*}) < \varepsilon_0/4$" is true for some $t_1 \in (0, 2\pi)$. Take $t_2 = \min\{t_1, \varepsilon_0\pi/(4MN), \delta_1 - \Delta, \delta_2 - \Delta, \ldots, \delta_N - \Delta\}$, which is in $(0, 2\pi)$. Define $I''_j = [\theta_j - \delta_j + \Delta + t_2/2, \theta_j + \delta_j - \Delta - t_2/2]$ for $j = 1, 2, \ldots, N$. Since $t_2 \leq \delta_j - \Delta$, we have $t_2/2 < \delta_j - \Delta$. Now for every $j = 1, 2, \ldots, N, \theta \in I''_j, r \in [r^*, 1), \alpha_1, \alpha_2 \in [\theta - t_2/2, \theta + t_2/2]$,

$$\begin{aligned}
& |u_r(e^{i\alpha_1}) - u_r(e^{i\alpha_2})| \\
\leq \ & |u_r(e^{i\alpha_1}) - u_{r^*}(e^{i\alpha_1})| + |u_{r^*}(e^{i\alpha_1}) - u_{r^*}(e^{i\alpha_2})| + |u_{r^*}(e^{i\alpha_2}) - u_r(e^{i\alpha_2})| \\
< \ & \varepsilon_0/8 + \omega(t_2; e^{i\theta}; u_{r^*}) + \varepsilon_0/8.
\end{aligned}$$

So $\omega(t_2; e^{i\theta}; u_r) < \varepsilon_0/4 + \omega(t_2; e^{i\theta}; u_{r^*})$. Now for $r \in [r^*, 1)$,

$$\tau(t_2; u_r) = \int_T \omega(t_2; z; u_r)d\sigma(z) = \frac{1}{2\pi}\int_{\theta=0}^{\theta=2\pi} \omega(t_2; e^{i\theta}; u_r)d\theta$$

$$
\begin{aligned}
&= \frac{1}{2\pi}\int_{[0,2\pi]\setminus\cup_{j=1}^{N} I_j'} \omega(t_2;e^{i\theta};u_r)d\theta + \frac{1}{2\pi}\int_{\cup_{j=1}^{N} I_j'} \omega(t_2;e^{i\theta};u_r)d\theta \\
&\le \frac{1}{2\pi}(2M)\frac{\pi\varepsilon_0}{4M} + \sum_{j=1}^{N}\frac{1}{2\pi}\int_{I_j'\setminus I_j''}\omega(t_2;e^{i\theta};u_r)d\theta + \sum_{j=1}^{N}\frac{1}{2\pi}\int_{I_j''}\omega(t_2;e^{i\theta};u_r)d\theta \\
&\le \frac{\varepsilon_0}{4} + \sum_{j=1}^{N}\frac{1}{2\pi}(2M)t_2 + \sum_{j=1}^{N}\frac{1}{2\pi}\int_{I_j''}\Big(\frac{\varepsilon_0}{4} + \omega(t_2;e^{i\theta};u_{r^*})\Big)d\theta \\
&\le \frac{\varepsilon_0}{4} + \sum_{j=1}^{N}\frac{M}{\pi}\cdot\frac{\varepsilon_0\pi}{4M\cdot N} + \sum_{j=1}^{N}\frac{\varepsilon_0}{8\pi}\int_{I_j''}d\theta + \sum_{j=1}^{N}\frac{1}{2\pi}\int_{I_j''}\omega(t_2;e^{i\theta};u_{r^*})d\theta \\
&= \frac{\varepsilon_0}{4} + \frac{\varepsilon_0}{4} + \frac{\varepsilon_0}{4}\Big(\frac{1}{2\pi}\int_{\cup_{j=1}^{N} I_j''}d\theta\Big) + \frac{1}{2\pi}\int_{\cup_{j=1}^{N} I_j''}\omega(t_2;e^{i\theta};u_{r^*})d\theta \\
&\le \frac{\varepsilon_0}{4} + \frac{\varepsilon_0}{4} + \frac{\varepsilon_0}{4} + \frac{1}{2\pi}\int_{\theta=0}^{\theta=2\pi}\omega(t_2;e^{i\theta};u_{r^*})d\theta \\
&< \frac{\varepsilon_0}{4} + \frac{\varepsilon_0}{4} + \frac{\varepsilon_0}{4} + \frac{\varepsilon_0}{4} = \varepsilon_0.
\end{aligned}
$$

So for all $t \in (0,t_2)$, we see $\tau^*(t;u) = \limsup_{r\to 1^-}\tau(t;u_r) \le \limsup_{r\to 1^-}\tau(t_2;u_r)$ $= \inf_{0<\delta<1}\sup_{1-\delta<r<1}\tau(t_2;u_r) \le \varepsilon_0$. Therefore, $\lim\limits_{t\to 0^+}\tau^*(t;u) = 0$.

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Received: July 23, 2019; Published: August 20, 2019