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The Invariance, Formulas for Solutions and Periodicity of Some Recurrence Equations

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Abstract

The explicit solution of a class of fifth order difference equations with variable coefficients via Lie analysis is presented. The relationship between the canonical coordinate and similarity variables that leads to the solutions is clearly stated. From these solutions, we provide necessary and sufficient conditions for existence of periodic solutions in some specific cases. This paper is an extension of a recent result in the literature.

Mathematics Subject Classification: 39A11, 39A05

Keywords: Ordinary difference equation, invariance analysis, periodicity

1 Introduction

In the second half of the nineteenth century, the prominent Norwegian Sophus Lie began to create a remarkable body of work that became the foundation of the theory of continuous groups of transformations that leave differential equations invariant. This development permits one to obtain solutions to differential equations systematically by performing symmetry analysis on them. Following this work, considerable research efforts were directed to understanding the elegant structure of Lie groups by P. Olver, G. Bluman, W. Killing, P. Hydon and many other. In 1918, the German Mathematician Emmy Noether proved her theorem which uncovered the most fundamental justification for

conservation laws, that is, conservation laws follow from the symmetry properties of nature [13].
The study of difference equations using the idea of Lie was investigated and a lot of progress has been made [7, 9]. Recent studies have shown that most of the major features are retained when using symmetries and conservation laws to investigate difference equations. The association of symmetry and conservation laws that leads to double reduction, a result known for differential equations, is now known to be true for difference equations as well, see [12]. Ordinary difference equations (OΔE's) can be constructed from ordinary differential equations (ODE's) in a manner that the algebra structure of Lie symmetry is not affected. What is striking is how closely ODE's and OΔE's correspond . The method of finding of continuous symmetries of autonomous systems of first order OΔE's was developed by Maeda [9]. Most recently, this method was used even in non-autonomous cases [4].
In this paper, using symmetry transformations and invariant methods, we derive solutions of the following difference equation

$$x_{n+1} = \frac{x_n x_{n-3}}{a_n x_n + b_n x_{n-4}} \tag{1}$$

for some arbitrary sequences of real numbers $\{a_n\}$ and $\{b_n\}$.
This paper is inspired by some finding in [8] where the authors obtained exact solutions of

$$x_{n+1} = \frac{x_n x_{n-3}}{x_n - x_{n-4}}. \tag{2}$$

For the sake of notation and definitions used in this paper, we shall consider the Kovalevskaya form of (1), that is,

$$u_{n+5} = \frac{u_{n+1} u_{n+4}}{A_n u_{n+4} + B_n u_n}, \tag{3}$$

where A_n and B_n are random real sequences. We will explain how solutions of (1) and (3) are related and will show that solutions in [8] for (2) are indeed special cases of the solutions that we present in this work. The set of calculations one deals with when applying Lie analysis method to difference equations can be cumbersome. Unlike ODE's, OΔE's do not have a computer package which can be utilized to perform symmetry analysis.

2 Preliminaries

In this section, we give definitions and technical concepts which are at the core of understanding the case study. For more details, see [7].

Consideration of only forward O∆E will be used. Assume that the fifth order O∆E is of the form

$$u_{n+5} = \omega(u_n, u_{n+1}, u_{n+4}) \tag{4}$$

with $\omega_{,u_n} = \partial\omega/\partial u_n \neq 0$ for each $n \in D \subset \mathbb{N}$, where D is a regular domain. Then the symmetry condition is

$$\hat{u}_{n+5} = w(n, \hat{u}_n, \cdots, \hat{u}_{n+4}), \tag{5}$$

when (4) holds, under some transformations. Substituting

$$\hat{n} = n, \hat{u}_{n+k} = u_{n+k} + \epsilon Q(n+k, u_{n+k}) + O(\epsilon^2), \tag{6}$$

for $k = 0, \cdots, 4$, into (4), one obtains the linearised symmetry condition (LSC):

$$Q(n+5, u_{n+5}) = \sum_{k=0}^{4} \omega_{,u_{n+k}}\, Q(n+k, u_{n+k}) \tag{7}$$

when (4) holds. Note that (6) is the group of point transformations and $Q(n, u_n)$ is called the characteristic of the local Lie group with respect to coordinates (n, u_n).

For reduction of a given order linear ODE, one needs to know a non zero solution of the associated homogeneous equation. Lie's method can be extended to non linear O∆E's provided that one can calculate a compatible canonical coordinate. Given a second order O∆E

$$u_{n+2} = \omega(n, u_n, u_{n+1}). \tag{8}$$

Suppose that we are able to find a characteristic $Q(n, u_n)$ for (8) and a compatible canonical coordinate s_n. Let

$$r_n = s_{n+1} - s_n = \int \frac{du_{n+1}}{Q(n+1, u_{n+1})} - \int \frac{du_n}{Q(n, u_n)}. \tag{9}$$

Applying the shift operator $S^{(j)} : n \to n + j$, for $j = 1$, one can replace u_{n+2} by ω and further substitute r_{n+1} as a function of n, u_n and u_{n+1}. Then,

$$\begin{aligned}\frac{\partial r_{n+1}}{\partial u_{n+1}} &= \frac{\omega_{,u_{n+1}}}{Q(n+2,\omega)} - \frac{1}{Q(n+1, u_{n+1})}\\ &= -\frac{\omega_{,u_n} Q(n, u_n)}{Q(n+1, u_{n+1})Q(n+2,\omega)}\\ &= -\frac{Q(n, u_n)}{Q(n+1, u_{n+1})}\frac{\partial r_{n+1}}{\partial u_n}.\end{aligned} \tag{10}$$

Regarding r_{n+1} as a function of n, s_n and s_{n+1}, equation (10) equates to

$$\frac{\partial r_{n+1}}{\partial s_{n+1}} + \frac{\partial r_{n+1}}{\partial s_n} = 0.$$

Notice that r_{n+1} depends on n and $s_{n+1} - s_n$ strictly. Therefore, the OΔE has been reduced to a first order OΔE, which is imperative in the forward form :

$$r_{n+1} = F(n, r_n). \tag{11}$$

If equation (11) can be solved, then the solution can be generalised to

$$r_n = f(n, c_1) \tag{12}$$

for some constant c_1.
For higher order, the expression of r_n given in (9) is not always the best choice. The choice of r_n in this paper is different and is more convenient.

3 Symmetry analysis and formula for solutions

Consider the difference equation (3). Assume the Lie point transformations are of the form (6) and let the corresponding symmetry generator of the characteristic Q be

$$X = Q(n, u_n)\frac{\partial}{\partial u_n} + S^{(1)}Q(n, u_n)\frac{\partial}{\partial u_{n+1}} + \cdots + S^{(4)}Q(n, u_n)\frac{\partial}{\partial u_{n+4}}. \tag{13}$$

The symmetry generator X satisfies the linearized symmetry condition (7), i.e.,

$$\begin{aligned} &S^{(5)}Q(n, u_n) - \tfrac{B_n u_n u_{n+1}}{(A_n u_{n+4} + B_n u_n)^2} S^{(4)}Q(n, u_n) - \\ &\tfrac{u_{n+4}}{A_n u_{n+4} + B_n u_n} S^{(1)}Q(n, u_n) + \tfrac{B_n u_{n+1} u_{n+4}}{(A_n u_{n+4} + B_n u_n)^2} Q(n, u_n) = 0. \end{aligned} \tag{14}$$

In order to solve for Q, we firstly eliminate the first term in the equation above, by applying the operator

$$L = \frac{\partial}{\partial u_n} + \frac{\partial u_{n+1}}{\partial u_n}\frac{\partial}{\partial u_{n+1}} \tag{15}$$

on (14), that is, we differentiate the functional equation (14) implicitly with respect to u_n by assuming u_{n+1} as a function of u_n, u_{n+4} and u_{n+5}. This yields the following (after simplification):

$$\begin{aligned} &-A_n Q(n+4, u_{n+4}) + \tfrac{A_n u_{n+4} + B_n u_n}{u_{n+1}} S^{(1)}Q(n, u_n) - (A_n u_{n+4} + B_n u_n) \\ &S^{(1)}Q'(n, u_n) - B_n Q(n, u_n) + (A_n u_{n+4} + B_n u_n) Q'(n, u_n) = 0. \end{aligned} \tag{16}$$

We differentiate the above equation with respect to u_n twice, keeping u_{n+1} constant, and obtain the following equation:

$$B_n Q''(n,u_n) + (A_n u_{n+4} + B_n u_n)Q'''(n,u_n) = 0 \tag{17}$$

whose solution is

$$Q(n,u_n) = M(n)u_n + N(n), \tag{18}$$

for some functions M and N to be found. We now substitute (18) in (14) and get the overdetermined equation

$$\begin{aligned}
&M(n+5)(A_n u_{n+1}u_{n+4}^2 + B_n u_n u_{n+1}u_{n+4}) + N(n+5)(A_n{}^2 u_{n+4}^2 + B_n{}^2 u_n{}^2 \\
&+ 2A_n B_n u_n u_{n+4}) - M(n+4)B_n u_n u_{n+1}u_{n+4} - N(n+4)B_n u_n u_{n+1} \\
&- M(n+1)(A_n u_{n+1}u_{n+4}^2 + B_n u_n u_{n+1}u_{n+4}) - N(n+1)(A_n u_{n+4}^2 \\
&+ B_n u_n u_{n+4}) + M(n)B_n u_n u_{n+1}u_{n+4} + N(n)B_n u_{n+1}u_{n+4} = 0.
\end{aligned} \tag{19}$$

which separation by monomials gives

$$\begin{aligned}
u_{n+1}u_{n+4}^2 &: \; M(n+5) - M(n+1) = 0, \\
u_n u_{n+1}u_{n+4} &: \; M(n+5) - M(n+4) - M(n+1) + M(n) = 0, \\
u_{n+4}^2 &: \; A_n N(n+5) - N(n+1) = 0, \\
u_n{}^2 &: \; N(n+5) = 0, \\
u_n u_{n+4} &: \; 2A_n N(n+5) - N(n+1) = 0, \\
u_n u_{n+1} &: \; N(n+4) = 0, \\
u_{n+1}u_{n+4} &: \; N(n) = 0.
\end{aligned} \tag{20}$$

We have

$$M(n+4) - M(n) = 0, \quad N(n) = 0. \tag{21}$$

We solve for M and obtain $M(n) = c_1 + c_2(-1)^n + c_3(i)^n + c_4(-i)^n$, where c_1, c_2, c_3 and c_4 are constants. Hence we obtain four characteristics

$$\begin{aligned}
Q_1(n,u_n) &= u_n, \\
Q_2(n,u_n) &= (-1)^n u_n, \\
Q_3(n,u_n) &= (i)^n u_n, \\
Q_4(n,u_n) &= (-i)^n u_n.
\end{aligned} \tag{22}$$

Each one of these characteristics leads to a symmetry, that is, we have as well four point symmetries

$$\begin{aligned}
X_1 &= u_n\frac{\partial}{\partial u_n} + u_{n+1}\frac{\partial}{\partial u_{n+1}} + u_{n+2}\frac{\partial}{\partial u_{n+2}} + u_{n+3}\frac{\partial}{\partial u_{n+3}} + u_{n+4}\frac{\partial}{\partial u_{n+4}}, \\
X_2 &= (-1)^n u_n\frac{\partial}{\partial u_n} - (-1)^n u_{n+1}\frac{\partial}{\partial u_{n+1}} + (-1)^n u_{n+2}\frac{\partial}{\partial u_{n+2}}
\end{aligned}$$

$$
\begin{aligned}
& -(-1)^n u_{n+3}\frac{\partial}{\partial u_{n+3}} + (-1)^n u_{n+4}\frac{\partial}{\partial u_{n+4}},\\
X_3 &= (i)^n u_n\frac{\partial}{\partial u_n} + (i)^{n+1}u_{n+1}\frac{\partial}{\partial u_{n+1}} - (i)^n u_{n+2}\frac{\partial}{\partial u_{n+2}}\\
& -(i)^{n+1}u_{n+3}\frac{\partial}{\partial u_{n+3}} + (i)^n u_{n+4}\frac{\partial}{\partial u_{n+4}},\\
X_4 &= (-i)^n u_n\frac{\partial}{\partial u_n} + (-i)^{n+1}u_{n+1}\frac{\partial}{\partial u_{n+1}} - (-i)^n u_{n+2}\frac{\partial}{\partial u_{n+2}}\\
& -(-i)^{n+1}u_{n+3}\frac{\partial}{\partial u_{n+3}} + (-i)^n u_{n+4}\frac{\partial}{\partial u_{n+4}}. \qquad (23)
\end{aligned}
$$

For some r_n, an invariant function of (14), we can write according to the theory

$$X_i r_n = 0, \quad i = 1, ..., 4. \qquad (24)$$

We therefore solve for r_n by introducing the canonical coordinate

$$s_n = \int \frac{du_n}{M(n)u_n} = \frac{1}{M(n)}\ln|u_n| \qquad (25)$$

and let

$$|r_n| = \exp[M(n)s_n - M(n+4)s_{n+4}]. \qquad (26)$$

We obtain one (new) independent variable (also an invariant) as follows

$$r_n = \frac{u_n}{u_{n+4}}, \qquad (27)$$

and

$$r_{n+1} = A_n + B_n r_n. \qquad (28)$$

The general form of (28) after iterations is

$$r_n = r_0\left(\prod_{k_1=0}^{n-1} B_{k_1}\right) + \sum_{l=0}^{n-1}\left(A_l \prod_{k_2=l+1}^{n-1} B_{k_2}\right), \qquad (29)$$

or simply

$$r_n = \frac{u_0}{u_4}\left(\prod_{k_1=0}^{n-1} B_{k_1}\right) + \sum_{l=0}^{n-1}\left(A_l \prod_{k_2=l+1}^{n-1} B_{k_2}\right). \qquad (30)$$

By going up the hierarchy created by the changes of variables, we find that

$$u_n = H_n \exp\left(-\frac{1}{4}\sum_{k=0}^{n-1}\left[1 + (-1)^{n-k} + 2\cos\left(\frac{n-k}{2}\pi\right)\right]\ln|r_k|\right), \qquad (31)$$

where the r_k's are given in (30) and H_n's are such that

$$H_0 = x_0,\ H_1 = x_1,\ H_2 = x_2,\ H_3 = x_3,\ H_{4n+j} = H_j. \qquad (32)$$

Equation (31) is the solution of (3) in a unified manner. The periodicity of H_n gives hint on the grouping of the solution and using (31) we obtain

$$u_{4n+j} = \frac{u_j}{\prod_{k_2=0}^{n-1} r_{4k_2+j}}, \quad j = 0, 1, 2, 3. \tag{33}$$

Note. Although Equation (33) is easily obtained using (27), we prefer using (31) because it gives the solution in a unified manner (this is one of the objectives of this paper) which is different from the common way many authors have been presenting their solutions.
We substitute the expression of r_{4k_2+j}, given in (30), in (33) to get

$$u_{4n+j} = \frac{u_j}{\prod_{k_2=0}^{n-1} \left[\frac{u_0}{u_4} \left(\prod_{k_1=0}^{4k_2+j-1} B_{k_1} \right) + \sum_{l=0}^{4k_2+j-1} \left(A_l \prod_{k_2=l+1}^{4k_2+j-1} B_{k_2} \right) \right]}. \tag{34}$$

It follows that (after shifting back four times) the formulas for solutions of (1) are given by

$$x_{4n+j-4} = \frac{x_{j-4}}{\prod_{k_2=0}^{n-1} \left[\frac{x_{-4}}{x_0} \left(\prod_{k_1=0}^{4k_2+j-1} b_{k_1} \right) + \sum_{l=0}^{4k_2+j-1} \left(a_l \prod_{k_2=l+1}^{4k_2+j-1} b_{k_2} \right) \right]}, \quad j = 0, 1, 2, 3. \tag{35}$$

If we let $a_n = a$ and $b_n = b$, we have

$$x_{4n+j-4} = \frac{x_{j-4}}{\prod_{k_2=0}^{n-1} \left(\frac{x_{-4}}{x_0} b^{4k_2+j} + a \sum_{l=0}^{4k_2+j-1} b^l \right)}, \quad j = 0, 1, 2, 3. \tag{36}$$

For $a = 1$ and $b = -1$, equations in (35) yield

$$\begin{aligned} x_{4n-4} &= \frac{x_0^n}{(x_{-4})^{n-1}}, \; x_{4n-3} = \frac{x_{-3} x_0^n}{(x_0 - x_{-4})^n}, \\ x_{4n-2} &= \frac{x_{-2} x_0^n}{(x_{-4})^n}, \; x_{4n-1} = \frac{x_{-1} x_0^n}{(x_0 - x_{-4})^n}, \end{aligned} \tag{37}$$

which appeared in [8] (see Theorem 3.2).

Theorem 3.1. *System*

$$x_{n+1} = \frac{x_n x_{n-3}}{2x_n - x_{n-4}} \tag{38}$$

has a periodic solution of period 4 if and only if $x_{-4} = x_0$.

Proof. Here, $a = 2$ and $b = -1$ and then (36) simplifies to

$$x_{4n+j-4} = \frac{x_{j-4}}{\prod_{k_2=0}^{n-1}\left(\frac{x_{-4}}{x_0}(-1)^{4k_2+j} + 2\sum_{l=0}^{4k_2+j-1}(-1)^l\right)}, \quad j = 0,1,2,3, \tag{39}$$

$$= \frac{x_{j-4}}{\prod_{k_2=0}^{n-1}\left(\frac{x_{-4}}{x_0}(-1)^{j} + 2\sum_{l=0}^{4k_2+j-1}(-1)^l\right)}, \quad j = 0,1,2,3, \tag{40}$$

$$= x_{j-4}, \quad j = 0,1,2,3. \tag{41}$$

□

For the sake of confirmation of the above theorem, we consider two numerical examples for system (38) with initial conditions $x_{-4} = 0.1, x_{-3} = 0.2, x_{-2} = -0.3, x_{-1} = 0.44, x_0 = 0.1$ (see Figure 1) and $x_{-4} = 0.1, x_{-3} = 0.5, x_{-2} = -0.3, x_{-1} = -0.75, x_0 = 0.1$ (see Figure 2).

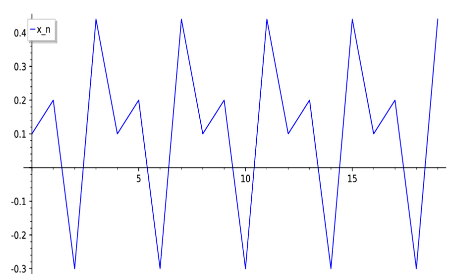

Figure 1: $a = 2, b = -1, x_{-4} = 0.1, x_{-3} = 0.2, x_{-2} = -0.3, x_{-1} = 0.44, x_0 = 0.1$.

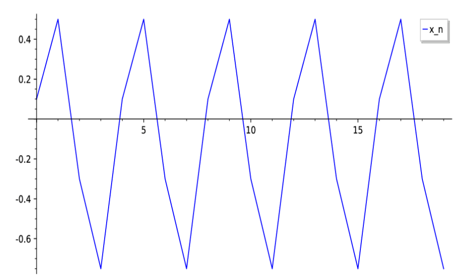

Figure 2: $a = 2, b = -1, x_{-4} = 0.1, x_{-3} = 0.5, x_{-2} = -0.3, x_{-1} = -0.75, x_0 = 0.1$.

Theorem 3.2. *System*

$$x_{n+1} = \frac{x_n x_{n-3}}{-2x_n - x_{n-4}} \tag{42}$$

has a periodic solution of period eight if and only if $x_{-4} = -x_0$.

Proof. Here, $a = -2$ and $b = -1$ and then (36) simplifies to

$$x_{8n+j-4} = \frac{x_{j-4}}{\prod_{k_2=0}^{2n-1}\left(\frac{x_{-4}}{x_0}(-1)^{4k_2+j} - 2\sum_{l=0}^{4k_2+j-1}(-1)^l\right)}, \quad j = 0,1,2,3, \tag{43}$$

$$= \frac{x_{j-4}}{\prod_{k_2=0}^{2n-1}\left(\frac{x_{-4}}{x_0}(-1)^{j} - 2\sum_{l=0}^{4k_2+j-1}(-1)^l\right)}, \quad j = 0,1,2,3, \tag{44}$$

$$= x_{j-4}, \quad j = 0,1,2,3. \tag{45}$$

□

Below are two numerical examples for system (42) with initial conditions $x_{-4} = 0.1, x_{-3} = 0.5, x_{-2} = -0.3, x_{-1} = -0.75, x_0 = -0.1$ (see Figure 3) and $x_{-4} = 0.1, x_{-3} = 0.5, x_{-2} = 0.3, x_{-1} = 0.75, x_0 = -0.1$ (see Figure 4).

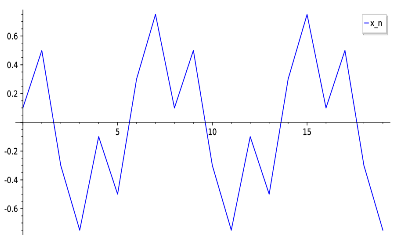

Figure 3: $a = -2, b = -1, x_{-4} = 0.1, x_{-3} = 0.5, x_{-2} = -0.3, x_{-1} = -0.75, x_0 = -0.1$.

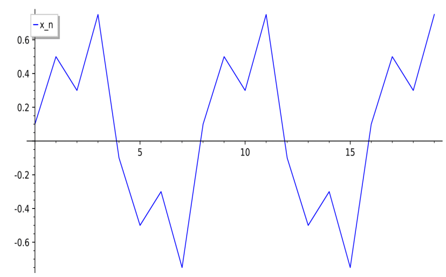

Figure 4: $a = -2, b = -1, x_{-4} = 0.1, x_{-3} = 0.5, x_{-2} = 0.3, x_{-1} = 0.75, x_0 = -0.1$.

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