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Teaching of Function Investigation for Engineering Students as a Model of Exploratory Thinking

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Abstract

Modern technologies have changed engineering training requirements, content of engineering disciplines and their teaching methods. However, teaching of base mathematics courses, and above all, the calculus course, remains mostly unchanged and does not take enough account to capabilities and requirements of modern technologies. This is due to several reasons. Fundamentals of mathematical language of engineering disciplines remained generally the same; however, the ways and possibilities of its using have changed significantly. Has disappeared not only the need to perform manually arithmetic operations, but routine symbolic procedures too, due to the numerous and ever evolving computer programs. This frees up time from the need to solve numerous algorithmic skill development exercises, towards a better understanding of the language of mathematics and development skills of the problem formulating in this language, data entry skills and interpretations of the obtained results abilities. Such changes in teaching requires first, the changes of teachers' approach to teaching of mathematics to engineering students. Note that many of the lecturers do not feel the need for the mention above changes in teaching, and continue to teach in the old manner, ignoring the nowadays demands and possibilities of using modern technology in the educational process.
Keywords: calculus course; engineering training requirements; mathematical language; algorithmic skills; function investigation; technical approach; qualitative method.

Considering these new requirements and capabilities of modern scientific calculators and computer applications, we developed the special approach for teaching calculus course and above all, the first and very essential its topic "elementary function investigation and sketching their graphs", which is intended to change the position described above. An important feature of our approach is a qualitative method investigation, combining the active use of scientific calculator and graphical applications, and requiring a minimum of routine technical work. Another feature of our approach is that it is new to the students (is different from the technical approach used in the high school) and does not depend on the differences in computing and algebraic skills of students. Our approach immediately introduces students to the basic concepts of calculus, such as limits asymptotes, precise and approximate calculations. Note, that the traditional method, based mostly on the function derivative roots' calculation, allows completing the investigation only in special simple cases and requires considerable time to algebraic calculations and use of numerical methods for more complicated functions. We also directed students to skillful use of scientific calculator in the process of function investigation, for better understanding of notions limits and asymptotes by instant calculations of function values, connected to related process. An important element of our approach is also acquaintance with data entry and storage in computer devices on the example of scientific calculator and graphical applications. Our goal is to change such situation, when the greatest efforts directed towards the development of technical skills and only a small share to enhancing of understanding and development of exploratory thinking of the students

Introduction

One of the main purposes of teaching calculus is to teach students to investigate the functions, and to sketch their graphs. Development of students' abilities to do this is very essential in formation of the so-called "function-sense" (Eisenberg & Dreyfus, [4]). As is known, the investigation of functions is a complex thought process, which includes recognizing basic properties of functions, solving equations and inequalities, performing calculations and symbolic manipulations, and returning to the world of visual perceptions of the analytical results by interpreting them with the aid of sketch of the function graph. This process is very important for developing students' theoretical thinking, and may be useful as a model of research processes in many domains of knowledge. It gives an excellent opportunity to most of the students to see mathematics "as a research activity which represents an exemplary human enterprise that aims at universalizing knowledge by development of uniformity, general accessibility and availability" (C. Keitel, [5]).

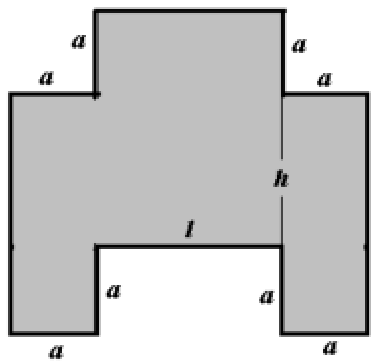
From our point of view, it is advantageous to concern to functions investigation not only as to formal mathematical task, but teach to think about the given problem as about certain real investigation process. Such approach, as it revealed by our experience, not only leads to better understanding, but also help to improve essentially the students' abilities to solve problems and check their solutions in the process of functions investigation. Using of metaphoric thinking makes this process more alive, more comprehensible, more motivated and more interesting for the students. In this way, the student advances from some primary ideas about the investigating object to more and more detailed knowledge and more precise model (graph) of it.

Learning difficulties and ways of overcoming them

In the course of teaching Calculus for novice engineering students in the last years, we face difficulties conditioned by the ever-increasing gap between the requirement of academic education and knowledge and skills acquired by students in high school. A similar situation observed in many universities in the world and marked by numerous publications. It manifests itself first that students are not ready for independent thinking since used to prescription training, and having met the task, which was not solved before, do not understand how to approach it and give up, although they have all the necessary knowledge to solve this problem. Therefore, we see one of the most important aids in changing such attitude to the problems of the new type, unfamiliar earlier. We try to develop a research approach in case of meeting new challenges and to form exploratory thinking skills of the students.

The first and very important theme of Calculus course is "function and their graphs". Therefore the initial problem of the initial homework of Calculus course is of a new (for the students) type to solve which do not need special knowledge yet not known to students. Here one of such problems:

It is required to paint the wall of some building having the shape shown in the Figure 1.



Need to describe the analytical dependence of the amount of paint on the height to which the wall is painted. Known that for painting a unit of area requires q liters of paint.

From year to year, we meet with the fact that most of students find it difficult to solve this problem and say: "We did not solve such problems earlier. How do you want that we solve it? Give us a similar example first and show what to do". It takes a lot of effort to explain that it is important to solve new problems and not just exercises, according to samples made before. We appeal to students: "Try to solve the problem based on your own knowledge, and don't wait for someone else do it and then explain to you". For example, after explaining the solution of the problem of painting the wall, student begins to understand that he had all the necessary knowledge from high school, however did not make enough effort to comprehend and solve it. Based on this and other similar problem we try to rebuild the attitude of students to study mathematics as a series of routine tasks and algorithms and start believing in the ability to solve problems based on existing to this moment their knowledge. We also explain that each exam will have some new tasks requires independent thinking and similar ones were not solved before.

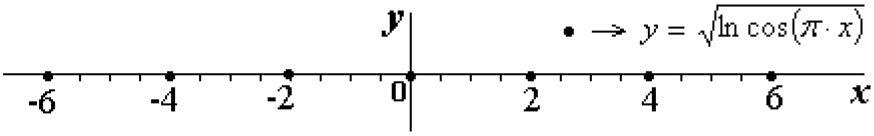
Our model of teaching calculus, based on constructivist learning theory, is from the very beginning of the course engage the students in the function investigation process on the current level of knowledge, and to form the concepts, necessary for more advanced exploration, as the course development. Our goal not only to improve the understanding of the particular topic of the calculus course but also to integrate STEM knowledge and skills in the calculus teaching process. Therefore, we explain this theme not as function investigation only, but as a process of exploration thinking, necessary for all science, with active use of modern technology, actively using engineering examples. Our research study, conducted in many groups of engineering students, demonstrated enhancing of students understanding of the theme and all notions connected with it. Note that since our research approach, we observed increasing of cognitive attitude to study of Calculus and improving the student's exam results too.

Our experience of teaching shows that many engineering students see mathematics as an abstract subject in which they must operate with special symbols, in accordance with artificial rules. They can learn to do these operations successfully, but it is difficult for them to go on from formal symbolic calculations to certain form of real-life representation (for example, graphic representation). As a result, many students fail at the stage of syntheses of formal symbolic results in order to construct visual models of investigating objects (functions). In order to reduce these difficulties and to give a real-life sense to the function investigation process, we use non-formal metaphoric schemes of function investigation, which assume answers on certain non-formal questions; and ask to give proper visual interpretations of each stage. The importance of metaphoric thoughts in teaching mathematics noted in a number educational research studies (J.I. Acevedo, V.Font, J. Gumenez, [1]). We use some metaphoric approaches, connected with investigations of real objects, and sometimes refer to different interpretations on different stages of function investigation process. In accordance with our metaphoric scheme, the student must give a meaning to every stage of function investigation process, by means of words and graphical representations. After each stage, the student must decide whether to complete it, or go on to the next stage. The differences between these two approaches illustrated by the following example of investigation of the function $f(x)=\sqrt{\ln\cos(\pi\cdot x)}$. This question address to 235 calculus students learning in accordance with standard approaches. Because of the use of the standard scheme of investigation, 65% of the students have performed all stages of function investigation and calculated the derivative of the function. Only 12% of them understood that investigation must be stopped at the first stage without going on to other stages. 20% of the students have began by calculating the function derivative, which in fact, does not exist in any point. On the contrary, in an experimental group (82 students) only 13% of the students calculated the derivative and 79% have understood at the first stage of investigation that the graph is only a series of isolated points on the *x*-axis. Here is the investigation process of the function $f(x)=\sqrt{\ln\cos(\pi x)}$.

Stage 1. Domain of definition (conditions for existence of the investigating object). It is a composite object, which requires series of sequential calculations:

$$1)x \to x_1 = \pi x;\ 2)x_1 \to x_2 = \cos x_1;\ 3)x_2 \to x_3 = \ln x_2;\ 4)x_3 \to x_4 = \sqrt{x_3}$$

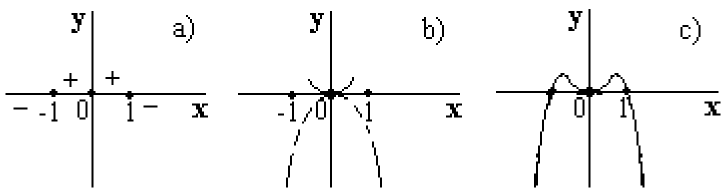
The fourth computation is possible only if $x_3 \ge 0$ thus $\ln x_2 \ge 0$. From here $x_2 \ge 1$, thus $\cos x_1 \ge 1$. But the last inequality is valid only in the form $\cos(\pi x) = 1$, which means that only isolated points $2k, k = 0, \pm 1, \pm 2, ...$ belong to the domain of $f(x)=\sqrt{\ln\cos(\pi \cdot x)}$. In all the above points, the value of the function is equals to zero. What are the graphical results of this stage of investigation? The graph is a set of points $P_k(2k,0), k = 0, \pm 1, \pm 2, ...$ on the *x*-axis.



Therefore, this is the end of our investigation. We have obtained complete information about the function and its graph. There is no sense to continue investigation and to go on to the next stages.

Comments. In fact, as mentioned above, many of the students of the control group (69%) have continued their investigation, including computation of the first derivative of the function and verifying existence of extreme points. The main reason of this is the habit (acquired from the school learning experience) to do first at all the technical stages of the given formal scheme and only after that try to analyze the obtained information and construct from it the searched graphical image. Note, the essential part of our approach is aimed to break this habit and to reform it to the habit to give a meaning to every stage of function investigation, to understand graphically the analytical results of algebraic computations and to sketch simultaneously the appropriate curves on the coordinate plane.

The important part of our approach is to cultivate the understanding of the fact that for investigation of the basic learning objects in the calculus course namely, elementary functions - it is crucial to know the primary elements, from which them



can be constructed, and to keep in the memory the proper graphical images. This enables using in the investigation process such sentences as "the graph of the function $y = x^4 - x^6$ for x values of x close to the origin is similar to the graph $y = x^4$ $\left[y = x^4(1 - x^2)\right]$ and for the large positive or negative values of x it is similar to $y = -x^6$ $\left[y = -x^6\left(1 - x^{-2}\right)\right]$. With the help of such considerations already, after two first stages of investigation the appropriate draft image (figure (c)) of the graph may be drown.

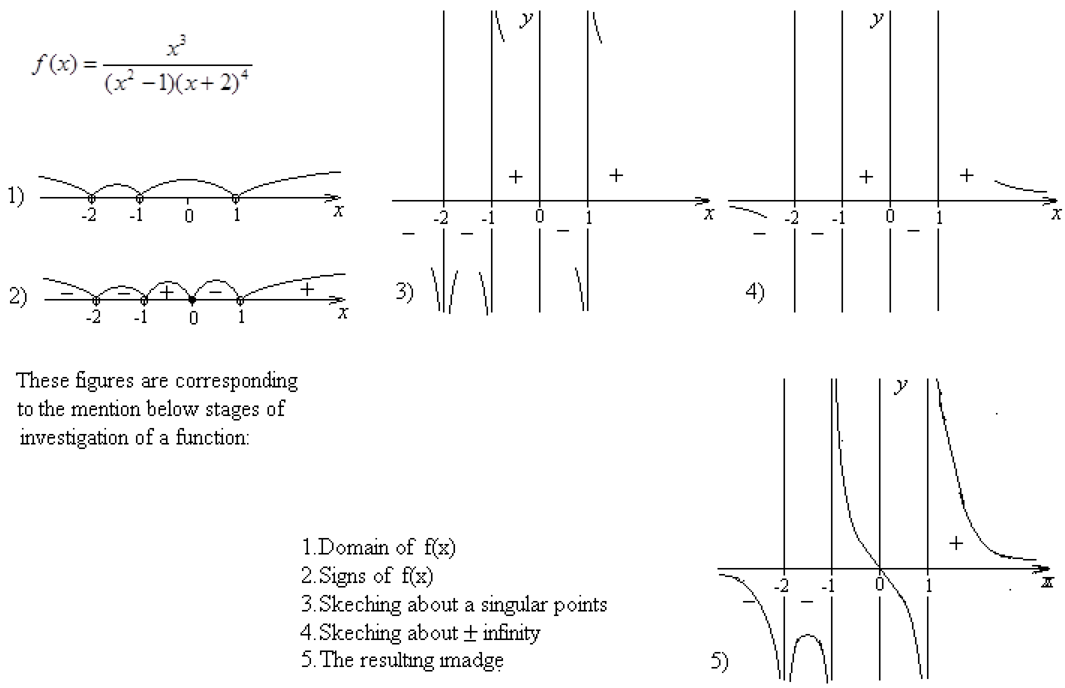
Metaphoric scheme of function investigation

We use the following scheme of function investigation including the formal stages, non-formal explanations of them and the graphical images, sketched after first stages.

Formal Stage of investigation	**Non-formal questions and graphical results of this stage of investigation**
1. Finding the domain of definitions.	What are conditions of existence of the given object? <u>Graphical result</u>: appropriate intervals on the *x*-axis.
2. Analysis of the formula structure. Checking equivalent forms of given formula of the function. Relation to primary functions.	What are primary elements from which investigating object consist of? Whether the object may represented in other equivalent form? What similar objects I know? What are the conditions for similarity of these objects to the investigating one?
3. Finding "the signs of the function".	Where is the object located? - The primary information. (Were the graph - over or under the *x*- axis?) <u>Graphical result</u>: to mark areas on the coordinate plane where points of the graph can be.
4. Clarifying the behavior of the function at singular points.	What is the state of the object with respect to its existence interval? How looks the graph near the singular points? <u>Graphical result</u>: sketch the graph near the singular points (vertical asymptotes if exist).
5. Clarifying the behavior of the function at "infinity".	What kind of behavior of the object in the distant future ($+\infty$) we can predict? The same question about the remote past ($-\infty$)? <u>Graphical result</u>: to sketch the graph at infinity (to mark horizontal asymptotes if exist)
6. Performance the initial drawing graph of the function.	What is our comprehension about the object based on the obtained information? How it related to the whole picture? <u>Graphical result</u>: to draw the whole sketch of the graph by assembling the obtained parts.
7. Using of the first derivative of the function (clarifying the extreme points, increasing or decreasing of the function).	Clarifying behavior of the investigated object by means of some test (which may be "expensive", needs time for its execution and at the end not give any results. <u>Graphical result</u>: to indicate exactly extreme points and intervals of the function increasing or decreasing
8. Using of the second derivative of the function (clarifying the inflection points, intervals convexity of the graph of the function).	More thoroughly clarifying behavior of the investigating object by means of more complicated special test. <u>Graphical result</u>: to mark the found inflection points and the intervals of convexity (concavity) of the function.
9. Final sketching of the function graph.	Improving the draft (obtained formerly) based on results of the tests 7-8, or in the contrary, checking the derivatives computations if they are in essential contradiction with this draft.

Principal features of experimental approach and some learning examples

From the beginning of the investigation process, we concentrate the students on the principal aims of the process - to study the properties of the given function and answer the question what the graph of the function looks like. We direct the students to think about various levels of information relating to the function. What known exactly about it? (i.e., proved conclusions) and what we may assume with more or less confidence? We recommend trying to sketch the graph from the first stages of investigation. If not all, then piece by piece. The following stages may confirm some of the previous assumptions, but may also reject or change them. In this way, the students see the function investigation as a real-life investigation process where, at every stage, we clarify some properties of the object. We teach that it is better to come to preliminary conclusions about the object (though not 100% trustworthy) than to wait for more complex analysis (by means of more complex and more expensive tools). It is important to understand that the derivative of the function is in many cases more complicated than the original function, needs own investigation (two first stages at least) and may not help us at all. As example of such situation, let us investigate the function $f(x)=\dfrac{x^3}{(x^2-1)(x+2)^4}$. Here are the short schemes of pre-derivative investigation and the sketch of the graph:



As a result, in the experimental group, 90% of the students sketch the suitable graphical image of the function. On the contrary, in the control group only 4% of the students sketch the proper graphical image. Most of the students in the control group spend a lot of time to symbolic calculation of the first derivative $f'(x) = -\frac{-x^2\left(3x^3 - 2x^2 - x + 6\right)}{(x^2-1)^2(x+2)^5}$ and try to solve the equation $f'(x) = 0$ without success. We note that in the experimental group, 85% of the students sketch the suitable graphic images in less than 5 minutes and only few of them tried to make use of the derivative, because most of the students understood beforehand that it will take many time (will be expensive) and may be pointless at the end. Only one student from the control group who nevertheless tried to calculate the second derivative, showing:

$$f''(x) = \frac{2x(6x^6 - 8x^5 - 3x^4 + 36x^3 + 5x^2 - 12x + 12)}{(x^2-1)^3(x+2)^6}.$$

For students in the experimental group, known that, in general, the derivatives are more complicated than the original function and so should not put much hope in it.

As concerning the finding of asymptotes, we recommend to follow the definition of asymptote of the function $f(x)$ at the $+\infty$ [$-\infty$] as a straight line $y = mx + n$, so that the function may be represented in the form $f(x) = mx + n + r(x)$, where $r(x) \underset{x\to+\infty}{\to} 0$ $\left[r(x) \underset{x\to-\infty}{\to} 0\right]$. (We recommend using known formula for finding the values of m and n by computation of the appropriate limits only in the cases where it is not obvious how to obtain the presentation of the function mentioned above). Below you can see additional example which illustrates this process.

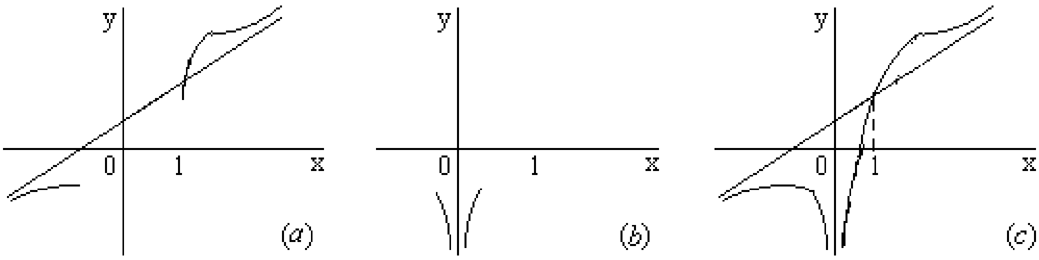
Example. Investigate the function $y = \frac{3x^5 + 2x^4 + x - 1}{x^4}$ and sketch its graph.

<u>1.</u> Domain of definition is all values of x, except $x = 0$.

<u>2-6</u> .The equivalent form of the primary formula is $y = 3x + 2 + \frac{x-1}{x^4}$

From the last formula we saw that the asymptote at the $+\infty$ and at the $-\infty$ is a straight-line y=3x+2, because $\left[r(x) = \frac{x-1}{x^4} \underset{x\to\pm\infty}{\to} 0\right]$. From this representation it is clear also that for x >1 the graph of the function is located above the asymptote y=3x+2, for x=1 the graph intersect it and for x<1 the graph is under the asymptote (this follows from the sign of $r(x)$).

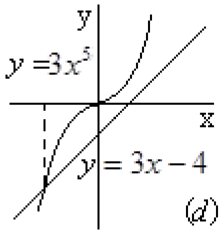
From the primary formula we can conclude that near x = 0 (which is the end point of the function domain), the curve is similar $y = -\frac{1}{x^4}$.



It is important to point out that in this example we have "jumped" over the stage (3), dealing with clarifying the sign of the function, because it was difficult to do this in algebraic way. Nevertheless, from the obtained draft we can suggest that the equation $3x^5+2x^4+x-1=0$ has only one real root (which is between 0 and 1) and the signs are as we can see on the figure (c). It proved easily on the first derivative stage.
At this stage it is preferable to use the next equivalent form the function representation $f(x)=3x+2+\frac{1}{x^3}-\frac{1}{x^4}$, from which follows:

$$f'(x)=3-\frac{3}{x^4}+\frac{4}{x^5}=\frac{3x^5-3x+4}{x^5}.$$

From the figure (d) it follows the equation



$3x^5-3x+4=0 \Leftrightarrow 3x^5=3x-4$ has only one real root, which is negative, and this prove our graphical assumption (c).

The second derivative $f''(x)=\frac{12}{x^5}-\frac{20}{x^6}=4\left(\frac{3x-5}{x^6}\right)$ give us the inflection point x=3 that also corresponds to our draft (c). Note that we also try to encourage the use of simple numerical methods accompanied by appropriate graphical considerations, for finding of extreme or inflection points of the function.
An experiment conducted in two engineering students groups. The students both of the groups studied in the preparatory course. The final grades scores tested in the t-test at a 95% significance level and was found that there was no significant difference between the averages. Group A of 60 students and Group B of 62 students.
Students in Group A studied the calculus course in the standard way and in Group B, the students learned according to the following model:
I. Frequent use of various sub-languages (analytical, literal, graphic and numerical)

II. Investigating a function according to the model presented in the article.
In the middle of semester, the two groups were tested the same test and at the end of the semester, as well all students (in both groups) were tested in the same test. The results (scores) were compared and the B group results were significantly better than the control group A results. In all tests conducted, we compared the results of the tests, the semester test results, the results of function investigation scores and the non-function investigation scores, and all found that the results of group B were significantly better than the results of group A.

Below are the findings

Comparison of two group math scores in preparatory course

	Group A	Group B
Number of students (n)	60	62
Average value (Y)	69.97	70.72
Standard deviation (S)	10.91	11.07

F- test checks whether the 2 group variances are equal or different.
95% significance level test, For example, states that if it exists

$$F\left(\frac{S_2^2}{S_1^2}, n_2 - 1, n_1 - 1\right) > 95\%$$

Then, the variances are significantly different. In this case F = 52% therefore the variances are equal and possible use common variance in the t -test.
The t- test between averages with shared variance calculate as follows:

$$z = \frac{Y_2 - Y_1}{S\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}}$$

If it exists $t(z, n_1 + n_2 - 2) > 95\%$, so the averages are significantly different. In this case t = 65%, so according to t-test there is no significant difference between the averages.

Comparison of two group math scores in the final exam

	Group A	Group B
Number of students (n)	60	62
Average value (Y)	68.85	84.03
Standard deviation (S)	19.16	12.23

The F- test checks whether the two group variances are equal or different.
95% significance level test, For example, states that if it exists

$$F\left(\frac{S_2^2}{S_1^2}, n_2 - 1, n_1 - 1\right) > 95\%$$

Then the variances are significantly different. In this case F = 96%, therefore the variances are different and it is not possible to use the common variance in the t-test.

The t- test between averages with different variances calculated as follows:

$$z = \frac{Y_2 - Y_1}{\sqrt{\frac{S_1^2}{n_1} + \frac{S_2^2}{n_2}}}$$

If it exists: $t(z, n_1 + n_2 - 2) > 95\%$

Then the averages are significantly different. In this case t = 100%, so according to the t- test, the difference between the averages is significant.

Comparison of two groups math scores in the mid-test

	Group A	Group B
Number of students (n)	59	62
Average value (Y)	66.2	75.6
Standard deviation (S)	20.6	14.8

Tested and found that the variances are different, the t- test between averages with different variances calculated as follows:

$$z = \frac{Y_2 - Y_1}{\sqrt{\frac{S_1^2}{n_1} + \frac{S_2^2}{n_2}}}$$

In our case t = 99.7%. So according to the t- test, the difference between the averages is significant.

Comparison of two group's math scores without function investigation

	Group A	Group B
Number of students (n)	60	62
Average value (Y)	53.3	66.1
Standard deviation (S)	16.7	10.0

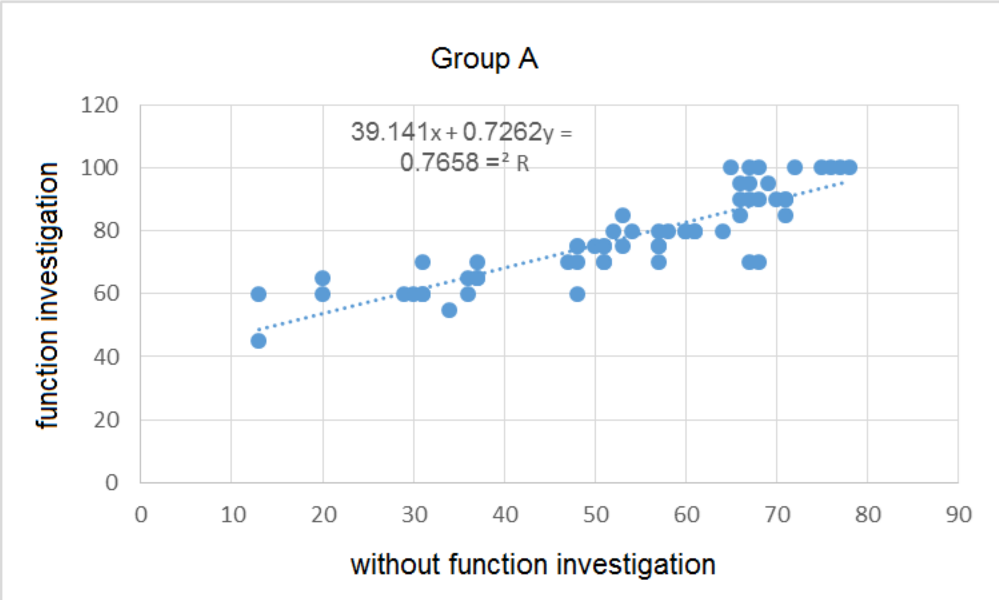
Tested and found that the variances are different. In this case t = 99.9%. According to the t-test, the difference between the averages is significant.

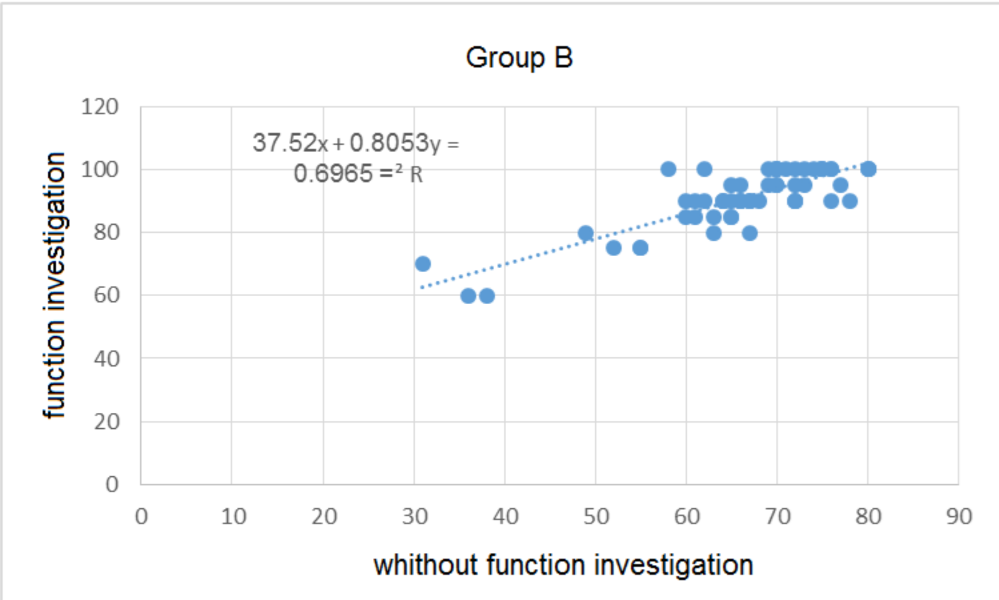
Comparison of two group's math scores for function investigation

	Group A	Group B
Number of students (n)	60	61
Average value (Y)	77.8	91
Standard deviation (S)	13.9	9.5

Tested and found that the variances are different, in this case t = 99.9%. So according to the t-test, the difference between the averages is significant.

Regression between results without scores function investigation and with scores function investigation





In both cases, there is a positive correlation between the scores obtained without function investigation and with scores function investigation.

Conclusions

The research results reveal that the students in the experimental group are doing the same task as in the standard one, but in a more self-supervisory way. They are more self-confident in their results and make fewer mistakes, not only in graphic representations, but also in formal operations, such as differentiation and limits

calculations. We observed the students of the experimental group correcting mistakes of such symbol operations, if they saw that their results were inconsistent with the previous sketching graph. On the other hand, the students in the control group were trying to repair the sketch of the graph in order to make it more corresponding to mistaken analytical results. Note that the students in the experimental group sketched the graph of the function more quickly as in the control group. Furthermore, the experimental study improves students' motivation in learning calculus, not only as an instrument of solution of special mathematical tasks, but also as a way of thinking about complicated problems. Nowadays in the era of high technologies is less important to perform numerical or symbolic calculations, while there is an urgent need to develop the qualitative thinking of students. Our approach promote this development in the learning process.
As a matter in the beginning, our method faced with a negative attitude of teachers. Many of them thought that at first a student need to get the necessary tools otherwise the investigation would be incomplete. It took a lot of effort to convince them that at the beginning of the study of any phenomena it is never complete and in many cases requires the development of a special apparatus for more detailed investigation. There was also resistance from students who wanted first a clear computational algorithm for function investigation. However, at the end both teachers and students were convinced of the effectiveness of our approach and became its active supporters. Note again that the essential features of our approach are in use of new problems, unpredictable and simple solution of previously known tasks and wide use of modern technology in the learning process.

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