International Journal of Contemporary Mathematical Sciences
Vol. 14, 2019, no. 4, 225 - 235
HIKARI Ltd, www.m-hikari.com
https://doi.org/10.12988/ijcms.2019.91023

An Algorithm for Equilibrium in a Dynamic Stochastic Monetary Economy

Maria B. Chiarolla

Dipartimento di Scienze dell'Economia
Centro Ecotekne, Università del Salento
73100 Lecce, Italy

Abstract

This paper establishes an algorithm for the equilibrium in a stochastic continuous time economy model, on a finite time interval, including a representative agent maximizing her expected total utility of consumption, leisure, and money, and a single firm that optimally produces the consumption good and maximizes its expected total profits based on employment rate and money held. First, under the assumption of equilibrium, a link between the firm's control problem and the representative agent's optimal expected total utility is obtained. Then such link is exploited to establish an algorithm for equilibrium.

Mathematics Subject Classification: 91B51, 91B62, 91B40, 60H30

Keywords: dynamic stochastic general equilibrium, financial market, wage rate, employment rate, consumption decisions, investment rate, convex analysis, stochastic analysis

1 Introduction

The nominal model with exogeneous endowment of Chiarolla and Haussmann (2001) [2] attempted to build a macroeconomic model, based on the actions of both the individual agents and the firms, that extented the stochastic dynamic equilibrium model of Karatzas, Lehockzy and Shreve (1990) [3] by endogenizing dividend and earning processes. The agents derived utility from consumption of both goods and leisure, hence there were two "prices", one for the good and one for leisure. There was a production function $R(t, L(t))$, that transformed labour L into the consumable good, and a wage process w. Then, profits were endogenously determined via the *static* firm's problem of maximizing the profit rate $\delta(t) = p(t)R(t, L(t)) - w(t)L(t)$ by the choice of employment rate process $L(t)$, hence they were distributed as dividends to the shareholders. As usual, the equilibrium price of the productive asset coincided with the expected value of the discounted future dividend stream.

The present model allows capital investment in the production process and it adds money in the economy for the purpose of facilitating the transactions of both the agent and the firm. The money supply is exogeneous and determined by a Central Bank. As in the setting of the "monetary" model of Basak and Gallmeyer (1999) [1] (itself based on the framework of Karatzas, Lehoczky and Shreve (1990) [3]), here money is thought as another good, hence another real quantity rather than a nominal one. The model includes an endogenous complete financial market consisting of three instruments, a real bond to hedge against inflation, a nominal bond to finance production, and the productive asset (incomplete markets can be completed by introducing auxiliary tradables as in Karatzas, Lehoczky and Shreve (1990) [3]). Contrary to what happens in Basak and Gallmeyer (1999) [1], and generalizing the setting of Chiarolla and Haussmann (2001) [2], here the productive asset pays an endogenous dividend stream (in real terms) $\tilde{\delta}(t)$ to the shareholders; that is, corporate profits are distributed as dividends at a rate $\tilde{\delta}(t) = \tilde{R}(t, C^{\nu}(t), L(t), \tilde{K}(t)) - \tilde{w}(t)L(t) - r(t)\tilde{K}(t) - \nu(t)$.

The market, the firm, and the representative agent's utility maximization problem are defined in Section 2. Under the assumption that the expected total discounted firm's profit $J(\nu, L, K)$ is strictly concave on the closed convex set $\mathcal{S}$ of stochastic controls (ν, L, K) consisting of real investment rate ν (which impacts the technology process), employment rate L and money to be borrowed K, the firm's maximum profit problem has a unique solution, if it exists, and can be identified from the condition $0 \in \partial J(\nu, L, K)$, the subgradient set of J (cf. Rockafellar 1970 [4]). The agent problem is solved in Section 2. Section 3 is concerned with equilibrium. The novelty of this paper is a link between the firm's control problem and the representative agent's optimal expected total utility. Such link is then exploited to establish an algorithm for the existence of equilibrium. The real interest rate, the nominal interest rate, the wage rate, the employment rate, the money balances, the consumption rate, the price of money, hence the price of the good and the price of the productive asset are all determined from equilibrium considerations. The only exogenous parameters in the model are those of the money supply (chosen by the Central Bank).

2 The Economy Model

An economy with finite horizon T is built on a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_{t\in[0,T]}, P)$ with the filtration $\{\mathcal{F}_t : t \in [0,T]\}$ completed with respect to the filtration generated by an exogenous two-dimensional Brownian motion $\{W(t), t \in [0,T]\}$. There is one firm producing a single kind of perishable consumption good which sells at price $p(t)$ units of local currency at time t, and there is one agent providing labour to the firm, consuming the good, and owning the productive asset. The price process $p(t)$ is *endogenous* and will be determined by equilibrium considerations.

If $q(t) = 1/p(t)$ is the price of money in units of the commodity, then multiplication by q changes nominal quantities into real quantities. Real variables are denoted using "tilde", hence in general $\tilde{x} = qx$. The superscript $^\top$ stands for transpose. It is reasonable to assume that money is worth nothing beyond the horizon T, then in equilibrium the *endogenous* process q will be shown to satisfy

$$\begin{cases} dq(t) = q(t)[\mu_q(t)dt + \sigma_q^\top(t)dW(t)], & t \in [0,T), \\ q(T) = 0. \end{cases} \tag{1}$$

The country's money supply $M(t)$ is *exogenous* and given by the Itô process

$$\begin{cases} dM(t) = M(t)[\mu_M(t)dt + \sigma_M^\top(t)dW(t)], & t \in (0,T], \\ M(0) = M_0. \end{cases} \tag{2}$$

The market is *endogenous* in this model and consists of a real bond, a nominal bond and a productive asset. The real bond is locally riskless in real terms and and its price is given by

$$\begin{cases} d\tilde{B}(t) = \tilde{r}(t)\tilde{B}(t)\,dt, & t \in (0,T], \\ \tilde{B}(0) \quad \text{endogenous} \end{cases} \tag{3}$$

where $\tilde{r}$ is the *real interest rate*. By writing $\tilde{B}$ in nominal terms (that is, after multiplication by q^{-1}) one obtains an inflation-indexed bond (or TIPS, i.e. Treasury inflation-protected security), that allows hedging against inflation.

The nominal bond is needed to finance production and its price is given by

$$\begin{cases} dB_0(t) = r(t)B_0(t)\,dt, & t \in (0,T], \\ B_0(0) = 1, \end{cases} \tag{4}$$

where r is the *nominal interest rate*.

The nominal price of the productive asset (there is a total of one divisible share) is the nonnegative $\mathcal{F}_t$-semimartingale

$$dA(t) + \delta(t)dt = \mu_A(t)\,dt + \sigma_A^\top(t)dW(t), \qquad t \in [0,T), \tag{5}$$

with $A(T) < +\infty$ almost surely. The process δ represents the rate at which dividends are paid out to the shareholders.

The real prices $\tilde{B}_0 = qB_0$ of the nominal bond and $\tilde{A} = qA$ of the productive asset satisfy, respectively,

$$\begin{cases} d\tilde{B}_0(t) = \tilde{B}_0(t)[(r(t) + \mu_q(t))dt + \sigma_q^\top(t)dW(t)], & t \in (0,T], \\ \tilde{B}_0(0) = q(0); \end{cases} \tag{6}$$

and

$$\begin{cases} d\tilde{A}(t) + \tilde{\delta}(t)dt = q(t)[\mu_{\tilde{A}}(t)\,dt + \sigma_{\tilde{A}}^\top(t)dW(t)], & t \in [0,T), \\ \tilde{A}(T) = 0, \end{cases} \tag{7}$$

where $\tilde{\delta}$ is the real dividend process and $\mu_{\tilde{A}} = \mu_q A + \mu_A + \sigma_A^\top \sigma_q,\ \sigma_{\tilde{A}} = A\sigma_q + \sigma_A$. Notice that $\tilde{A}(t)$ represents the real value of the productive asset at time t, and hence it is zero at the terminal time T.

The measurable, $\mathcal{F}_t$-adapted real-valued drift processes $\tilde{r}, r, \mu_q, \mu_A$ and vector-valued diffusion process σ_q and σ_A will be determined endogenously by the equilibrium arguments. To avoid technicalities, these coefficients are assumed to be integrable or bounded as required. This can be verified in specific examples.

2.1 The Risk-Neutral Probability Measure

Set

$$\Sigma(t) := \begin{pmatrix} \sigma_q^\top(t) \\ \sigma_{\tilde{A}}^\top(t) \end{pmatrix} \tag{8}$$

and assume

$$\Sigma(t) \ \text{ invertible for all } \ t \in [0,T], \text{ a.s.} \tag{9}$$

Then the market is complete (i.e. the agent can *hedge* all the risk) and the *market price of risk* is well defined as the unique solution θ of

$$\Sigma(t)\theta(t) = \begin{pmatrix} \mu_q(t) + r(t) - \tilde{r}(t) \\ \mu_{\tilde{A}}(t) - \tilde{r}(t)A(t) \end{pmatrix}. \tag{10}$$

Notice that from the first row the *Fisher Equation* follows

$$\mu_q(t) + r(t) - \tilde{r}(t) = \sigma_q^\top(t)\theta(t). \tag{11}$$

Assume that the exponential process

$$Z(t) = \exp\Big[-\int_0^t \theta(s)\cdot dW(s) - \frac{1}{2}\int_0^t \|\theta(s)\|^2 ds \Big], \qquad t \in [0,T]$$

is a martingale. Then the probability measure $P^\circ(A) := E\{Z(T)\mathbb{1}_A\},\ A \in \mathcal{F}_T$, is the *risk-neutral* probability measure equivalent to P with Radon-Nikodym derivative $\frac{dP^\circ}{dP}\Big|_{\mathcal{F}_t} = Z(t), \quad t \in [0,T]$. The process $\zeta(t) := \tilde{B}^{-1}(t)Z(t)$ is the *real state-price density* (or *deflator*) and satisfies

$$\begin{cases} d\zeta(t) = \zeta(t)[-\tilde{r}(t)dt - \theta^\top(t)dW(t)], & t \in (0,T], \\ \zeta(0) = \tilde{B}(0)^{-1}. \end{cases} \tag{12}$$

Finally, $W^\circ(t) := W(t) + \int_0^t \theta(s)ds$ is a standard Brownian motion under P°.

Under the risk-neutral measure P° one has

$$\begin{aligned} &dq(t) = q(t)[(\tilde{r}(t) - r(t))dt + \sigma_q^\top dW^\circ(t)], \\ &d\tilde{B}_0(t) = \tilde{B}_0(t)[\tilde{r}(t)\,dt + \sigma_q^\top dW^\circ(t)], \\ &d\tilde{A}(t) + \tilde{\delta}(t)dt = \tilde{r}(t)\tilde{A}(t)\,dt + q(t)\sigma_{\tilde{A}}^\top(t)dW^\circ(t), \\ &d\tilde{M}(t) = \tilde{M}(t)[(\tilde{r}(t) + \eta(t) - r(t))dt + (\sigma_M^\top(t) + \sigma_q^\top(t))dW^\circ(t)], \end{aligned} \tag{13}$$

with $\eta(t) := \mu_M(t) - \sigma_M^\top(t)(\theta(t) - \sigma_q)$. In particular, under the assumption $E\{\int_0^T \zeta(t)q^2(t)dt\} < \infty$, one can integrate $\tilde{B}^{-1}q$ from t to T to obtain the representation

$$\begin{aligned} q(t) &= \tilde{B}(t)E^\circ\Big\{\int_t^T \tilde{B}^{-1}(s)r(s)q(s)ds\Big|\mathcal{F}_t\Big\} \\ &= \frac{1}{\zeta(t)}E\Big\{\int_t^T \zeta(s)r(s)q(s)ds\Big|\mathcal{F}_t\Big\}. \end{aligned} \tag{14}$$

Similarly the following representation is obtained for the real money supply

$$\tilde{M}(t) = \frac{1}{\zeta(t)}E\Big\{\int_t^T \zeta(s)(r(s) - \eta(s))\tilde{M}(s)ds\Big|\mathcal{F}_t\Big\}, \tag{15}$$

and the price of the productive asset is expressed as the expected present value of future dividends, that is

$$\tilde{A}(t) = \frac{1}{\zeta(t)}E\Big\{\int_t^T \zeta(s)\tilde{\delta}(s)ds\Big|\mathcal{F}_t\Big\}. \tag{16}$$

2.2 The Firm

The firm produces the consumption good at rate $R(t, C^\nu(t), L(t), \tilde{K}(t))$ when it employs $L(t)$ units of labour, it borrows nominal capital $K(t)$ with $K(0) = \phi_0(0)B_0(0) = \phi_0(0)$, and it retains some real earnings $\nu(t) \geq 0$ to improve its capacity C^ν. Labour, at time t, costs $w(t)$ units of currency per unit of time, and is provided by the single (i.e. representative) agent present in the economy, so $0 \leq L(t) \leq 1$. On the other hand, the firm borrows the nominal capital $K(t) = \tilde{K}(t)/q(t)$ and pays interests at rate $r(t)$.

Corporate profits are distributed as dividends to the shareholder at a rate (in real terms)

$$\tilde{\delta}(t) := R(t, C^\nu(t), L(t), \tilde{K}(t)) - \tilde{w}(t)L(t) - r(t)\tilde{K}(t) - \nu(t), \quad t \in [0,T]. \tag{17}$$

For each time t the *production function* $R(t, C, L, \tilde{K})$ is assumed to be measurable, continuous on $[0, +\infty) \times [0, 1] \times [0, +\infty)$, strictly concave, increasing in $C, L, \tilde{K}$, continuously differentiable on $(0, +\infty) \times (0, 1) \times (0, +\infty)$, with $0 \leq R(t, C, L, \tilde{K}) \leq \kappa_R(1 + C + L + \tilde{K})$.

The nonnegative, measurable, $\mathcal{F}_t$-adapted *wage process* $\{w(t) : t \in [0, T]\}$ is *endogenous* and will be determined by equilibrium considerations.

The manager of the firm optimally chooses labour $L \geq 0$, capital $K \geq 0$ (in real terms, $\tilde{K}$), and real investment $\nu \geq 0$ so as to maximize the expected total discounted real profit

$$J(\nu, L, K) := E\Big\{\int_0^T \zeta(t)[R(t, C^\nu(t), L(t), \tilde{K}(t)) - \tilde{w}(t)L(t) - r(t)\tilde{K}(t) - \nu(t)]dt\Big\} \quad (18)$$

over the closed convex set

$$\mathcal{S} := \Big\{(\nu, L, K) : \nu, L, K \geq 0 \text{ a.s.}, \mathcal{F}_t\text{-adapted}, \zeta\nu, L, K \in L^1((0, T) \times \Omega)\Big\}. \quad (19)$$

That is, the manager's optimal profit problem is

$$\max_{(\nu, L, K) \in \mathcal{S}} J(\nu, L, K). \quad (20)$$

Under the assumption of $J(\nu, L, K) < \infty$ and J strictly concave on $\mathcal{S}$, the solution of (20) is unique, if it exists, and can be identified from the condition $0 \in \partial J(\nu, L, K)$, the subgradient set of J (cf. [4]).

2.3 The Agent

The agent selects a personal consumption, labour (or leisure), and money holding strategy by optimizing her utility. Initially the agent owns the totality of the productive asset and an amount of money $m(0) = M(0) - K(0)$, where $K(0)$ is held by the firm's manager. The agent chooses her *consumption process* $\{c(t) : t \in [0, T]\}$ measured in units of the commodity with $c(t) \geq 0$ and $\sup_{t \in [0,T]} c(t) < +\infty$ a.s.; her *money-holding process* $\{m(t) : t \in [0, T]\}$ measured in local currency with $m(t) \geq 0$ and $\sup_{t \in [0,T]} m(t) < +\infty$ a.s. She also chooses her *leisure process* $\{l(t) : t \in [0, T]\}$, a measurable, $\mathcal{F}_t$-adapted process with $l(t) \in [0, 1]$ for all $t \in [0, T]$, a.s.

In order to finance her consumption and labour/leisure strategy she must pick her *productive asset share process* $\{\pi(t) : t \in [0, T]\}$ (with $\pi(0) = 1$), her *financial asset portfolio process* of real bond and nominal bond $\{(\phi, \phi_0)(t) : t \in [0, T]\}$ with $\phi(0) = 0$ and $\phi_0(0) = K(0)$, such that $\sup_{t \in [0,T]} |\pi(t)| < +\infty$, $\int_0^T [|\phi(t)|^2 + |\phi_0(t)|^2]dt < +\infty$ almost surely. (Notice that $M(0) = m(0) + \phi_0(0)$.)

The components of the portfolio processes are measured in numbers of shares and may be either positive or negative, i.e. short selling and borrowing are allowed. Here $1 - l(t)$ denotes the intensity with which the agent works. Her *earnings process* is given by the measurable, $\mathcal{F}_t$-adapted, nonnegative, bounded process $w(t)(1-l(t))$, $t \in [0,T]$, measured in units of the local currency (here $w(t)$ is the *wage rate process*). Money transfers from the government to the agent take place at the *welfare rate* $w_g(t)$, measured in units of the local currency.

The government is **assumed** to finance its net payments to agents by printing money, that is $dw_g(t) = dM(t)$. The wealth of the agent at time t is $X(t) := \pi(t)A(t) + \phi(t)q^{-1}(t)\tilde{B}(t) + \phi_0(t)B_0(t) + m(t)$, so the wealth lies in the holding of the productive asset, of the financial assets, and of the cash. We require that the following transaction balance holds

$$\begin{aligned} &q^{-1}(t)\tilde{B}(t)d\phi(t) + B_0(t)d\phi_0(t) + A(t)d\pi(t) + dm(t) \\ &= [(1-l(t))w(t) + \pi(t)\delta(t) - q^{-1}(t)c(t)]dt + dw_g(t); \end{aligned} \tag{21}$$

that is, changes in the portfolio and cash position are financed by income net of consumption. Therefore the above transaction balance may be written in real terms as

$$\begin{aligned} &\tilde{B}(t)d\phi(t) + \tilde{B}_0(t)d\phi_0(t) + \tilde{A}(t)d\pi(t) + q(t)dm(t) + d[q,m](t) \\ &= [(1-l(t))\tilde{w}(t) + \pi(t)\tilde{\delta}(t) - c(t)]dt + q(t)dM(t) + d[q,M](t) \end{aligned} \tag{22}$$

where the last two terms account for the change in wealth due to real money transfers. After changing Brownian motion, we obtain the agent's revised *real budget equation* (with $\eta(t) := \mu_M(t) - \sigma_M^\top(t)(\theta(t) - \sigma_q(t))$)

$$\begin{aligned} d\tilde{X}(t) &= \Big[\tilde{r}(t)\tilde{X}(t) + (1-l(t))\tilde{w}(t) - r(t)q(t)m(t) - c(t) + q(t)M(t)\eta(t)\Big]dt \\ &+ \Big(\phi_0(t)\tilde{B}_0(t) + q(t)m(t)\,,\, q(t)\pi(t)\Big)\,\Sigma(t)dW^\circ(t) + q(t)M(t)\sigma_M^\top(t)dW^\circ(t). \end{aligned} \tag{23}$$

Given interest rates $\tilde{r}, r$, a money price q, a productive asset price A, a dividend rate δ and a wage rate w, the strategy $(c, l, m, \pi, \phi, \phi_0)$ is *feasible* for the agent if her real wealth $\tilde{X}$ satisfies $\zeta(t)\tilde{X}(t) \geq -k$ for all t, for some finite constant $k = k(c, l, m, \pi, \phi, \phi_0)$, and if $\tilde{X}(T) \geq 0$ almost surely. That is, the debt the agent may incurr at any time is "limited" and at the terminal time, when all debt must be liquidated, bankruptcy does not occur.

The *utility function* $U(t, c, l, qm) : [0,T] \times [0,+\infty) \times [0,1] \times [0,+\infty) \to [-\infty, \infty)$ represents the agent's utility of consumption rate $c \geq 0$, of leisure rate $0 \leq l \leq 1$, and cash holdings in real term $qm \geq 0$, at time t. For each t, the function $U(t,\cdot,\cdot,\cdot)$ is continuous, concave on its domain, i.e. where it is finite, and, on $\mathrm{dom}(\partial U(t,\cdot,\cdot,\cdot))$ (i.e. where the subgradient set is non-empty, cf. [2]), $U(t,\cdot,\cdot,\cdot)$ is strictly concave, non-decreasing and twice continuously differentiable.

The agent aims to maximize her expected total utility; that is, to solve

$$\max_{\mathcal{H}} E\Big\{ \int_0^T U(t, c(t), l(t), q(t)m(t))dt \Big\} \tag{24}$$

with the set $\mathcal{H}$ given by
$\mathcal{H} := \Big\{(c, l, m, \pi, \phi, \phi_0)$ feasible: $E\{\int_0^T U^-(t, c(t), l(t), q(t)m(t))dt\} < +\infty\Big\}$.

Assume $E\Big\{\int_0^T \zeta(t)[\tilde{w}(t) + \tilde{\delta}(t) + r(t)\tilde{M}(t)]dt\Big\} < \infty$ and set

$$\bar{c}(t) := (c(t), l(t), q(t)m(t)), \qquad \bar{p}(t) := (1, \tilde{w}(t), r(t)), \tag{25}$$

$$\xi := E\Big\{\int_0^T \zeta(t)\Big[\tilde{w}(t) + \tilde{\delta}(t) + r(t)\tilde{M}(t)\Big]dt\Big\}. \tag{26}$$

Then as in (4.5) of [2] (also, compare with Proposition 2.1 of [1], and with Lemma 9.1 and Theorem 9.2 of [3]), by using the budget equation (23), the Fisher equation (11), the initial condition $M(0) = m(0)+\phi_0(0)$, the money supply representation (15) (which, in particular, implies $\tilde{M}(0) = E\{\int_0^T \zeta(s)(r(s)-\eta(s))\tilde{M}(s)ds\}$), and the martingale representation theorem, respectively, it follows that any feasible strategy $(c, l, m, \pi, \tilde{\phi}, \phi_0)$ satisfies

$$E\Big\{\int_0^T \zeta(t)\,\bar{c}(t)\cdot\bar{p}(t)dt\Big\} \leq \xi. \tag{27}$$

Similarly, given processes c, l, m such that (27) holds, then

$$\exists \text{ a portfolio}(\pi, \tilde{\phi}, \phi_0) \text{ such that the strategy } (c, l, m, \pi, \tilde{\phi}, \phi_0) \text{ is feasible.} \tag{28}$$

Notice that, due to the presence of money, in this model as well as in the monetary model of [1], π is determined by the martingale representation and cannot, a priori, be taken identically equal to one, as happens in [2] and [3]. However, this will be the case in equilibrium.

The agent problem is then equivalent to the *static* problem

$$\max E\Big\{\int_0^T U(t, c(t), l(t), q(t)m(t))dt\Big\} \tag{29}$$

subject to $\max E\Big\{\int_0^T \zeta(t)\,\bar{c}(t)\cdot\bar{p}(t)dt\Big\} \leq \xi$. This problem is solved by using convex analysis to treat utility functions depending on several variables, in fact the main difficulty concerns inverting $\nabla_{\bar{c}}U$ (see [2] for more details). It follows that there exists a continuous function $I(t,\cdot,\cdot,\cdot) : \mathbb{R}^3_+ \mapsto [0,\infty[\times[0,1]\times[0,\infty[$ which extends $(\nabla_{\bar{c}}U(t,\cdot,\cdot,\cdot))^{-1}$, and it is continuously differentiable on the image of $\nabla_{\bar{c}}U(t,\cdot,\cdot,\cdot)$. Summing up, one has

Proposition 2.1 *If the utility function U satisfies some regularity conditions, then there exists a unique optimal consumption-leisure-real money holding strategy* $\hat{\bar{c}}$ *for the agent given by*

$$\hat{\bar{c}}(t) = I(t, \zeta(t)\bar{p}(t)), \qquad t \in [0, T]. \tag{30}$$

The corresponding productive asset share and financial asset portfolio are given by (28) *above.*

3 An Algorithm for Equilibrium

Equilibrium requires that the agent acts optimally, that the manager of the firm chooses real investment, labour, and nominal capital to maximize the expected total discounted value of output, that profits are distributed as dividends, that the goods market, the labour market, the money market, the stock market, and the financial market clear.

Definition 3.1 The market is in *equilibrium* if there exist a nominal interest rate process r, a wage process w, a money price process q, a dividend process $\tilde{\delta}$, all suitably integrable, and a real interest rate $\tilde{r}$, a real investment process $\hat{\nu}$, a labour process $\hat{L}$, a nominal capital process $\hat{K}$, and a strategy $\left(\hat{c}, \hat{l}, \hat{m}, \hat{\pi}, \hat{\tilde{\phi}}, \hat{\phi}_0\right)$ such that

$$\left(\hat{c}, \hat{l}, \hat{m}, \hat{\pi}, \hat{\tilde{\phi}}, \hat{\phi}_0\right) \quad \text{is optimal for the agent relative to } w, r, \tilde{r}, q, \tilde{\delta} \tag{31}$$

$$(\hat{\nu}(\cdot), \hat{L}(\cdot), \hat{\tilde{K}}(\cdot)) \in \arg\max\{J(\nu, L, K) : (\nu, L, K) \in \mathcal{S}) \tag{32}$$

$$\tilde{\delta}(t) = R(t, C^{\hat{\nu}}(t), \hat{L}(t), \hat{\tilde{K}}(t)) - \tilde{w}(t)\hat{L}(t) - r(t)\hat{\tilde{K}}(t) - \hat{\nu}(t), \text{ a.e. } t \in [0,T] \tag{33}$$

$$\hat{c}(t) = R(t, C^{\hat{\nu}}(t), \hat{L}(t), \hat{\tilde{K}}(t)) - \hat{\nu}(t), \quad \text{a.e. } t \in [0,T], \quad \text{a.s.} \tag{34}$$

$$\hat{l}(t) = 1 - \hat{L}(t), \quad \text{a.e. } t \in [0,T], \quad \text{a.s.} \tag{35}$$

$$\hat{m}(t) = M(t) - \hat{K}(t), \quad \text{a.e. } t \in [0,T], \quad \text{a.s.} \tag{36}$$

$$\hat{\pi}(t) = 1, \quad \text{a.e. } t \in [0,T], \quad \text{a.s.} \tag{37}$$

$$\hat{\tilde{\phi}}(t) = 0, \quad \text{a.e. } t \in [0,T], \quad \text{a.s.} \tag{38}$$

$$\hat{\phi}_0(t) = B_0^{-1}(t)\hat{K}(t), \quad \text{a.e. } t \in [0,T], \quad \text{a.s.} \tag{39}$$

Notice that, if an equilibrium exists, it is possible to establish a link between the agent's utility maximization problem and the firm's optimal capacity problem as follows. Recall that (cf. (30) and (25)) $\hat{\bar{c}}(t) = I(t, \zeta(t)\bar{p}(t))$ is optimal for the agent, that is $\nabla_{\bar{c}} U(t, \hat{c}(t), \hat{l}(t), q(t)\hat{m}(t)) = (\zeta(t), \zeta(t)\tilde{w}(t), \zeta(t)r(t))$. Hence, with

$$\begin{cases} \hat{U}(t, \nu(t), L(t), \tilde{K}(t)) \\ := U(t, R(t, C^{\nu}(t), L(t), \tilde{K}(t)) - \nu(t), 1 - L(t), q(t)M(t) - \tilde{K}(t)) \\ (\hat{U}_1(t, \nu(t), L(t), \tilde{K}(t)),\ \hat{U}_2(t, \nu(t), L(t), \tilde{K}(t)),\ \hat{U}_3(t, \nu(t), L(t), \tilde{K}(t))) \\ := \nabla_{\bar{c}} U(t, R(t, C^{\nu}(t), L(t), \tilde{K}(t)) - \nu(t), 1 - L(t), q(t)M(t) - \tilde{K}(t)) \end{cases} \tag{40}$$

in equilibrium (cf. (34), (35), (36)) it holds

$$\begin{cases} \zeta(t) = \hat{U}_1(t, \hat{\nu}(t), \hat{L}(t), \hat{\tilde{K}}(t)), \\ \zeta(t)\tilde{w}(t) = \hat{U}_2(t, \hat{\nu}(t), \hat{L}(t), \hat{\tilde{K}}(t)), \\ \zeta(t)r(t) = \hat{U}_3(t, \hat{\nu}(t), \hat{L}(t), \hat{\tilde{K}}(t)). \end{cases} \tag{41}$$

These together with (14) imply that the equilibrium price of money is

$$q(t) = \left(\hat{U}_1(t, \hat{\nu}(t), \hat{L}(t), \hat{\tilde{K}}(t))\right)^{-1} E\left\{ \int_t^T q(s)\hat{U}_3(s, \hat{\nu}(s), \hat{L}(s), \hat{\tilde{K}}(s))ds \Big| \mathcal{F}_t \right\}. \quad (42)$$

Now define on $\mathcal{S}$ the functional $\bar{J}(\nu, L, \tilde{K}) := E\{\int_0^T \hat{U}(t, \nu(t), L(t), \tilde{K}(t))dt\}$, then $\bar{J}(\nu, L, \tilde{K})$ is strictly concave since $U(t, \cdot, \cdot, \cdot)$ is so on $\mathrm{dom}(\partial U(t, \cdot, \cdot, \cdot))$. Hence the solution of the problem

$$\max_{\mathcal{S}} \bar{J}(\nu, L, \tilde{K}), \quad (43)$$

if it exists, is identified from the condition $0 \in \partial \bar{J}$.

The main point now is that one can easily check that J and $\bar{J}$ have the same subgradient, and hence must have the same point of max! But (43) is independent of all but one of the market parameters, it depends only on qM, thus the optimal $(\hat{\nu}, \hat{L}, \hat{\tilde{K}})$ is a function only of qM. Since M is exogenous, it can be assumed to be Markovian (after possibly augmenting it by other state variables). Therefore q may be obtained by proceeding as in [1], Proposition 3.3; that is, (42) implies, under appropriate regularity conditions, that

$$q(t, M(t))\hat{U}_1(t, q(t, M(t))M(t)) + \int_0^t q(s, M(s))\hat{U}_3(s, q(s, M(s))M(s))ds$$

is a P-martingale, hence its drift must be zero. It follows that $q(t, M(t))$ solves

$$\begin{cases} \left([\frac{\partial}{\partial t} + \mathcal{L}]q(t, \cdot)\hat{U}_1(t, q(t, \cdot)\cdot) + q(t, \cdot)\hat{U}_3(t, q(t, \cdot)\cdot)\right)(M(t)) = 0 \\ q(T, M(T)) = 0 \end{cases} \quad (44)$$

(here $\mathcal{L}$ denotes the differential operator associated to M).

In conclusion, the following algorithm provides an equilibrium:
- solve (44) to find q,
- qM is then known, so replace it in (40) and hence in (43),
- then solve (43) to find the optimal $(\hat{\nu}, \hat{L}, \hat{\tilde{K}})$.

Hence existence of an equilibrium follows (cf. ([2]), Proposition 7.1) and the result below holds.

Theorem 3.2 *Find q as the solution of (44). Assume that (43) has a solution for this q and denote it by $(\hat{\nu}, \hat{L}, \hat{\tilde{K}})$. Define $\zeta, \tilde{w}, r$ from (41), and $\tilde{\delta}, \hat{c}, \hat{l}, \hat{m}, \hat{\pi}, \hat{\tilde{\phi}}, \hat{\phi}_0$ from (33)-(39). Obtain $\tilde{r}$ and θ as the growth rate and volatility of ζ. $\tilde{A}$, and hence A, μ_A, σ_A are determined from (16). This choice of parameters provides an equilibrium.*

References

[1] S. Basak and M. Gallmeyer, Currency prices, the nominal exchange rate, and security prices in a two-country dynamic monetary equilibrium, *Mathematical Finance,* **9** (1999), 1 - 30. https://doi.org/10.1111/1467-9965.00061

[2] M.B. Chiarolla and U.G. Haussmann, Equilibrium in a stochastic model with consumption, wages and investment, *J. Math. Economics,* **35** (2001), 311 - 346. https://doi.org/10.1016/s0304-4068(00)00070-7

[3] I. Karatzas, J.P. Lehoczky and S.E. Shreve, Existence and uniqueness of multi-agent equilibrium in a stochastic, dynamic consumption/investment model, *Mathematics of Operations Research,* **15** (1990), 80 - 128. https://doi.org/10.1287/moor.15.1.80

[4] R. T. Rockafellar, *Convex Analysis,* Princeton University Press, Princeton, NJ, 1970.

Received: November 7, 2019; Published: November 29, 2019