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Power Sums of Arithmetic Progressions and Bernoulli Polynomials

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Abstract

For $k \in \mathbb{N}_0$, we consider the polynomial $\mathcal{S}_k(x)$ associated with the sum of powers of natural numbers $S_k(n) = 0^k + 1^k + 2^k + \cdots + (n-1)^k$. Expressing $\mathcal{S}_k(x)$ in Faulhaber's form, and using the link between $\mathcal{S}_k(x)$ and the Bernoulli polynomials $B_k(x)$, we explicitly derive $B_k(x)$ as a sum of even or odd powers of $x - \frac{1}{2}$, according as k is even or odd. Then, elaborating on previous work by Bazsó et al., we determine the coefficients of the generalized polynomial

$$\mathcal{S}_{a,b}^k(x) = b^k + (a+b)^k + (2a+b)^k + \cdots + \big(a(x-1)+b\big)^k,$$

when it is expressed in terms of the variable $x + \frac{b}{a} - \frac{1}{2}$.

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1 Introduction

For $k \in \mathbb{N}_0 = \{0, 1, 2, \dots\}$, the classical Bernoulli polynomial $B_k(x)$ in the real variable x is defined by (see, for instance, [1])

$$B_k(x) = \sum_{j=0}^{k} \binom{k}{j} B_{k-j} x^j,$$

$B_0(x) = 1$
$B_1(x) = x - \frac{1}{2}$
$B_2(x) = x^2 - x + \frac{1}{6}$
$B_3(x) = x^3 - \frac{3}{2}x^2 + \frac{1}{2}x$
$B_4(x) = x^4 - 2x^3 + x^2 - \frac{1}{30}$
$B_5(x) = x^5 - \frac{5}{2}x^4 + \frac{5}{3}x^2 - \frac{1}{6}x$
$B_6(x) = x^6 - 3x^5 + \frac{5}{2}x^4 - \frac{1}{2}x^2 + \frac{1}{42}$
$B_7(x) = x^7 - \frac{7}{2}x^6 + \frac{7}{2}x^5 - \frac{7}{6}x^3 + \frac{1}{6}x$
$B_8(x) = x^8 - 4x^7 + \frac{14}{3}x^6 - \frac{7}{3}x^4 + \frac{2}{3}x^2 - \frac{1}{30}$
$B_9(x) = x^9 - \frac{9}{2}x^8 + 6x^7 - \frac{21}{5}x^5 + 2x^3 - \frac{3}{10}x$

Table 1: The Bernoulli polynomials $B_k(x)$ for $0 \leq k \leq 9$.

where $B_0, B_1, B_2, \ldots$ are the (rational) Bernoulli numbers given by $B_0 = 1$ and the recursion

$$B_k = -\frac{1}{k+1}\sum_{j=0}^{k-1}\binom{k+1}{j}B_j, \quad k \geq 1.$$

Table 1 lists $B_k(x)$ for $0 \leq k \leq 9$. Note that $B_k = B_k(0)$.

Let us denote by $\mathbb{C}[x]$ the ring of polynomials in the variable x with complex coefficients, and let us recall the following definitions: A polynomial $F(x) \in \mathbb{C}[x]$ is nontrivially decomposable if there are $G_1(x)$ and $G_2(x)$ in $\mathbb{C}[x]$, both of degrees at least two, such that $F(x) = G_1(G_2(x))$. A polynomial is indecomposable if it is not decomposable. Furthermore, two decompositions $F(x) = G_1(G_2(x))$ and $F(x) = H_1(H_2(x))$ are said to be equivalent if there exists a linear polynomial $L(x) \in \mathbb{C}[x]$ such that $G_1(x) = L(H_1(x))$ and $G_2(x) = L(H_2(x))$.

Of particular interest for this work is Theorem 1.1 below, which is due to Bilu et al. [5, Theorem 4.1].

Theorem 1.1. *The polynomial $B_k(x)$ is indecomposable for odd $k \in \mathbb{N}_0$. For $k = 2m$, any nontrivial decomposition of $B_{2m}(x)$ is equivalent to the following decomposition:*

$$B_{2m}(x) = \widetilde{B}_{2m}\left(\left(x - \frac{1}{2}\right)^2\right), \tag{1}$$

where $\widetilde{B}_{2m}(x)$ is a indecomposable polynomial of degree m.

Define the sum of powers of the first n natural numbers (starting with 0) as

$$S_k(n) = 0^k + 1^k + 2^k + \cdots + (n-1)^k, \quad k \in \mathbb{N}_0.$$

By convention, we take $S_k(0) = 0$ for all k. Also, assuming that $0^0 = 1$, we have that $S_0(n) = n$. As is well known, there is a close relationship between $S_k(n)$ and the Bernoulli polynomials, namely,

$$S_k(n) = \frac{1}{k+1}[B_{k+1}(n) - B_{k+1}]. \tag{2}$$

Using (2), one can naturally extend $S_k(n)$ to a polynomial $\mathcal{S}_k(x)$ in which x takes any real value as follows

$$\mathcal{S}_k(x) = \frac{1}{k+1}[B_{k+1}(x) - B_{k+1}], \quad (x \in \mathbb{R}; k \in \mathbb{N}_0). \tag{3}$$

In [20], Rakaczki proved that the polynomial defined in (3) is indecomposable for even k. He also observed that, for odd $k = 2m-1$, all the decompositions of $\mathcal{S}_{2m-1}(x)$ are equivalent to the following decomposition:

$$\mathcal{S}_{2m-1}(x) = \widetilde{S}_{2m-1}\left(\left(x - \frac{1}{2}\right)^2\right), \tag{4}$$

where $\widetilde{S}_{2m-1}(x)$ is an (indecomposable) polynomial of degree m.

Bazsó et al. [2] considered the more general power sum

$$S^k_{a,b}(n) = b^k + (a+b)^k + (2a+b)^k + \cdots + \big(a(n-1)+b\big)^k, \tag{5}$$

where $a \neq 0$ and b are coprime integers. As noted in [2] (see also [12, 15]), $S^k_{a,b}(n)$ and $B_k(n)$ are related by

$$S^k_{a,b}(n) = \frac{a^k}{k+1}\left[B_{k+1}\left(n + \frac{b}{a}\right) - B_{k+1}\left(\frac{b}{a}\right)\right],$$

so that one can in turn extend the definition of the polynomial $S^k_{a,b}(n)$ to hold true for any real value x as follows:

$$\mathcal{S}^k_{a,b}(x) = \frac{a^k}{k+1}\left[B_{k+1}\left(x + \frac{b}{a}\right) - B_{k+1}\left(\frac{b}{a}\right)\right]. \tag{6}$$

Relying on Theorem 1.1, Bazsó et al. [2] proved that the polynomial $\mathcal{S}^k_{a,b}(x)$ is indecomposable for even k, and that, for odd $k = 2m-1$, any nontrivial decomposition of $\mathcal{S}^k_{a,b}(x)$ is equivalent to the following decomposition:

$$\mathcal{S}^{2m-1}_{a,b}(x) = \widetilde{S}^{2m-1}_{a,b}\left(\left(x + \frac{b}{a} - \frac{1}{2}\right)^2\right), \tag{7}$$

where $\widetilde{S}^{2m-1}_{a,b}(x)$ is an (indecomposable) polynomial of degree m.

The rest of the paper is organized as follows. In Section 2, we first recast the polynomial $\mathcal{S}_k(x)$ in the form of the so-called Faulhaber polynomial, namely, as an even or odd polynomial in $x-\frac{1}{2}$, and give the corresponding coefficients in terms of $B_k(\frac{1}{2})$, that is, the value of the Bernoulli polynomial $B_k(x)$ at the point $x=\frac{1}{2}$. Then, in Section 3, we employ the Faulhaber form of $\mathcal{S}_k(x)$ established in Section 2 to refine Theorem 1.1 in the following sense. On the one hand, we explicitly determine the coefficients of the polynomial $\widetilde{B}_{2m}(x)$ in (1). On the other hand, we show that, for odd $k=2m+1$, the polynomial $B_k(x)$ can be expressed as the product of $B_1(x)$ by a decomposable polynomial of the form $\widetilde{B}_{2m+1}\left(\left(x-\frac{1}{2}\right)^2\right)$, where $\widetilde{B}_{2m+1}(x)$ is an indecomposable polynomial of degree m (see Theorem 3.1 below). In Section 4, we apply Theorem 3.1 to explicitly derive, via equation (6), the coefficients of the generalized polynomial $\mathcal{S}^k_{a,b}(x)$ when it is expressed as a function of the variable $x+\frac{b}{a}-\frac{1}{2}$. Finally, in Section 5, we obtain an equivalent decomposition of both the Bernoulli polynomials and the generalized polynomial $\mathcal{S}^k_{a,b}(x)$ in terms of the quadratic polynomial $\mathcal{S}_1(x)=\frac{1}{2}x(x-1)$.

2 Faulhaber polynomials

Since the pioneering work of the German mathematician Johann Faulhaber (1580-1635) [16], it is well known that the polynomial $\mathcal{S}_k(x)$ is expressible as a sum of even or odd powers of $x-\frac{1}{2}$, according as k is odd or even, respectively (see, for instance, [4, 6, 7, 8, 11, 13, 17]). Specifically, for odd $k=2m-1$ $(m\geq 1)$, $\mathcal{S}_{2m-1}(x)$ can be written as

$$\mathcal{S}_{2m-1}(x)=\sum_{j=0}^{m} f_j^{(2m-1)}\left(x-\frac{1}{2}\right)^{2j}, \tag{8}$$

where the coefficients $f_j^{(2m-1)}$ are given by [8]

$$f_0^{(2m-1)}=\mathcal{S}_{2m-1}\left(\frac{1}{2}\right)=\frac{1}{2m}\left[B_{2m}\left(\frac{1}{2}\right)-B_{2m}\right], \tag{9}$$

and, for $i=1,\dots,m$,

$$f_i^{(2m-1)}=\frac{1}{2m}\binom{2m}{2i}B_{2m-2i}\left(\frac{1}{2}\right), \tag{10}$$

with $B_{2m}(\frac{1}{2})=(2^{1-2m}-1)B_{2m}$. On the other hand, for even $k=2m$ $(m\geq 1)$, $\mathcal{S}_{2m}(x)$ has the form

$$\mathcal{S}_{2m}(x)=\sum_{j=0}^{m} f_j^{(2m)}\left(x-\frac{1}{2}\right)^{2j+1}=\left(x-\frac{1}{2}\right)\sum_{j=0}^{m} f_j^{(2m)}\left(x-\frac{1}{2}\right)^{2j}, \tag{11}$$

where the coefficients $f_j^{(2m)}$ are given by [8]

$$f_j^{(2m)} = \frac{1}{2j+1}\binom{2m}{2j} B_{2m-2j}\left(\frac{1}{2}\right), \tag{12}$$

for $j = 0, 1, \ldots, m$.

Note that Rakaczki's result (4) follows directly from the form of the Faulhaber polynomial (8), from which we get that $\widetilde{S}_{2m-1}(x) = \sum_{j=0}^{m} f_j^{(2m-1)} x^j$. Furthermore, in view of (11), $\mathcal{S}_{2m}(x)$ can be written in the form

$$\mathcal{S}_{2m}(x) = \left(x - \frac{1}{2}\right) \widetilde{S}_{2m}\left(\left(x - \frac{1}{2}\right)^2\right),$$

where $\widetilde{S}_{2m}(x) = \sum_{j=0}^{m} f_j^{(2m)} x^j$ is an (indecomposable) polynomial of degree m.

3 Explicit decomposition of the Bernoulli polynomials

In this section, we derive the following theorem showing up the explicit decomposition of the Bernoulli polynomials.

Theorem 3.1. *For $k = 2m$ ($m \geq 0$), the polynomial $B_k(x)$ can be expressed as*

$$B_{2m}(x) = \widetilde{B}_{2m}\left(\left(x - \frac{1}{2}\right)^2\right) = \sum_{j=0}^{m} \widetilde{b}_j^{(2m)} \left(x - \frac{1}{2}\right)^{2j}, \tag{13}$$

where, for $j = 0, 1, \ldots, m$, the coefficients $\widetilde{b}_j^{(2m)}$ are given by

$$\widetilde{b}_j^{(2m)} = \binom{2m}{2j} B_{2m-2j}\left(\frac{1}{2}\right). \tag{14}$$

For $k = 2m+1$ ($m \geq 0$), the polynomial $B_k(x)$ can be expressed as

$$B_{2m+1}(x) = \left(x - \frac{1}{2}\right) \widetilde{B}_{2m+1}\left(\left(x - \frac{1}{2}\right)^2\right) = \left(x - \frac{1}{2}\right) \sum_{j=0}^{m} \widetilde{b}_j^{(2m+1)} \left(x - \frac{1}{2}\right)^{2j}, \tag{15}$$

where, for $j = 0, 1, \ldots, m$, the coefficients $\widetilde{b}_j^{(2m+1)}$ are given by

$$\widetilde{b}_j^{(2m+1)} = \binom{2m+1}{2j+1} B_{2m-2j}\left(\frac{1}{2}\right). \tag{16}$$

Furthermore, the polynomials $\widetilde{B}_{2m}(x) = \sum_{j=0}^{m} \widetilde{b}_j^{(2m)} x^j$ and $\widetilde{B}_{2m+1}(x) = \sum_{j=0}^{m} \widetilde{b}_j^{(2m+1)} x^j$ are indecomposable.

Proof. For $k = 2m-1$ ($m \geq 1$), it follows from (3) that

$$B_{2m}(x) = 2m\mathcal{S}_{2m-1}(x) + B_{2m}. \tag{17}$$

Therefore, substituting (8) into (17), and using (9) and (10), yields

$$\begin{aligned} B_{2m}(x) &= B_{2m}\left(\frac{1}{2}\right) + 2m\sum_{i=1}^{m} f_i^{(2m-1)}\left(x-\frac{1}{2}\right)^{2i} \\ &= \sum_{j=0}^{m}\binom{2m}{2j}B_{2m-2j}\left(\frac{1}{2}\right)\left(x-\frac{1}{2}\right)^{2j}. \end{aligned} \tag{18}$$

Clearly, the formula (18) also holds for $m = 0$. Similarly, for $k = 2m$ ($m \geq 1$), from (3) we have

$$B_{2m+1}(x) = (2m+1)\mathcal{S}_{2m}(x), \tag{19}$$

since $B_{2m+1} = 0$ for $m \geq 1$. Thus, substituting (11) into (19), and using (12), we obtain

$$\begin{aligned} B_{2m+1}(x) &= \left(x-\frac{1}{2}\right)\sum_{j=0}^{m}\frac{2m+1}{2j+1}\binom{2m}{2j}B_{2m-2j}\left(\frac{1}{2}\right)\left(x-\frac{1}{2}\right)^{2j} \\ &= \left(x-\frac{1}{2}\right)\sum_{j=0}^{m}\binom{2m+1}{2j+1}B_{2m-2j}\left(\frac{1}{2}\right)\left(x-\frac{1}{2}\right)^{2j}. \end{aligned} \tag{20}$$

Note that the formula (20) reduces to $B_1(x) = x - \frac{1}{2}$ when $m = 0$.

On the other hand, the indecomposability of both $\widetilde{B}_{2m}(x)$ and $\widetilde{B}_{2m+1}(x)$ follows from the symmetry properties of the Bernoulli polynomials. Indeed, from (17) and (19), and the well-known symmetry properties of the power sum polynomials $\mathcal{S}_k(x)$ [13, 17, 19], it is deduced that, for $m \geq 0$ (see also [1]),

$$B_{2m}(-x+1) = B_{2m}(x), \tag{21}$$

and

$$B_{2m+1}(-x+1) = -B_{2m+1}(x). \tag{22}$$

In words, the above equations (21) and (22) mean, respectively, that $B_{2m}(x)$ is symmetric about the vertical line at $\frac{1}{2}$, and that $B_{2m+1}(x)$ is symmetric about the point $(\frac{1}{2}, 0)$. These symmetry conditions by themselves imply that the Bernoulli polynomials $B_{2m}(x)$ and $B_{2m+1}(x)$ are *necessarily* of the form in (13) and (15), respectively, and that, in particular, $\widetilde{B}_{2m}(x)$ and $\widetilde{B}_{2m+1}(x)$ are indecomposable. See [17, Lemma] for a more thorough analysis.[1]

[1] Another way of seeing this, is to invoke the elementary result according to which a polynomial function $P(x)$ has a line of symmetry at $x = r$ if and only if $P(x) = f\big(g(x-r)\big)$, where $g(x)$ is a polynomial of degree 2 and $f(x)$ is any polynomial. Moreover, if a polynomial function is reflective symmetric then its derivative is point symmetric, and vice versa. In fact, the Bernoulli polynomials satisfy the well-known property that $B_k'(x) = kB_{k-1}(x)$, $k \geq 1$ [1].

For the sake of completeness, next we show that $B_{2m+1}(x)$ cannot be expressed in the form

$$C_{2m+1}(x) = \left(x - \frac{1}{2}\right) \sum_{t=0}^{r} \widetilde{c}_t^{(2m+1)} \left(x - \frac{1}{2}\right)^{kt}, \tag{23}$$

for any choice of the coefficients $\widetilde{c}_t^{(2m+1)}$ $(t = 0, 1, \dots, r)$, where r and k are positive integers fulfilling $rk = 2m$ and $k \geq 3$. As we shall presently see, assuming the form (23) leads to a contradiction with the Bernoulli polynomial

$$\begin{aligned} B_{2m+1}(x) &= \sum_{j=0}^{2m+1} \binom{2m+1}{j} B_j x^{2m+1-j} \\ &= x^{2m+1} - \left(m + \frac{1}{2}\right) x^{2m} + \frac{1}{6} m\,(2m+1)\, x^{2m-1} - \cdots . \end{aligned} \tag{24}$$

Indeed, employing the binomial theorem, the polynomial $C_{2m+1}(x)$ in (23) can be expanded as

$$C_{2m+1}(x) = \sum_{t=0}^{r} \widetilde{c}_t^{(2m+1)} \left[x^{kt+1} - \frac{1}{2}(kt+1)x^{kt} + \frac{1}{8} kt(kt+1)x^{kt-1} - \cdots \right].$$

Since $B_{2m+1}(x)$ is a monic polynomial, we will take, without loss of generality, $\widetilde{c}_r^{(2m+1)} = 1$. Thus, under the said conditions $rk = 2m$ and $k \geq 3$, we obtain

$$C_{2m+1}(x) = x^{2m+1} - \left(m + \frac{1}{2}\right) x^{2m} + \frac{1}{4} m\,(2m+1)\, x^{2m-1} + C_{2m-2}(x), \tag{25}$$

where $C_{2m-2}(x)$ is a polynomial in x of degree $2m-2$. Note that the coefficients of x^{2m-1} in (24) and (25) are different, and then $C_{2m+1}(x) \neq B_{2m+1}(x)$. □

As an example, using (18) and (20), the polynomials $B_8(x)$ and $B_9(x)$ in Table 1 can be expressed as follows:

$$B_8(x) = \left(x - \frac{1}{2}\right)^8 - \frac{7}{3}\left(x - \frac{1}{2}\right)^6 + \frac{49}{24}\left(x - \frac{1}{2}\right)^4 - \frac{31}{48}\left(x - \frac{1}{2}\right)^2 + \frac{127}{3840},$$

and

$$\begin{aligned} B_9(x) = \left(x - \frac{1}{2}\right) \Bigg[\left(x - \frac{1}{2}\right)^8 - 3\left(x - \frac{1}{2}\right)^6 + \frac{147}{40}\left(x - \frac{1}{2}\right)^4 \\ - \frac{31}{16}\left(x - \frac{1}{2}\right)^2 + \frac{381}{1280} \Bigg]. \end{aligned}$$

It is important to note that the coefficients $\widetilde{b}_j^{(2m+1)}$ and $\widetilde{b}_j^{(2m)}$ are related by $\widetilde{b}_j^{(2m+1)} = \frac{2m+1}{2j+1} \widetilde{b}_j^{(2m)}$, $j = 0, 1, \dots, m$, so that one can determine the polynomial $\widetilde{B}_{2m+1}(x)$ from $\widetilde{B}_{2m}(x)$, and vice versa. Let us further observe that, from (20), we immediately obtain that $B_{2m+1}\left(\frac{1}{2}\right) = 0$ for all $m \geq 0$.

4 Explicit decomposition of the generalized polynomial $\mathcal{S}_{a,b}^k(x)$

In this section, we apply Theorem 3.1 to explicitly derive all the possible decompositions of the polynomial $\mathcal{S}_{a,b}^k(x)$. In the first place, for $k = 2m-1$ ($m \geq 1$), we have from (6)

$$\mathcal{S}_{a,b}^{2m-1}(x) = \frac{a^{2m-1}}{2m}\left[B_{2m}\left(x+\frac{b}{a}\right) - B_{2m}\left(\frac{b}{a}\right)\right].$$

Then, using (13), we obtain

$$\mathcal{S}_{a,b}^{2m-1}(x) = \frac{a^{2m-1}}{2m}\sum_{j=0}^{m}\widetilde{b}_j^{(2m)}\left[\left(x+\frac{b}{a}-\frac{1}{2}\right)^{2j} - \left(\frac{b}{a}-\frac{1}{2}\right)^{2j}\right],$$

or, equivalently,

$$\mathcal{S}_{a,b}^{2m-1}(x) = \mathcal{S}_{a,b}^{2m-1}\left(\frac{1}{2}-\frac{b}{a}\right) + \frac{a^{2m-1}}{2m}\sum_{i=1}^{m}\widetilde{b}_i^{(2m)}\left(x+\frac{b}{a}-\frac{1}{2}\right)^{2i}, \tag{26}$$

where

$$\mathcal{S}_{a,b}^{2m-1}\left(\frac{1}{2}-\frac{b}{a}\right) = -\frac{a^{2m-1}}{2m}\sum_{i=1}^{m}\widetilde{b}_i^{(2m)}\left(\frac{b}{a}-\frac{1}{2}\right)^{2i}, \tag{27}$$

and where the coefficients $\widetilde{b}_i^{(2m)}$ are given in (14). (Notice that the coefficient $\widetilde{b}_0^{(2m)}$ does not appear in either (26) or (27).) The polynomial representation (26) for $\mathcal{S}_{a,b}^{2m-1}(x)$ is of course consistent with the polynomial decomposition in (7). Furthermore, the polynomial (26) reduces to the Faulhaber polynomial (8) when $a = 1$ and $b = 0$.

On the other hand, for $k = 2m$ ($m \geq 1$), starting from (6) and using (15), it is easily seen that

$$\mathcal{S}_{a,b}^{2m}(x) = \mathcal{S}_{a,b}^{2m}\left(\frac{1}{2}-\frac{b}{a}\right) + \frac{a^{2m}}{2m+1}\left(x+\frac{b}{a}-\frac{1}{2}\right)\sum_{j=0}^{m}\widetilde{b}_j^{(2m+1)}\left(x+\frac{b}{a}-\frac{1}{2}\right)^{2j}, \tag{28}$$

where

$$\mathcal{S}_{a,b}^{2m}\left(\frac{1}{2}-\frac{b}{a}\right) = -\frac{a^{2m}}{2m+1}\left(\frac{b}{a}-\frac{1}{2}\right)\sum_{j=0}^{m}\widetilde{b}_j^{(2m+1)}\left(\frac{b}{a}-\frac{1}{2}\right)^{2j}, \tag{29}$$

and where the coefficients $\widetilde{b}_j^{(2m+1)}$ are given in (16). Analogously, the polynomial (28) reduces to the Faulhaber polynomial (11) when $a = 1$ and $b = 0$.

However, as shown in [2], the polynomial for $\mathcal{S}_{a,b}^{2m}(x)$ is definitely not decomposable, in contrast to the polynomial for $\mathcal{S}_{a,b}^{2m-1}(x)$.

As an example illustrating the above formulas, let us explicitly determine the polynomial defining the sum of powers of the first n odd positive integers

$$S_{2,1}^{k}(n) = 1^k + 3^k + 5^k + \cdots + (2n-1)^k.$$

For $k = 2m-1$ ($m \geq 1$), we have from (27) that $S_{2,1}^{2m-1}(0) = 0$. Hence, using (26) (with $a = 2, b = 1$) and (14), we get

$$S_{2,1}^{2m-1}(n) = \frac{2^{2m}}{4m} \sum_{i=1}^{m} \binom{2m}{2i} B_{2m-2i}\left(\frac{1}{2}\right) n^{2i}.$$

In particular, for $m = 1$, we retrieve the well-known identity $S_{2,1}^{1}(n) = 1 + 3 + 5 + \cdots + (n-1) = n^2$. Similarly, for $k = 2m$ ($m \geq 1$), we have from (29) that $S_{2,1}^{2m}(0) = 0$, and then, using (28) (with $a = 2, b = 1$) and (16), we find that

$$S_{2,1}^{2m}(n) = \frac{2^{2m}}{2m+1} \sum_{j=0}^{m} \binom{2m+1}{2j+1} B_{2m-2j}\left(\frac{1}{2}\right) n^{2j+1}.$$

Consider next the concrete sum

$$S_{13,7}^{6}(n) = \sum_{j=0}^{n-1} (7 + 13j)^6 .$$

In general, as shown by Bazsó and Mező [3], for all $n > 1$, $k, a > 0$, and $b \geq 0$, the power sum (5) can be explicitly expressed as a polynomial in n as follows:

$$S_{a,b}^{k}(n) = \sum_{i=0}^{k+1} \left(\sum_{j=0}^{k} \frac{a^j W_{a,b}(k,j)}{j+1} S_1(j+1, i) \right) n^i, \tag{30}$$

where $S_1(n,k)$ are the (signed) Stirling numbers of the first kind, and $W_{m,r}(n,k)$ are the r-Whitney numbers of the second kind, which are defined by [18]

$$W_{m,r}(n,k) = \frac{1}{m^k k!} \sum_{i=0}^{k} \binom{k}{i} (-1)^{k-i} \big(mi + r\big)^n.$$

Applying the formula (30) for $a = 13, b = 7$, and $k = 6$, we get

$$\begin{aligned} S_{13,7}^{6}(n) = \frac{4826809}{7} n^7 + \frac{371293}{2} n^6 - \frac{2370563}{2} n^5 - 230685 n^4 \\ + \frac{4116671}{6} n^3 + 80535 n^2 - \frac{4545323}{42} n. \end{aligned}$$

On the other hand, applying the formulas (28), (29), and (16) (with $a = 13, b = 7, m = 3$), we obtain the equivalent (indecomposable) representation

$$S^6_{13,7}(n) = \frac{4826809}{7}\left(n+\frac{1}{26}\right)^7 - \frac{4826809}{4}\left(n+\frac{1}{26}\right)^5 + \frac{33787663}{48}\left(n+\frac{1}{26}\right)^3 - \frac{149631079}{1344}\left(n+\frac{1}{26}\right) + \frac{55147}{13}.$$

By way of comparison, we note that, for example, the polynomial $S^5_{13,7}(n)$ admits the decomposition (see equation (7)), $S^5_{13,7}(n) = \widetilde{S}^5_{13,7}\left(\left(n+\frac{1}{26}\right)^2\right)$, where

$$\widetilde{S}^5_{13,7}(x) = \frac{371293}{6}x^3 - \frac{1856465}{24}x^2 + \frac{2599051}{96}x - \frac{66361}{1664}.$$

For related work concerning Faulhaber's theorem on power sums of an arithmetic progression, the reader is referred to Refs. [2, 9, 10, 14].

5 Concluding remarks

Let $\mathcal{S}_1(x)$ denote the quadratic polynomial $\mathcal{S}_1(x) = \frac{1}{2}x(x-1)$. Thus, by using the relation

$$\left(x-\frac{1}{2}\right)^2 = \frac{1}{4}\left(1+8\mathcal{S}_1(x)\right),$$

one can deduce from Theorem 3.1 that, for $m \geq 0$,

$$B_{2m}(x) = \widehat{B}_{2m}\left(\mathcal{S}_1(x)\right) = \sum_{j=0}^{m} \widehat{b}_j^{(2m)}\left(\mathcal{S}_1(x)\right)^j, \tag{31}$$

where, for $j = 0, 1, \ldots, m$,

$$\widehat{b}_j^{(2m)} = 8^j \sum_{k=j}^{m} \frac{1}{4^k}\binom{2m}{2k}\binom{k}{j} B_{2m-2k}\left(\frac{1}{2}\right). \tag{32}$$

Furthermore, for $m \geq 0$, it can be shown that

$$B_{2m+1}(x) = \left(x-\frac{1}{2}\right)\widehat{B}_{2m+1}\left(\mathcal{S}_1(x)\right) = \left(x-\frac{1}{2}\right)\sum_{j=0}^{m} \widehat{b}_j^{(2m+1)}\left(\mathcal{S}_1(x)\right)^j, \tag{33}$$

where, for $j = 0, 1, \ldots, m$,

$$\widehat{b}_j^{(2m+1)} = 8^j \sum_{k=j}^{m} \frac{1}{4^k}\binom{2m+1}{2k+1}\binom{k}{j} B_{2m-2k}\left(\frac{1}{2}\right). \tag{34}$$

Note that, since $B_k(0) = B_k$, we must have $\widehat{b}_0^{(2m)} = B_{2m}$ for $m \geq 0$, and $\widehat{b}_0^{(2m+1)} = 0$ for $m \geq 1$. Hence, from (32) and (34), we get the following identities

$$\sum_{k=0}^{m} \frac{1}{4^k} \binom{2m}{2k} B_{2m-2k}\left(\frac{1}{2}\right) = B_{2m}, \quad m \geq 0,$$

and

$$\sum_{k=0}^{m} \frac{1}{4^k} \binom{2m+1}{2k+1} B_{2m-2k}\left(\frac{1}{2}\right) = 0, \quad m \geq 1,$$

respectively. Incidentally, since $\binom{2m+1}{2k+1} = \binom{2m}{2k} + \binom{2m}{2k+1}$, from the last two identities a third one is obtained, namely

$$\sum_{k=0}^{m-1} \frac{1}{4^k} \binom{2m}{2k+1} B_{2m-2k}\left(\frac{1}{2}\right) = -B_{2m}, \quad m \geq 1.$$

Furthermore, from (31) and (33) we readily obtain that $B_{2m}(1) = \widehat{b}_0^{(2m)} = B_{2m}$ $(m \geq 0)$, and $B_{2m+1}(1) = \frac{1}{2}\widehat{b}_0^{(2m+1)} = 0$ $(m \geq 1)$. Moreover, it is easily verified from (31) and (33) that $B_k(1-x) = (-1)^k B_k(x)$, $k \geq 0$, in accordance with equations (21) and (22). Let us also mention that, for $m \geq 2$, the linear term in the Bernoulli polynomials $B_{2m}(x)$ is zero. This means that, for $m \geq 2$, we necessarily have $\widehat{b}_1^{(2m)} = 0$ and, therefore,

$$\sum_{k=1}^{m} \frac{k}{4^k} \binom{2m}{2k} B_{2m-2k}\left(\frac{1}{2}\right) = 0, \quad m \geq 2.$$

On the other hand, the Bernoulli polynomials fulfill the difference equation $B_k(x+1) - B_k(x) = kx^{k-1}$, $k \geq 1$ [1]. We have therefore

$$B_{2m}(2) = 2m + B_{2m}, \quad m \geq 0, \tag{35}$$

and

$$B_{2m+1}(2) = 2m+1, \quad m \geq 1. \tag{36}$$

Thus, taking into account that $\mathcal{S}_1(2) = 1$, from (35) and (31) it follows that

$$\sum_{j=1}^{m} \widehat{b}_j^{(2m)} = 2m, \quad m \geq 1, \tag{37}$$

and, similarly, from (36) and (33) it follows that

$$\sum_{j=1}^{m} \widehat{b}_j^{(2m+1)} = \frac{2}{3}(2m+1), \quad m \geq 1. \tag{38}$$

Substituting the coefficients in (32) [(34)] into equation (37) [(38)], one can derive the identities

$$\sum_{k=1}^{m} \left(\frac{3}{2}\right)^{2k} \binom{2m}{2k} B_{2m-2k}\left(\frac{1}{2}\right) = 2m + 2\big(1-2^{-2m}\big)B_{2m}, \tag{39}$$

and

$$\sum_{k=1}^{m} \left(\frac{3}{2}\right)^{2k} \binom{2m+1}{2k+1} B_{2m-2k}\left(\frac{1}{2}\right) = (2m+1)\left(\frac{2}{3} + B_{2m} - 2^{1-2m}B_{2m}\right), \tag{40}$$

respectively, where $m \geq 1$. It is to be noted, incidentally, that one can also obtain the identities (39) and (40) by simply putting $x = 2$ into the Bernoulli polynomials (13) and (15), respectively, and then using the relations (35) and (36).

To conclude, we note that, from equation (6) and the decompositions in equations (31) and (33), we can alternatively express the generalized polynomial $\mathcal{S}_{a,b}^{k}(x)$ as follows. For $k = 2m-1$, $m \geq 1$, we have

$$\mathcal{S}_{a,b}^{2m-1}(x) = \mathcal{S}_{a,b}^{2m-1}\left(-\frac{b}{a}\right) + \frac{a^{2m-1}}{2m}\sum_{i=1}^{m} \widehat{b}_i^{(2m)} \left(\mathcal{S}_1\left(x+\frac{b}{a}\right)\right)^i,$$

where

$$\mathcal{S}_{a,b}^{2m-1}\left(-\frac{b}{a}\right) = -\frac{a^{2m-1}}{2m}\sum_{i=1}^{m} \widehat{b}_i^{(2m)} \left(\mathcal{S}_1\left(\frac{b}{a}\right)\right)^i,$$

and where the coefficients $\widehat{b}_i^{(2m)}$ are given in (32). On the other hand, for $k = 2m$, $m \geq 1$, we have

$$\mathcal{S}_{a,b}^{2m}(x) = \mathcal{S}_{a,b}^{2m}\left(-\frac{b}{a}\right) + \frac{a^{2m}}{2m+1}\left(x+\frac{b}{a}-\frac{1}{2}\right)\sum_{i=1}^{m} \widehat{b}_i^{(2m+1)} \left(\mathcal{S}_1\left(x+\frac{b}{a}\right)\right)^i,$$

where

$$\mathcal{S}_{a,b}^{2m}\left(-\frac{b}{a}\right) = -\frac{a^{2m}}{2m+1}\left(\frac{b}{a}-\frac{1}{2}\right)\sum_{i=1}^{m} \widehat{b}_i^{(2m+1)} \left(\mathcal{S}_1\left(\frac{b}{a}\right)\right)^i,$$

and where the coefficients $\widehat{b}_i^{(2m+1)}$ are given in (34). In particular, when $a = 1$ and $b = 0$, we retrieve the Faulhaber form (cf. [8, 11, 17])

$$\mathcal{S}_{2m-1}(x) = \frac{1}{2m}\sum_{i=1}^{m} \widehat{b}_i^{(2m)} \left(\mathcal{S}_1(x)\right)^i, \quad m \geq 1,$$

and

$$\mathcal{S}_{2m}(x) = \frac{1}{2m+1}\left(x-\frac{1}{2}\right)\sum_{i=1}^{m} \widehat{b}_i^{(2m+1)} \left(\mathcal{S}_1(x)\right)^i, \quad m \geq 1.$$

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