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The Inverse Weibull Inverse Exponential Distribution with Application

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Abstract

A new model named the *inverse Weibull inverse exponential* (IWIE) distribution, is introduced. We calculate the density (pdf), distribution function (cdf), survival function (sf), hazard function (hrf), reversed hazard function (rhrf), cumulative hazard function (chrf), quantile function, skewness and kurtosis, *r*th moment and order statistics. Maximum Likelihood (ML) method to estimate the IWIE distribution parameters are mentioned. One real data set is applied to show the importance of the IWIE model compared with some distributions.

Keywords: Inverse exponential; distribution; Odd inverse Pareto -G family; Moments, Order statistics

1. Introduction

In the recent years, many researchers are interested to expand generating family in order to obtain better fit for data analyzing. Some popular-known generating family

are: The the beta-G by Eugene et al. (2002), odd Frechet-G proposed by Haq and Elgarhy (2018), Muth-G studied by Almarashi and Elgarhy (2018), Elbatal et al. (2018) proposed a new alpha power transformation family of distributions, Hassan and Nassr (2018) introduced inverse Weibull-G (IW-G) among others.
The cdf and pdf of IW-G is

$$F(x) = e^{-\theta^{\beta}\left(\frac{G(x)}{1-G(x)}\right)^{-\beta}}, \quad x \in R, \quad \theta, \beta > 0, \tag{1}$$

$$f(x) = \beta\theta^{\beta} \frac{g(x)G(x)^{-\beta-1}}{\left[1-G(x)\right]^{-\beta+1}} e^{-\theta^{\beta}\left(\frac{G(x)}{1-G(x)}\right)^{-\beta}}, x \in R, \theta, \beta > 0,$$

where $G(.)$ is cdf of any baseline distribution.

Keller and Kamath (1982) studied the inverse exponential (IE) distribution. It has been identified and discussed by Lin et al. (1989) as a lifetime distribution. If X is a non-negative random variable (rv), then the distribution of a rv $Y = {}^{1}/_{X}$ follows an IE distribution. Hence, if X denotes a rv the pdf and cdf of the IE distribution with a scale parameter α are given by

$$g(x;\alpha) = \frac{\alpha}{x^2} e^{-\frac{\alpha}{x}} \quad , \quad x, \alpha > 0,$$

and

$$G(x;\alpha) = e^{-\frac{\alpha}{x}} \quad , \quad x, \alpha > 0. \tag{2}$$

This article is aimed to study a new three parameter distribution based on the IW-G family. The IWIE model provides more flexible model. The reminder of this paper can be arranged as follows: In the following section, the IWIE distribution is defined. Section 3 gives some statistical properties of the IWIE distribution. The ML method is implemented to obtain the estimators of the parameters in Section 4. Application to a real data illustrating the performance of the IWIE distribution is given in Section 5. At the end of the paper, Conclusions mention in the Section 6.

2. The New Model

A random variable is said to has the IWIE model with vector parameters φ, where $\varphi=(\theta,\beta,\alpha)$, by inserting (2) in (1) then, the cdf of IWIE is

$$F(x;\varphi)=e^{-\theta^{\beta}\left(\frac{e^{-\frac{\alpha}{x}}}{1-e^{-\frac{\alpha}{x}}}\right)^{-\beta}},\ \alpha,\beta,\theta,x>0. \tag{3}$$

The crossponding pdf to (3) is

$$f(x;\varphi)=\alpha\beta\theta^{\beta}x^{-2}\frac{e^{\frac{\alpha\beta}{x}}}{\left[1-e^{-\frac{\alpha}{x}}\right]^{-\beta+1}}e^{-\theta^{\beta}\left(\frac{e^{-\frac{\alpha}{x}}}{1-e^{-\frac{\alpha}{x}}}\right)^{-\beta}}. \tag{4}$$

The sf, the hrf, rhrf and chrf are, respectively, given by

$$R(x;\varphi)=1-e^{-\theta^{\beta}\left(\frac{e^{-\frac{\alpha}{x}}}{1-e^{-\frac{\alpha}{x}}}\right)^{-\beta}},$$

$$h(x;\varphi)=\frac{\alpha\beta\theta^{\beta}x^{-2}\frac{e^{\frac{\alpha\beta}{x}}}{\left[1-e^{-\frac{\alpha}{x}}\right]^{-\beta+1}}e^{-\theta^{\beta}\left(\frac{e^{-\frac{\alpha}{x}}}{1-e^{-\frac{\alpha}{x}}}\right)^{-\beta}}}{1-e^{-\theta^{\beta}\left(\frac{e^{-\frac{\alpha}{x}}}{1-e^{-\frac{\alpha}{x}}}\right)^{-\beta}}},$$

$$\tau(x;\varphi)=\alpha\beta\theta^{\beta}x^{-2}\frac{e^{\frac{\alpha\beta}{x}}}{\left[1-e^{-\frac{\alpha}{x}}\right]^{-\beta+1}},$$

and

$$H(x;\varphi) = -\ln\left[1 - e^{-\theta^{\beta}\left(\frac{e^{-\frac{\alpha}{x}}}{1-e^{-\frac{\alpha}{x}}}\right)^{-\beta}}\right].$$

Figure 1 show plots of pdf and hrf for specific parameter choices of φ .

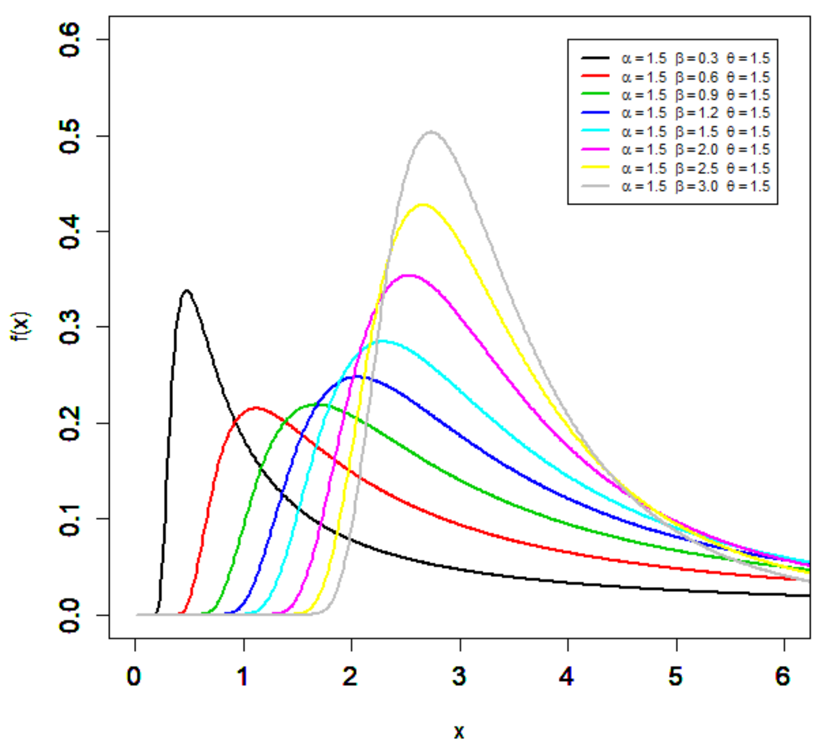

Figure 1. Plots of the pdf of the *IWIE* distribution for some different values of parameter.

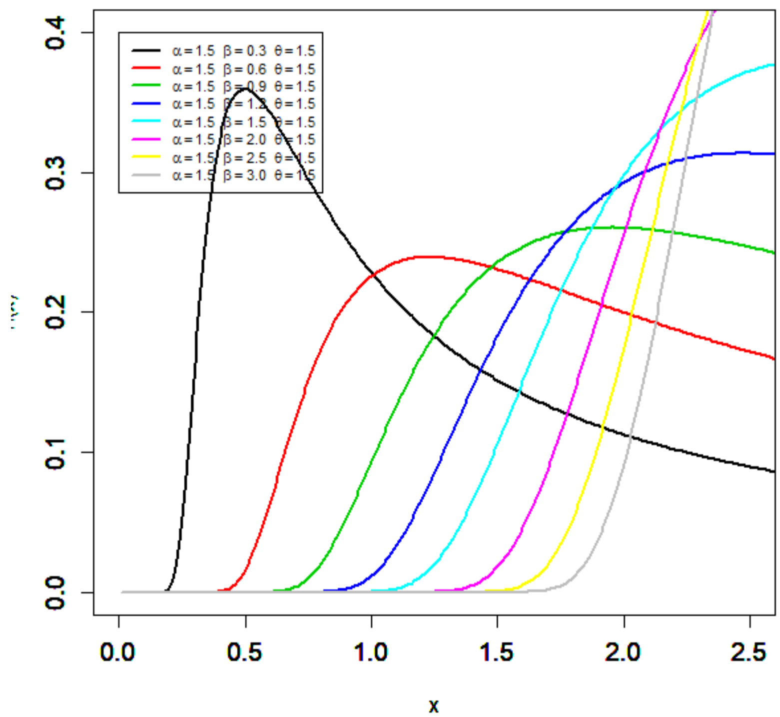

Figure 2. Plots of the hrf of the *IWIE* distribution for some different values of parameter.

As seen from Figure1, of IWIE model can be uni-model and right skewed. And from Figure 2, we can see the hrf of IWIE model can be J- shaped, and increasing.

3. Fundamental Statistical Properties

This section deals with some fundamental statistical properties of IWIE model.

3.1 Quantile and Median

The quantile function of the IWIE can be generated by inverting cdf (3) as

$$Q(u)=\frac{\alpha}{\ln\left[1+\left(\frac{-1}{\theta^{\beta}}\ln u\right)^{\frac{1}{\beta}}\right]}. \tag{5}$$

where U a uniform variate in the unit interval (0,1).

The median can be calculated by inserting $u=0.5$ in (5). Then, the median (M) is given by

$$M=\frac{\alpha}{\ln\left[1+\left(\frac{-1}{\theta^{\beta}}\ln 0.5\right)^{\frac{1}{\beta}}\right]}.$$

The Bowley skewness (see Kenney and Keeping (1962)), denoted by B, is defined by

$$B=\frac{X_{\frac{3}{4}}-2X_{\frac{1}{2}}+X_{\frac{1}{4}}}{X_{\frac{3}{4}}-X_{\frac{1}{4}}}.$$

The Moors kurtosis (see Moors (1988)), denoted by *K*, can be defined as follows

$$K=\frac{X_{\frac{7}{8}}-X_{\frac{5}{8}}+X_{\frac{3}{8}}-X_{\frac{1}{8}}}{X_{\frac{6}{8}}-X_{\frac{2}{8}}}.$$

The Bowley skewness and Moors kurtosis measures do not depend on the moments of the distribution and are almost insensitive to outliers. Plots of the skewness and kurtosis for some different values of the parameter θ as function of β are shown in Figure 3.

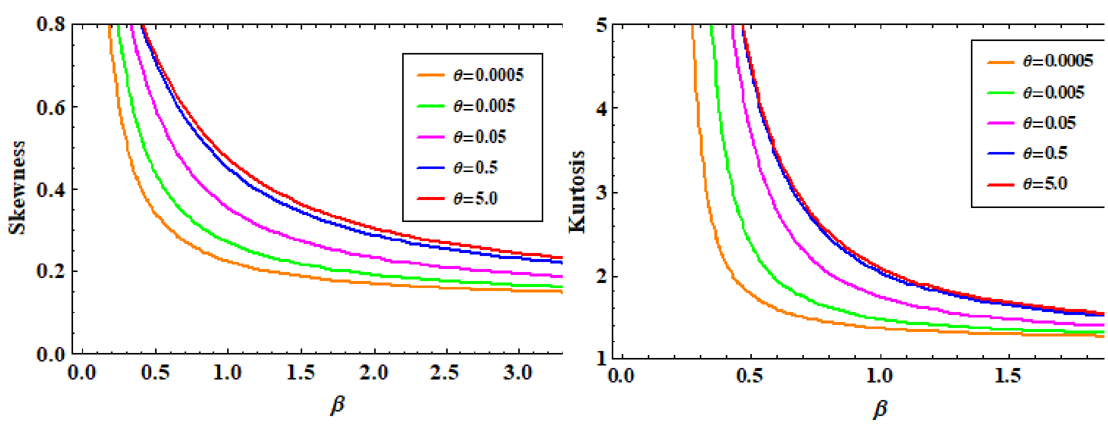

Figure 3. Bowley skewness and Moors kurtosis of the IWIE distribution as function of β for some values of θ and α=1.

3.2 Important expansion

In this subsection expansions of the pdf for IWIE distribution are derived.
By using the the exponential series, we get

$$e^{-\theta^{\beta}\left(\frac{e^{-\frac{\alpha}{x}}}{1-e^{-\frac{\alpha}{x}}}\right)^{-\beta}} = \sum_{i=0}^{\infty}\frac{(-1)^i\theta^{\beta i}}{i!}\left(\frac{e^{-\frac{\alpha}{x}}}{1-e^{-\frac{\alpha}{x}}}\right)^{-\beta i}.$$

Then,

$$f(x;\varphi)=\alpha\beta x^{-2}\sum_{i=0}^{\infty}\frac{(-1)^i\theta^{\beta(i+1)}}{i!}\frac{e^{\frac{\alpha\beta(i+1)}{x}}}{\left[1-e^{-\frac{\alpha}{x}}\right]^{-(\beta(i+1)-1)}}. \tag{6}$$

We can rewrite the equation (6) as

$$f(x;\varphi)=\alpha\beta x^{-2}\sum_{i=0}^{\infty}\frac{(-1)^i\theta^{\beta(i+1)}}{i!}\frac{\left[1-\left(1-e^{\frac{-\alpha}{x}}\right)\right]^{-\beta(i+1)}}{\left[1-e^{-\frac{\alpha}{x}}\right]^{-(\beta(i+1)-1)}}. \tag{7}$$

By using the generalized binomial theorem, for $\beta>0$ and $|z|<1$,

$$(1-z)^{-\beta}=\sum_{j=0}^{\infty}\binom{\beta+j-1}{j}z^j. \tag{8}$$

Then, by applying (8) in (7), the pdf of IWIE distribution becomes

$$f(x;\varphi)=\alpha\beta x^{-2}\sum_{i,j=0}^{\infty}\frac{(-1)^i\theta^{\beta(i+1)}}{i!}\binom{\beta(i+1)+j-1}{j}\left[1-e^{-\frac{\alpha}{x}}\right]^{j+(\beta(i+1)-1)}. \tag{9}$$

Using the generalized binomial theorem, for $\beta>0$ and $|z|<1$,

$$(1-z)^{\beta-1}=\sum_{k=0}^{\infty}(-1)^k\binom{\beta-1}{k}z^k. \tag{10}$$

Then, by applying (10) in (9), the pdf of IWIE is

$$f(x;\varphi) = x^{-2}\sum_{i=0}^{\infty}\eta_i e^{-\frac{k\alpha}{x}}. \tag{11}$$

where $\eta_i = \alpha\beta \sum_{j,k=0}^{\infty} \frac{(-1)^{i+k}\theta^{\beta(i+1)}}{i!} \binom{\beta(i+1)+j-1}{j}\binom{j+(\beta(i+1)-1)}{k}$.

3.3 Moments

The rth moment of rv X can be calculated from

$$\mu'_r = \int_0^{\infty} x^r f(x;\varphi)dx \tag{12}$$

Inserting (11) in (12), leads to:

$$\mu'_r = \sum_{i=0}^{\infty}\eta_i \int_0^{\infty} x^{r-2} e^{-\frac{k\alpha}{x}} dx.$$

Setting $y = \frac{\alpha k}{x}$ then,

$$\mu'_r = (\alpha k)^{r-1}\sum_{i=0}^{\infty}\eta_i \int_0^{\infty} y^{-r} e^{-y} dy.$$

Then the rth moment of IWIE becomes

$$\mu'_r = \sum_{i=0}^{\infty}\frac{\eta_i \Gamma(1-r)}{(\alpha k)^{1-r}}, \ r<1.$$

So, we cannot use the moments to get the mean and variance. The moment generating function (mgf) of the IWIE distribution is

$$M_x(t) = E(e^{tX}) = \int_0^{\infty} e^{tx} f(x;\varphi)\, dx = \int_0^{\infty}\sum_{r=0}^{\infty}\frac{t^r}{r!}x^r f(x;\varphi)\, dx = \sum_{r=0}^{\infty}\frac{t^r}{r!}E(X^r)$$

$$M_x(t) = \sum_{i,r=0}^{\infty}\frac{t^r \eta_i \Gamma(1-r)}{(\alpha k)^{1-r} r!}.$$

3.4 Order statistics

The density of the k^{th} order statistic, for r = 1,..., n from iid random variables $X_1, X_2, ..., X_n$ is given by

$$f_{X_{(k)}}(x) = \frac{1}{B(k, n-k+1)} f(x) F(x)^{k-1}[1-F(x)]^{n-k}. \tag{13}$$

Inserting (3) and (4) in (13), then

$$f_{X_{(k)}}(x) = \frac{\alpha\beta\theta^{\beta} x^{-2} e^{\frac{\alpha\beta}{x}}}{B(k, n-k+1)\left[1-e^{-\frac{\alpha}{x}}\right]^{-\beta+1}} e^{-k\theta^{\beta}\left(\frac{e^{-\frac{\alpha}{x}}}{1-e^{-\frac{\alpha}{x}}}\right)^{-\beta}} [1-e^{-\theta^{\beta}\left(\frac{e^{-\frac{\alpha}{x}}}{1-e^{-\frac{\alpha}{x}}}\right)^{-\beta}}]^{n-k}. \tag{14}$$

Setting $k = 1$ and $k = n,$ in (14), we obtain the pdf of the smallest and largest order statistics of the IWIE distribution.
The joint pdf of x_j and x_k (for $x_j < x_k$) is given by:

$$f_{X_j,X_k(x_j,x_k)} = \frac{n!}{(j-1)!(k-j-1)!(n-k)!}[F(x_j)]^{j-1}[F(x_k)-F(x_j)]^{k-j-1}[1-F(x_k)]^{n-k} f(x_j) f(x_k)$$

$$f_{X_j,X_k(x_j,x_k)} = \frac{n!\alpha^2\beta^2\theta^{2\beta}(x_k x_j)^{-2} e^{\frac{\alpha\beta}{x_j}} e^{\frac{\alpha\beta}{x_k}} e^{-j\theta^{\beta}\left(\frac{e^{-\frac{\alpha}{x_j}}}{1-e^{-\frac{\alpha}{x_j}}}\right)^{-\beta}} e^{-\theta^{\beta}\left(\frac{e^{-\frac{\alpha}{x_k}}}{1-e^{-\frac{\alpha}{x_k}}}\right)^{-\beta}}}{(j-1)!(k-j-1)!(n-k)!\left[1-e^{-\frac{\alpha}{x_j}}\right]^{-\beta+1}\left[1-e^{-\frac{\alpha}{x_k}}\right]^{-\beta+1}}$$
$$\times\left\{1-e^{-\theta^{\beta}\left(\frac{e^{-\frac{\alpha}{x_k}}}{1-e^{-\frac{\alpha}{x_k}}}\right)^{-\beta}}\right\}^{n-k} \times\left\{e^{-\theta^{\beta}\left(\frac{e^{-\frac{\alpha}{x_k}}}{1-e^{-\frac{\alpha}{x_k}}}\right)^{-\beta}} - e^{-\theta^{\beta}\left(\frac{e^{-\frac{\alpha}{x_j}}}{1-e^{-\frac{\alpha}{x_j}}}\right)^{-\beta}}\right\}^{k-j-1}.$$

3.5 Mode

The mode for the IWIE distribution can be calculated by differentiating f*(x)* with respect to *x* as follows

$$f(x;\varphi)=\alpha\beta\theta^{\beta}x^{-2}\frac{e^{\frac{\alpha\beta}{x}}}{\left[1-e^{-\frac{\alpha}{x}}\right]^{-\beta+1}}e^{-\theta^{\beta}\left(\frac{e^{-\frac{\alpha}{x}}}{1-e^{-\frac{\alpha}{x}}}\right)^{-\beta}}.$$

$$\frac{df(x;\varphi)}{dx}=f(x;\varphi)\left(-2x^{-1}-\alpha\beta x^{-2}-\frac{\alpha(\beta-1)x^{-2}e^{-\frac{\alpha}{x}}}{\left[1-e^{-\frac{\alpha}{x}}\right]}+\alpha\beta\theta^{\beta}x^{-2}\left(e^{\frac{\alpha}{x}}-1\right)^{\beta}\right.$$

By equating the previous with zero, we get

$$f(x;\varphi)\left(-2x^{-1}-\alpha\beta x^{-2}-\frac{\alpha(\beta-1)x^{-2}e^{-\frac{\alpha}{x}}}{\left[1-e^{-\frac{\alpha}{x}}\right]}+\alpha\beta\theta^{\beta}x^{-2}\left(e^{\frac{\alpha}{x}}-1\right)^{\beta-1}e^{\frac{\alpha}{x}}\right)=0 \tag{15}$$

Then, the mode of the IWIE model can be solve numerically by solving (15).

3.6 Rényi entropy

The Rényi entropy is calculated by

$$I_{\delta}(X)=\frac{1}{1-\delta}\log\int_{-\infty}^{\infty}f(x;\varphi)^{\delta}\,dx,\qquad \delta>0 \text{ and } \delta\neq 1.$$

By applying (8) and (10) in the pdf (4), then the pdf $f(x;\varphi)^{\delta}$ can be rewritten as

$$f(x;\varphi)^{\delta}=x^{-2\delta}\sum_{i=0}^{\infty}W_i e^{-\frac{k\alpha}{x}},$$

where

$$W_i=(\alpha\beta)^{\delta}\delta^{i}\sum_{j,k=0}^{\infty}\frac{(-1)^{i+k}\theta^{\beta(i+\delta)}}{i!}\binom{\beta(i+\delta)+j-1}{j}\binom{j+\beta(i+\delta)-\delta}{k}.$$

Therefore, the Rényi entropy of *IWIE* distribution is

$$I_{\delta}(X)=\frac{1}{1-\delta}\log\left[\sum_{i=0}^{\infty}W_i\int_{0}^{\infty}x^{-2\delta}e^{-\frac{k\alpha}{x}}dx\right].$$

then,

$$I_\delta(X) = \frac{1}{1-\delta}\log\left[\sum_{i=0}^{\infty}\frac{W_i\Gamma(1-2\delta)}{[k\alpha]^{2\delta-1}}\right].$$

4. ML Estimators

Let $X_1, X_2, \dots, X_n$ be a simple random sample (RS) from the IWIE distribution with set of parameters $\varphi = (\theta, \beta, \alpha)$ .The log likelihood (LL) function based on the observed RS of size n from pdf (4) is:

$$\ln \ell = n\ln\alpha + n\ln\beta + n\beta\ln\theta - 2\sum_{i=1}^{n}\ln(x_i) + \alpha\beta\sum_{i=1}^{n}\frac{1}{x_i}$$
$$+ (\beta-1)\sum_{i=1}^{n}\ln(1-e^{-\frac{\alpha}{x_i}}) - \theta^{\beta}\sum_{i=1}^{n}\left(e^{\frac{\alpha}{x_i}}-1\right)^{\beta}.$$

The first partial derivatives of the LL function, say $\ln\ell$, with respect to the parameters are:

$$\frac{\partial\ln\ell}{\partial\theta} = \frac{n\beta}{\theta} - \beta\theta^{\beta-1}\sum_{i=1}^{n}\left(e^{\frac{\alpha}{x_i}}-1\right)^{\beta},$$

$$\frac{\partial\ln\ell}{\partial\beta} = \frac{n}{\beta} + n\ln\theta + \alpha\sum_{i=1}^{n}\frac{1}{x_i} + \sum_{i=1}^{n}\ln(1-e^{-\frac{\alpha}{x_i}}) - \theta^{\beta}\ln\theta\sum_{i=1}^{n}\left(e^{\frac{\alpha}{x_i}}-1\right)^{\beta}$$
$$- \theta^{\beta}\sum_{i=1}^{n}\left(e^{\frac{\alpha}{x_i}}-1\right)^{\beta}\ln\left(e^{\frac{\alpha}{x_i}}-1\right),$$

and,

$$\frac{\partial\ln\ell}{\partial\alpha} = \frac{n}{\alpha} + \beta\sum_{i=1}^{n}\frac{1}{x_i} + (\beta-1)\sum_{i=1}^{n}\frac{1}{x_i(e^{\frac{\alpha}{x_i}}-1)} - \beta\theta^{\beta}\sum_{i=1}^{n}\frac{1}{x_i}e^{\frac{\alpha}{x_i}}\left(e^{\frac{\alpha}{x_i}}-1\right)^{\beta-1}.$$

The ML estimators of the distribution parameters are calculated by solving numerically the non-linear equations $\partial\ln\ell/\partial\theta = 0$, $\partial\ln\ell/\partial\beta = 0$, and $\partial\ln\ell/\partial\alpha = 0$, simultaneously.

5. Application

The importance of IWIE distribution is examined using one real data set. We compared the IWIE with some main five models; inverse Weibull Weibull (IWW; Hassan and

Nassr (2018)), additive Weibull (AW; Almalki and Yuan (2013)), new modified Weibull (NMW; Doostmoradi et al. (2014)), Weibull Weibull (WW; Abouelmagd et al. (2017)) and Weibull (W). The model selection is carried out using -2 log-likelihood function ($-2\ln L$), Akaike information criterion (AIC), the correct Akaike information criterion (CAIC) and Hannan-Quinn information criterion (HQIC). However, the better model has the smallest values of $-2\ln L$, AIC, CAIC and HQIC criteria.

The data set is mentioned in Murthy et al. (2004) about time between failures for repairable item.

In Table (1), we list the values of -2LnL, AIC, CAIC and the HQIC statistics. We observe that the IWIE model has the smallest -2LnL, AIC, CAIC and the HQIC as compared with those values of the other distributions. So, the IWIE model seems to be a very competitive model to this data.

Table 1: Measures of goodness of fit for the data set.

Model	-2LogL	AIC	CAIC	HQIC
IWIE	68.776	74.776	75.699	76.121
IWW	79.21	87.21	88.81	89.003
NMW	242.501	250.051	251.651	251.845
AW	159.642	167.642	169.242	169.435
WW	80.276	88.276	89.876	90.069
W	92.751	96.751	97.196	97.648

6. Concluding Remarks

This article, proposed a new three-parameter model called the IWIE distribution. We derive explicit expressions for the moments, quantile, mgf, order statistics, mode and Rényi entropy. We discuss the ML estimation of the model parameters. One application illustrate that the IWIE model provides consistently better fit than other distributions.

References

[1] Abouelmagd, T. H. M., SaeedAl-mualim, Elgarhy, M., Ahmed, Z. Afify and Munir Ahmad, Properties of the four-parameter Weibull distribution and its applications, *Pakistan Journal of Statistics*, **33** (2017), 449-466.

[2] Almalki, S. J. and Yuan, J., A new modified Weibull distribution, *Reliability Engineering and System Safety*, **111** (2013), 164-170. https://doi.org/10.1016/j.ress.2012.10.018

[3] Almarashi, A. M. and Elgarhy, M., A new muth generated family of distributions with applications, *Journal of Nonlinear Science and Applications*, **11** (2018), 1171-1184. https://doi.org/10.22436/jnsa.011.10.06

[4] Doostmoradi, A., Zadkarami, M. R. and Roshani Sheykhabad, A., A new modified Weibull distribution and its applications, *Journal of Statistical Research*, **11** (2014), 97-118. https://doi.org/10.18869/acadpub.jsri.11.1.97

[5] Elbatal, I., Ahmad, Z., Elgarhy, B. M, & Almarashi, A. M., A New Alpha Power Transformed Family of Distributions: Properties and Applications to the Weibull Model, *Journal of Nonlinear Science and Applications*, **12**(1) (2018), 1-20. https://doi.org/10.22436/jnsa.012.01.01

[6] Eugene, N., Lee, C., Famoye, F., Beta-normal distribution and its applications, *Communications in Statistics Theory and Methods*, **31** (2002), 497-512. https://doi.org/10.1081/sta-120003130

[7] Haq, M. and Elgarhy, M., The odd Frèchet-G family of probability distributions, *Journal of Statistics Applications & Probability*, **7** (2018), 189-203. https://doi.org/10.18576/jsap/070117

[8] Hassan, A. S. and Nassr, S. G., The inverse Weibull generator of distributions: properties and applications, *Journal of Data Science*, **16**(4) (2018), 723-742.

[9] Keller, A. Z. and Kamath, A. R., Reliability analysis of CNC machine tools, *Reliability Engineering*, **3** (1982), 449-473. https://doi.org/10.1016/0143-8174(82)90036-1

[10] Kenney, J. F. and Keeping, E., *Mathematics of Statistics*, D. Van Nostr and Company, 1962.

[11] Lin, C. T., Duran, B. S. and Lewis, T. O., Inverted Gamma as life distribution, *Micro electron Reliability*, **29**(4) (1989), 619-626.
https://doi.org/10.1016/0026-2714(89)90352-1

[12] Moors, J. J. A., A quantile alternative for kurtosis, *Journal of the Royal Statistical Society, Series D (The Statistician)*, **37**(1) (1988), 25-32.
https://doi.org/10.2307/2348376

[13] Murthy, D. N. P., Xie, M. and Jiang, R., *Weibull Models*, John Wiley and Sons, New Jersey, 2004.

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