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Characterization of the Distance-k Independent Dominating Sets of the n-Path

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Abstract

The distance between two vertices u and v in a graph G equals the length of a shortest path from u to v. A set I of vertices is distance-k independent if every vertex in I is at distance at least $k+1$ to any other vertex of I. A set I of vertices is distance-k dominating if every vertex not belonging to I is at distance at most k of a vertex in I . A set of vertices is a distance-k independent dominating set if and only if this set is a distance-k independent set and a distance-k dominating set. Note that in general counting the number of independent dominating sets in a graph is NP-complete [2]. In this paper, we want to characterize all the distance-k independent dominating sets of the path P_n. Besides, we calculate the number of the distance-k independent dominating sets of P_n.

Mathematics Subject Classification: 05C69

Keywords: dominating set, independent dominating set, distance-k independent dominating set, path

1 Introduction

One of the fastest growing areas within graph theory is the study of domination and related subset problems. The theory of independent domination was proposed by Berge [1] and Ore [4] in 1962. The importance of an independent dominating set in the context of clustering wireless networks has been widely acknowledged. The independent domination number of a graph is the minimum size of an independent dominating set of vertices. The independent dominating set (IDS) problem asks for the independent domination number in a graph. Gary and Johnson [2] showed that its NP-complete for general graphs. Goddard and Henning [3] offered a survey of selected recent results on independent domination in graphs.

Although there are many papers studying the independent dominating set (IDS) problem, we prefer to consider the number of the distance-k independent dominating sets in a graph. For a graph G, the set of all the distance-k independent dominating sets of a graph G is denoted by $\mathscr{I}_k(G)$ and its cardinality by $i_k(G)$. Denote P_n a path of order n. In this paper, we want to characterize the sets $\mathscr{I}_k(P_n)$ and calculate the numbers $i_k(P_n)$, where $k \geq 1$ and P_n is the n-path.

2 Characterization

In this section, we provide a constructive characterization of $\mathscr{I}_k(P_n)$, where $n \geq 1$ and $V(P_n) = \{1, 2, \ldots, n\}$. Besides, we calculate the number $i_k(P_n)$. Besides, we calculate the number of the distance-k independent dominating sets of P_n. In order to give a constructive characterization of $\mathscr{I}_k(P_n)$, we introduce the following sets.

For $k \geq 1$ and $1 \leq i \leq k+1$, let $\mathcal{T}_1, \mathcal{T}_2 =, \ldots, \mathcal{T}_{2k+1}$ be the following sets.

$$
\begin{aligned}
\mathcal{T}_1 &= \{\{1\}\}. \\
\mathcal{T}_2 &= \{\{1\}, \{2\}\}. \\
&\vdots \\
\mathcal{T}_i &= \{\{1\}, \ldots, \{i\}\}. \\
&\vdots \\
\mathcal{T}_{k+1} &= \{\{1\}, \{2\}, \ldots, \{k+1\}\}. \\
\mathcal{T}_{k+2} &= \{\{1, k+2\}, \{2\}, \ldots, \{k+1\}\}. \\
&\vdots \\
\mathcal{T}_{k+i} &= \{\{1, k+i\}, \{2, k+i\}, \ldots, \{i-1, k+i\}, \\
&\qquad \{1, k+i-1\}, \{2, k+i-1\}, \ldots, \{i-2, k+i-1\}, \\
&\qquad\quad \vdots \\
&\qquad \{1, k+2\}, \\
&\qquad \{i\}, \{i+1\}, \ldots, \{k+1\}\}. \\
&\vdots
\end{aligned}
$$

$$\begin{aligned}\mathcal{T}_{2k+1} = \{&\{1,2k+1\},\{2,2k+1\},\ldots,\{k,2k+1\},\\ &\{1,2k\},\{2,2k\},\ldots,\{k-1,2k\},\\ &\vdots\\ &\{1,k+2\},\\ &\{k+1\}\}.\end{aligned}$$

Observation 1. For $1 \le n \le 2k+1$, $\mathscr{I}_k(P_n) = \mathcal{T}_n$

Observation 2. For $1 \le n \le 2k+1$,

$$|T_n| = \begin{cases} n, & \text{if } 1 \le n \le k+1\,,\\ k+1+\frac{s(s+1)}{2}, & \text{if } k+2 \le n = (k+2)+s \le 2k+1\,.\end{cases}$$

Let $\mathcal{A}$ be a collection of sets. We define that $\mathcal{A} \oplus a = \{A \cup \{a\} : A \in D\}$. For $n \ge 2k+2$, let

$$\mathcal{T}_n = \bigcup_{i=1}^{k+1} \mathcal{T}_{n-k-i} \oplus \{n-i+1\}.$$

We want to show that $\mathscr{I}_k(P_n) = \mathcal{T}_n$. First, we prove the following lemma.

Lemma 2.1. *For $k \ge 1$ and $n \ge 2k+2$, $\mathcal{T}_n \subseteq \mathscr{I}_k(P_n)$.*

Proof. We prove this lemma by induction on n, where $n \ge 2k+2$. For $n = 2k+2$, then

$$\begin{aligned}\mathcal{T}_{2k+2} &= \bigcup_{i=1}^{k+1} \mathcal{T}_{(2k+2)-k-i} \oplus \{(2k+2)-i+1\}\\ &= \{\{1,2k+2\},\{2,2k+2\},\ldots,\{k+1,2k+2\},\\ &\quad \{1,2k+1\},\{2,2k+1\},\ldots,\{k,2k+1\},\\ &\quad \vdots\\ &\quad \{1,k+2\}\}.\end{aligned}$$

Every set in $\mathcal{T}_{2k+2}$ is a distance-k independent dominating set of P_n, so it's true for $n = 2k+2$. Assume that it's true for all $n' < n$ and let $I \in \mathcal{T}_n$, where $n \ge 2k+3$. Suppose a is the largest number in I, then $a = n-i+1$ for $i = 1,\ldots,k+1$. Let $I' = I - \{a\}$. Since $d(a, n-k-i) = (n-i+1)-(n-k-i) = k+1$, this means that $I' \in \mathcal{T}_{n-k-i}$ for $i = 1,\ldots,k+1$. By the induction hypothesis, $I' \in \mathscr{I}(P_{n-k-i})$. Note that $d(a, n-k-i) = k+1$, this means that I is a distance-k independent set of P_n. Now we want to show that I is a distance-k dominating set of P_n. Let $1 \le j \le n$ and $j \notin I$. We consider three cases.

<u>**Case 1**</u>. $1 \le j \le n-k-i$. Note that $I' \in \mathscr{I}_k(P_{n-k-i})$, then j is at distance at most k to some vertex in I'.

Case 2. $n-k-i+1 \leq j \leq a-1$. Note that $d(a,j) \leq (n-i+1)-(n-k-i+1) = k$, then j is at distance at most k to the vertex a.
Case 3. $a+1 \leq j \leq n$. Note that $d(a,j) \leq n-(a+1) = n-(n-i+1+1) = i-2 < k$, then j is at distance at most k to the vertex a.

By Case 1, Case 2 and Case 3, we can see that j is at distance at most k to some vertex in I, where $I = I' \cup \{a\}$. Thus I is a distance-k dominating set of P_n. Hence $I \in \mathscr{I}_k(P_n)$, we complete the proof. $\square$

In the following theorem, we will show that $\mathcal{T}_n$ is the characterization of $\mathscr{I}_k(P_n)$.

Theorem 2.2. *For $n \geq 2k+2$, $\mathscr{I}_k(P_n) = \mathcal{T}_n$.*

Proof. By Lemma 2.1, we obtain that $\mathcal{T}_n \subseteq \mathscr{I}_k(P_n)$. Now we want to show that $\mathscr{I}_k(P_n) \subseteq \mathcal{T}_n$ and prove it by induction on n, where $n \geq 2k+2$. We can see that

$$\begin{aligned}\mathscr{I}_k(P_{2k+2}) &= \{\{1,2k+2\},\{2,2k+2\},\ldots,\{k+1,2k+2\},\\ &\quad \{1,2k+1\},\{2,2k+1\},\ldots,\{k,2k+1\},\\ &\quad \vdots\\ &\quad \{1,k+2\}\}\\ &= \bigcup_{i=1}^{k+1} \mathcal{T}_{(2k+2)-k-i} \oplus \{(2k+2)-i+1\}\\ &= \mathcal{T}_{2k+2}.\end{aligned}$$

So it's true for $n = 2k+2$. Assume that it's true for all $n' < n$, where $n \geq 2k+3$. Suppose $I \in \mathscr{I}_k(P_n)$ and a is the largest number in I. Thus a is at distance at most k to the vertex n, say $a = n-i+1$ for some $i \in \{1,\ldots,k+1\}$. Let $I' = I - \{a\}$. Note that $a-(k+1) = (n-i+1)-(k+1) = n-k-i$, then $I' \subseteq \{1,2,\ldots,n-k-i\}$. Since I is a distance-k independent set of P_n, we can see that I' is a distance-k independent set of P_{n-k-i}. Since I is a distance-k dominating set of P_n and $d(a,j) \geq k+1$ for $j \in \{1,2,\ldots,n-k-i\}$, we have that I' is a distance-k dominating set of P_{n-k-i}. Hence $I' \in \mathscr{I}_k(P_{n-k-i})$, by the induction hypothesis, $I' \in \mathcal{T}_{n-k-i}$. This means that $I = I' \cup \{n-i+1\} \in \mathcal{T}_n$. So it's true for n and $\mathscr{I}_k(P_n) \subseteq \mathcal{T}_n$. We complete the proof. $\square$

For $k \geq 1$ and $n \geq 2k+2$, we provide a constructive characterization of $\mathscr{I}_k(P_n)$, where $\mathcal{T}_1, \mathcal{T}_1, \ldots, \mathcal{T}_{2k+1}$ are the initial conditions. Now we calculate the number of the distance-k independent dominating sets of P_n. For $k \geq 1$, $n \geq 2k+2$ and $1 \leq i \leq k+1$, let $\mathcal{T}_n^{(i)}$ be the collection of all the distance-k independent dominating sets of $\mathcal{T}_n$ which contain the vertex $n-i+1$. In the following lemma, we calculate the number $i_k(P_n)$.

Lemma 2.3. *For $n \geq 2k+2$, we have the following results.*
(i) *For $1 \leq i_1 < i_2 \leq k+1$, $\mathcal{T}_n^{(i_1)} \cap \mathcal{T}_n^{(i_2)} = \emptyset$.*
(ii) $i_k(P_n) = \sum_{i=1}^{k+1} i_k(P_{n-k-i})$.

Proof. (i) Suppose that $\mathcal{T}_n^{(i_1)} \cap \mathcal{T}_n^{(i_2)} \neq \emptyset$, where $1 \leq i_1 < i_2 \leq k+1$, and let $I \in \mathcal{T}_n^{(i_j)}$ for $j = 1$ and 2. Then we obtain that $n - i_1 + 1 \in I$ and $n - i_2 + 1 \in I$, by the definition, $i_2 - i_1 = (n - i_1 + 1) - (n - i_2 + 1) \geq k + 1$. This contradiction that $1 \leq i_1 < i_2 \leq k+1$. Hence $\mathcal{T}_n^{(i_1)} \cap \mathcal{T}_n^{(i_2)} = \emptyset$ when $1 \leq i_1 < i_2 \leq k+1$.

(ii)By (i) and Theorem 2.2, we have that $i_k(P_n) = |\mathcal{T}_n| = \sum_{i=1}^{k+1} |\mathcal{T}_n^{(i)}| = \sum_{i=1}^{k+1} |\mathcal{T}_{n-k-i}| = \sum_{i=1}^{k+1} i_k(P_{n-k-i})$. □

3 Instantiation

In this section, we characterize the sets $\mathscr{I}_3(P_n)$ for $1 \leq n \leq 15$. Besides, we calculate the numbers $i_k(P_n)$ for $1 \leq k \leq 10$ and $1 \leq n \leq 15$.

For $k = 3$, we show the sets $\mathscr{I}_3(P_1), \mathscr{I}_3(P_2), \ldots, \mathscr{I}_3(P_{15})$.

$\mathscr{I}_3(P_1) = \{\{1\}\}$.
$\mathscr{I}_3(P_2) = \{\{1\}, \{2\}\}$.
$\mathscr{I}_3(P_3) = \{\{1\}, \{2\}, \{3\}\}$.
$\mathscr{I}_3(P_4) = \{\{1\}, \{2\}\}, \{3\}, \{4\}\}$.
$\mathscr{I}_3(P_5) = \{\{1,5\}, \{2\}\}, \{3\}, \{4\}\}$.
$\mathscr{I}_3(P_6) = \{\{1,6\}, \{2,6\}, \{1,5\}, \{3\}, \{4\}\}$.
$\mathscr{I}_3(P_7) = \{\{1,7\}, \{2,7\}, \{3,7\}, \{1,6\}, \{2,6\}, \{1,5\}, \{4\}\}$.
$\mathscr{I}_3(P_8) = \{\{1,\underline{8}\}, \{2,\underline{8}\}, \{3,\underline{8}\}, \{4,\underline{8}\}, \{1,\underline{7}\}, \{2,\underline{7}\}, \{3,\underline{7}\}, \{1,\underline{6}\}, \{2,\underline{6}\}, \{1,\underline{5}\}\}$.
$\mathscr{I}_3(P_9) = \{\{1,5,\underline{9}\}, \{2,\underline{9}\}\{3,\underline{9}\}, \{4,\underline{9}\}, \{1,\underline{8}\}, \{2,\underline{8}\}, \{3,\underline{8}\}, \{4,\underline{8}\},$
$\{1,\underline{7}\}, \{2,\underline{7}\}, \{3,\underline{7}\}, \{1,\underline{6}\}, \{2,\underline{6}\}\}$.
$\mathscr{I}_3(P_{10}) = \{\{1,6,\underline{10}\}, \{2,6,\underline{10}\}, \{1,5,\underline{10}\}, \{3,\underline{10}\}, \{4,\underline{10}\}, \{1,5,\underline{9}\}, \{2,\underline{9}\}, \{3,\underline{9}\},$
$\{4,\underline{9}\}, \{1,\underline{8}\}, \{2,\underline{8}\}, \{3,\underline{8}\}, \{4,\underline{8}\}, \{1,\underline{7}\}, \{2,\underline{7}\}, \{3,\underline{7}\}\}$.
$\mathscr{I}_3(P_{11}) = \{\{1,7,\underline{11}\}, \{2,7,\underline{11}\}, \{3,7,\underline{11}\}, \{1,6,\underline{11}\}, \{2,6,\underline{11}\}, \{1,5,\underline{11}\}, \{4,\underline{11}\}\}$
$\{1,6,\underline{10}\}, \{2,6,\underline{10}\}, \{1,5,\underline{10}\}, \{3,\underline{10}\}, \{4,\underline{10}\}, \{1,5,\underline{9}\}, \{2,\underline{9}\}, \{3,\underline{9}\},$
$\{1,\underline{8}\}, \{2,\underline{8}\}, \{3,\underline{8}\}, \{4,\underline{8}\}\}$.
$\mathscr{I}_3(P_{12}) = \{4,\underline{9}\}, \{\{1,8,\underline{12}\}, \{2,8,\underline{12}\}, \{3,8,\underline{12}\}, \{4,8,\underline{12}\}, \{1,7,\underline{12}\}, \{2,7,\underline{12}\},$
$\{3,7,\underline{12}\}, \{1,6,\underline{12}\}, \{2,6,\underline{12}\}, \{1,5,\underline{12}\}, \{1,7,\underline{11}\}, \{2,7,\underline{11}\},$
$\{3,7,\underline{11}\}, \{1,6,\underline{11}\}, \{2,6,\underline{11}\}, \{1,5,\underline{11}\}, \{4,\underline{11}\}\}, \{1,6,\underline{10}\}, \{2,6,\underline{10}\},$
$\{1,5,\underline{10}\}, \{3,\underline{10}\}, \{4,\underline{10}\}, \{1,5,\underline{9}\}, \{2,\underline{9}\}, \{3,\underline{9}\}, \{4,\underline{9}\}\}$.

$$\begin{aligned}\mathscr{I}_3(P_{13}) = \{&\{1,5,9,\underline{13}\},\{2,9,\underline{13}\}\{3,9,\underline{13}\},\{4,9,\underline{13}\},\{1,8,\underline{13}\},\{2,8,\underline{13}\},\\ &\{3,8,\underline{13}\},\{4,8,\underline{13}\},\{1,7,\underline{13}\},\{2,7,\underline{13}\},\{3,7,\underline{13}\},\{1,6,\underline{13}\},\\ &\{2,6,\underline{13}\},\{1,8,\underline{12}\},\{2,8,\underline{12}\},\{3,8,\underline{12}\},\{4,8,\underline{12}\},\{1,7,\underline{12}\},\\ &\{2,7,\underline{12}\},\{3,7,\underline{12}\},\{1,6,\underline{12}\},\{2,6,\underline{12}\},\{1,5,\underline{12}\},\{1,7,\underline{11}\},\\ &\{2,7,\underline{11}\},\{3,7,\underline{11}\},\{1,6,\underline{11}\},\{2,6,\underline{11}\},\{1,5,\underline{11}\},\\ &\{4,\underline{11}\}\},\{1,6,\underline{10}\},\{2,6,\underline{10}\},\{1,5,\underline{10}\},\{3,\underline{10}\},\{4,\underline{10}\}\}.\end{aligned}$$

$$\begin{aligned}\mathscr{I}_3(P_{14}) = \{&\{1,6,10,\underline{14}\},\{2,6,10,\underline{14}\},\{1,5,10,\underline{14}\},\{3,10,\underline{14}\},\{4,10,\underline{14}\},\\ &\{1,5,9,\underline{14}\},\{2,9,\underline{14}\},\{3,9,\underline{14}\},\{4,9,\underline{14}\},\{1,8,\underline{14}\},\{2,8,\underline{14}\},\\ &\{3,8,\underline{14}\},\{4,8,\underline{14}\},\{1,7,\underline{14}\},\{2,7,\underline{14}\},\{3,7,\underline{14}\},\{1,5,9,\underline{13}\},\\ &\{2,9,\underline{13}\}\{3,9,\underline{13}\},\{4,9,\underline{13}\},\{1,8,\underline{13}\},\{2,8,\underline{13}\},\{3,8,\underline{13}\},\\ &\{4,8,\underline{13}\},\{1,7,\underline{13}\},\{2,7,\underline{13}\},\{3,7,\underline{13}\},\{1,6,\underline{13}\},\{2,6,\underline{13}\},\\ &\{1,8,\underline{12}\},\{2,8,\underline{12}\},\{3,8,\underline{12}\},\{4,8,\underline{12}\},\{1,7,\underline{12}\},\{2,7,\underline{12}\},\\ &\{3,7,\underline{12}\},\{1,6,\underline{12}\},\{2,6,\underline{12}\},\{1,5,\underline{12}\},\{1,7,\underline{11}\},\{2,7,\underline{11}\},\\ &\{3,7,\underline{11}\},\{1,6,\underline{11}\},\{2,6,\underline{11}\},\{1,5,\underline{11}\},\{4,\underline{11}\}\}.\end{aligned}$$

$$\begin{aligned}\mathscr{I}_3(P_{15}) = \{&\{1,7,11,\underline{15}\},\{2,7,11,\underline{15}\},\{3,7,11,\underline{15}\},\{1,6,11,\underline{15}\},\{2,6,11,\underline{15}\},\\ &\{1,5,11,\underline{15}\},\{4,11,\underline{15}\},\{1,6,10,\underline{15}\},\{2,6,10,\underline{15}\},\{1,5,10,\underline{15}\},\\ &\{3,10,\underline{15}\},\{4,10,\underline{15}\},\{1,5,9,\underline{15}\},\{2,9,\underline{15}\},\{3,9,\underline{15}\},\{4,9,\underline{15}\},\\ &\{1,8,\underline{15}\},\{2,8,\underline{15}\},\{3,8,\underline{15}\},\{4,8,\underline{15}\},\{1,6,10,\underline{14}\},\{2,6,10,\underline{14}\},\\ &\{1,5,10,\underline{14}\},\{3,10,\underline{14}\},\{4,10,\underline{14}\},\{1,5,9,\underline{14}\},\{2,9,\underline{14}\},\{3,9,\underline{14}\},\\ &\{4,9,\underline{14}\},\{1,8,\underline{14}\},\{2,8,\underline{14}\},\{3,8,\underline{14}\},\{4,8,\underline{14}\},\{1,7,\underline{14}\},\\ &\{2,7,\underline{14}\},\{3,7,\underline{14}\},\{1,5,9,\underline{13}\},\{2,9,\underline{13}\}\{3,9,\underline{13}\},\{4,9,\underline{13}\},\\ &\{1,8,\underline{13}\},\{2,8,\underline{13}\},\{3,8,\underline{13}\},\{4,8,\underline{13}\},\{1,7,\underline{13}\},\{2,7,\underline{13}\},\\ &\{3,7,\underline{13}\},\{1,6,\underline{13}\},\{2,6,\underline{13}\},\{1,8,\underline{12}\},\{2,8,\underline{12}\},\{3,8,\underline{12}\},\\ &\{4,8,\underline{12}\},\{1,7,\underline{12}\},\{2,7,\underline{12}\},\{3,7,\underline{12}\},\{1,6,\underline{12}\},\{2,6,\underline{12}\},\\ &\{1,5,\underline{12}\}\}.\end{aligned}$$

We end with the following table for summarizing the numbers $i_k(P_n)$ for $1 \le k \le 10$ and $1 \le n \le 15$.

Table 1: The numbers $i_k(P_n)$ for $1 \leq k \leq 10$ and $1 \leq n \leq 15$.

n	$i_1(P_n)$	$i_2(P_n)$	$i_3(P_n)$	$i_4(P_n)$	$i_5(P_n)$	$i_6(P_n)$	$i_7(P_n)$	$i_8(P_n)$	$i_9(P_n)$	$i_{10}(P_n)$
1	1	1	1	1	1	1	1	1	1	1
2	2	2	2	2	2	2	2	2	2	2
3	2	3	3	3	3	3	3	3	3	3
4	3	3	4	4	4	4	4	4	4	4
5	4	4	4	5	5	5	5	5	5	5
6	5	6	5	5	6	6	6	6	6	6
7	7	8	7	6	6	7	7	7	7	7
8	9	10	10	8	7	7	8	8	8	8
9	12	13	13	11	9	8	8	9	9	9
10	16	18	16	15	12	10	9	9	10	10
11	21	24	20	19	16	13	11	10	10	11
12	28	31	26	23	21	17	14	12	11	11
13	37	41	35	28	26	22	18	15	13	12
14	49	55	46	35	31	28	23	19	16	14
15	65	73	59	45	37	34	29	24	20	17
16	86	96	75	59	45	40	36	30	25	21
17	114	127	97	76	56	47	43	37	31	26
18	151	169	127	96	71	56	50	45	38	32
19	200	224	166	120	91	68	58	53	46	39
20	265	296	215	150	115	84	68	61	55	47
21	351	392	277	190	143	105	81	70	64	56
22	465	520	358	243	176	132	98	81	73	66
23	616	689	465	311	216	164	120	95	83	76
24	816	912	605	396	266	201	148	113	95	86
25	1081	1208	785	501	331	244	183	136	110	97

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