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Logistic-Modified Weibull Distribution and Parameter Estimation

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Abstract

The Logistic-Modified Weibull distribution is introduced as a new lifetime distribution based on the T-X family. Some properties of the new distribution are studied. Also, the estimation of the parameters is discussed using different methods such as maximum likelihood and Bayesian. The asymptotic variance-covariance matrix is obtained. Finally, we provide an application to real data.

Keywords: The Logistic-Modified Weibull distribution; maximum likelihood estimates; asymptotic variance-covariance matrix; Bayesian estimation

1 Introduction

Familier distributions are modified and/or generalized using different directions. These directions are interested in deriving new families of univariate continuous distributions by introducing one or more additional shape parameter(s)to the baseline distribution. Some of well-known families are: "the beta-G family (Eugene et al.,2002; Jones,2004); the gamma-G family(type 1)(Zografos and Balakrishanan, 2009); the Kumaraswamy-G (Kw-G)family(Cordeiro and de Castro,2011); the gamma-G type 2 family (Ristić and Balakrishanan, 2012); the McDonald - G (Mc-G) family (Alexander et al., 2012); the gamma-G family type 3(Torabi and Montazari, 2012); the log-gamma-G family (Amini et al., 2012);exponentiated generalized-G family (Cordeiro et al., 2013); the transformed - transformer T-X family (Alzaatreh et al., 2013); the exponentiated T-X family (Alzaghal et al., 2013); the Weibull-G family (Bourguignon et al., 2014); the gamma-X family (Alzaatreh et al., 2014); the logistic-G (Torabi and Montazari, 2014); the logistic-X family (Tahir et al., 2014); the T-normal family (Alzaatreh et al., 2014); the T-X family using quantile functions (Aljarrah et al., 2014); the T-X family using the logit function (Al-Aqtash et al., 2015) and more.

Suppose that we have a random variable (r.v.) $T \in [a, b]$ such that $-\infty < a < b < \infty$ with the probability distribution(pdf), say v(t). Also, let Q[G(x)] be a function of the cumulative distribution function (cdf) of a random variable X. Alzaatreh et al. (2013) introduced the T-X family of distributions with the following formula:

$$F(x) = \int_a^{Q[G(x)]} v(t)dt \tag{1}$$

where Q[G(x)] satisfies the following conditions:

$$\begin{cases} \text{Q[.]with support}[a, b], \\ Q[.] \text{ is differentiable and monotonically non-decreasing}, and \\ Q[.] \rightarrow a \text{ as x} \rightarrow -\infty \text{ and } Q[.] \rightarrow \text{b as x} \rightarrow \infty \end{cases}$$

The pdf corresponding to (1) is given by

$$f(x) = v(Q(G(x)))\frac{d}{dx}Q(G(x)) \tag{2}$$

Tahir et al. (2014) studied the case when T follows the logistic distribution with shape parameter $a > 0$, its cdf is

$$(1 + e^{-at})^{-1}, -\infty < t < \infty, \tag{3}$$

The corresponding pdf take the form

$$v(t) = ae^{-at}((1 + e^{-at})^{-1}, -\infty < t < \infty. \tag{4}$$

and they studied the case when $Q[G(x)] = log\{-[log[1 - G(x)]\}$(Aljarrah et al., 2014) expressed this function as a quantile function). So the new family of continuous distributions generated from a logistic r.v. called the logistic-X family takes the following cdf and pdf, respectively

$$F(x) = [1 + \{-log[1 - G(x)]\}^{-a}]^{-1}, \tag{5}$$

$$f(x) = \frac{ag(x)}{1-G(x)}\{-log[1-G(x)]\}^{-(a+1)}[1+\{-log[1-G(x)]\}^{-a}]^{-2}. \quad (6)$$

where g(x) is the pdf of the baseline distribution. In this article we consider X follows Modified Weibull (Sarhan and Zain-Din, 2009) r.v. and introduced the logistic-Modified Weibull distribution as a new distribution in the following section.

2 Logistic-Modified Weibull Distribution

The Modified Weibull distribution has pdf and cdf given by
$g(x) = (\alpha + c\lambda x^{c-1})e^{-(\alpha x+\lambda x^c)}$, $G(x) = 1 - e^{-(\alpha x+\lambda x^c)}$ respectively. Then from (5) we can obtain the cdf of logistic-Modified Weibull distribution as follow

$$F_{LMW}(x) = \frac{1}{[1+(\alpha x+\lambda x^c)^{-a}]}, \ a, c, \lambda, \alpha > 0, x > 0, \quad (7)$$

so, the survival function is

$$S_{LMW}(x) = \frac{(\alpha x+\lambda x^c)^{-a}}{[1+(\alpha x+\lambda x^c)^{-a}]}, \ a, c, \lambda, \alpha > 0, x > 0, \quad (8)$$

and the corresponding pdf takes the form

$$f_{LMW}(x) = \frac{a\,(\alpha+\lambda c x^{c-1})(\alpha x+\lambda x^c)^{-(a+1)}}{[1+(\alpha x+\lambda x^c)^{-a}]^2}, a, c, \lambda, \alpha > 0, x > 0. \quad (9)$$

One can note the following special cases:
for $\lambda = 0$ or $(\alpha = 0, c = 1)$, we have the Logistic-Exponential (LE).
for $\alpha = 0$, we have Logistic-Weibull (LW).
for $\alpha = 0$, $c = 2$, we have the Logistic-Rayleigh (LR).
for $c = 2$, we have Logistic-Linear Failure Rate distribution$(LLFR)$.

Plots of the pdf of the LMW for different parameter values are given in figure (1). From (8) and (9) the hazard function of the LMW distribution is given by

$$h(x) = \frac{a(\alpha+\lambda c x^{c-1})}{(\alpha x+\lambda x^c)[1+(\alpha x+\lambda x^c)^{-a}]} \quad (10)$$

and figure (2) displays some shapes of the hazard function for selected parameter values.

The r^{th} raw moments for the LMW distribution are obtained using the following formula:

$$E(X^k) = \int_0^\infty \frac{ax^k(\alpha+\lambda x^{c-1})(\alpha x+\lambda x^c)^{-(a+1)}}{[1+(\alpha x+\lambda x^c)^{-a}]^2}dx \quad (11)$$

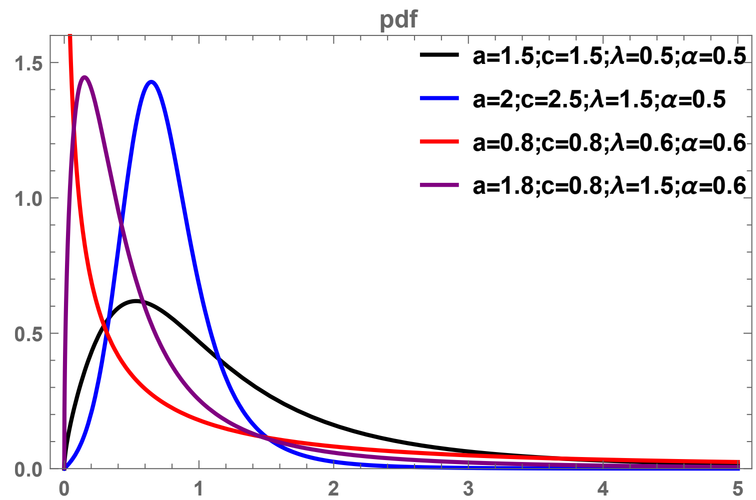

Figure 1: Probability density function of LMW distribution for different values of the parameters

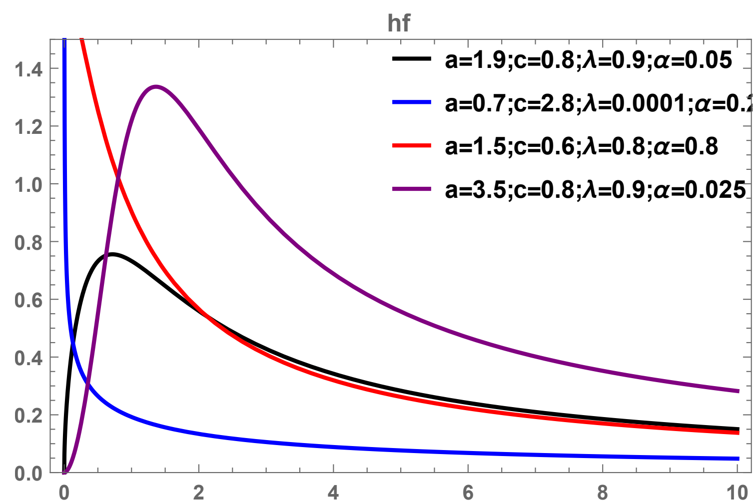

Figure 2: Hazard function of LMW distribution for different values of the parameters

one can use numerical integration procedures to calculate (11).
The r^{th} raw moments are obtained for the LW distribution as follows

$$\begin{aligned} E(X^k) &= \frac{ac}{\lambda^a}\int_0^\infty \frac{x^{k-(ac+1)}}{[1+(\lambda x^c)^{-a}]^2}dx \\ &= \frac{k\pi}{ac}(\lambda^a)^{\frac{-k}{ac}}\csc[\frac{\pi(k)}{ac}] \end{aligned} \quad (12)$$

where $0 < \frac{k}{ac} < 2$,(see formula 3.194-6 page 316 in Gradshteyn and Ryzhik

(2007)).

3 Parameter Estimation

Here, two methods of estimation; maximum likelihood and Bayesian Estimation are discussed.

3.1 Maximum Likelihood Estimation

Let

$$L = a^n \prod_{i=1}^{n}(\alpha + \lambda c x_i^{c-1}) \prod_{i=1}^{n}(\alpha x_i + \lambda x_i^c)^{-(a+1)} \prod_{i=1}^{n}[1 + (\alpha x_i + \lambda x_i^c)^{-a}]^{-2} \quad (13)$$

The log-likelihood function is given by

$$lnL = nlna + \sum_{i=1}^{n} ln(\alpha + \lambda c x_i^{c-1}) - (a+1)\sum_{i=1}^{n} ln(\alpha x_i + \lambda x_i^c) - 2\sum_{i=1}^{n} ln[1 + (\alpha x_i + \lambda x_i^c)^{-a}]. \quad (14)$$

The first derivatives of the log-likelihood function are given as follows

$$\begin{aligned}
\frac{\partial lnL}{\partial a} &= \frac{n}{a} - \sum_{i=1}^{n} ln(\alpha x_i + \lambda x_i^c) + 2\sum_{i=1}^{n} \frac{ln(\alpha x_i + \lambda x_i^c)}{[1 + (\alpha x_i + \lambda x_i^c)^a]} \\
\frac{\partial lnL}{\partial \lambda} &= \sum_{i=1}^{n} \frac{c x_i^{c-1}}{[\alpha + \lambda c x_i^{c-1}]} - (a+1)\sum_{i=1}^{n} \frac{x_i^c}{[\alpha x_i + \lambda x_i^c]} + 2a\sum_{i=1}^{n} \frac{x_i^c(\alpha x_i + \lambda x_i^c)^{-(a+1)}}{[1 + (\alpha x_i + \lambda x_i^c)^{-a}]} \quad (15) \\
\frac{\partial lnL}{\partial c} &= \sum_{i=1}^{n} \frac{\lambda x_i^{c-1}(1 + clnx_i)}{[\alpha + \lambda c x_i^{c-1}]} - (a+1)\sum_{i=1}^{n} \frac{\lambda x_i^{c-1} lnx_i}{[\alpha x_i + \lambda x_i^c]} + 2a\lambda \sum_{i=1}^{n} \frac{x_i^c lnx_i(\alpha x_i + \lambda x_i^c)^{-(a+1)}}{[1 + (\alpha x_i + \lambda x_i^c)^{-a}]} \\
\frac{\partial lnL}{\partial \alpha} &= \sum_{i=1}^{n} \frac{1}{[\alpha + \lambda c x_i^{c-1}]} - (a+1)\sum_{i=1}^{n} \frac{1}{[\alpha x_i + \lambda x_i^c]} + 2a\sum_{i=1}^{n} \frac{x_i(\alpha x_i + \lambda x_i^c)^{-(a+1)}}{[1 + (\alpha x_i + \lambda x_i^c)^{-a}]}
\end{aligned}$$

Equating equations in (15) to zero and solving them numerically, one can obtain the estimates of the unknown parameters. Now, the second order derivative of log-likelihood function are obtained as follows

$$\begin{aligned}
\frac{\partial^2 lnL}{\partial a^2} &= -\frac{n}{a^2} + 2\sum_{i=1}^{n} \frac{(\alpha x_i + \lambda x_i^c)^{-2a} ln(\alpha x_i + \lambda x_i^c)^2}{[1 + (\alpha x_i + \lambda x_i^c)^{-a}]^2} - 2\sum_{i=1}^{n} \frac{(\alpha x_i + \lambda x_i^c)^{-a} ln(\alpha x_i + \lambda x_i^c)^2}{[1 + (\alpha x_i + \lambda x_i^c)^{-a}]}, \\
\frac{\partial^2 lnL}{\partial a \partial \lambda} &= -\sum_{i=1}^{n} \frac{x_i^c}{\alpha x_i + \lambda x_i^c} + 2\sum_{i=1}^{n} \frac{x_i^c(\alpha x_i + \lambda x_i^c)^{-a-1}(1 - aln(\alpha x_i + \lambda x_i^c)}{[1 + (\alpha x_i + \lambda x_i^c)^{-a}]} \\
&\quad + 2a\sum_{i=1}^{n} \frac{x_i^c(\alpha x_i + \lambda x_i^c)^{-2a-1} ln(\alpha x_i + \lambda x_i^c)}{[1 + (\alpha x_i + \lambda x_i^c)^{-a}]^2},
\end{aligned}$$

$$\begin{aligned}
\frac{\partial^2 lnL}{\partial a c} &= -\sum_{i=1}^{n}\frac{\lambda x_i^c log[x_i]}{\alpha x_i+\lambda x_i^c} + 2\sum_{i=1}^{n}\frac{\lambda x_i^c(\alpha x_i+\lambda x_i^c)^{-a-1}log[x_i](1-aln(\alpha x_i+\lambda x_i^c))}{[1+(\alpha x_i+\lambda x_i^c)^{-a}]} \\
&-2a\lambda\sum_{i=1}^{n}\frac{x_i^c(\alpha x_i+\lambda x_i^c)^{-2a-1}ln[x_i]ln(\alpha x_i+\lambda x_i^c)}{[1+(\alpha x_i+\lambda x_i^c)^{-a}]^2}, \\
\frac{\partial^2 lnL}{\partial a \alpha} &= -\sum_{i=1}^{n}\frac{x_i}{\alpha x_i+\lambda x_i^c} + 2\sum_{i=1}^{n}\frac{x_i(\alpha x_i+\lambda x_i^c)^{-a-1}(1-aln(\alpha x_i+\lambda x_i^c))}{[1+(\alpha x_i+\lambda x_i^c)^{-a}]} \\
&+2a\sum_{i=1}^{n}\frac{x_i(\alpha x_i+\lambda x_i^c)^{-2a-1}ln(\alpha x_i+\lambda x_i^c)}{[1+(\alpha x_i+\lambda x_i^c)^{-a}]^2},
\end{aligned}$$

$$\begin{aligned}
\frac{\partial^2 lnL}{\partial \lambda^2} &= -\sum_{i=1}^{n}\frac{c^2x_i^{2c-2}}{(\alpha+c\lambda x_i^{c-1})^2} + \sum_{i=1}^{n}\frac{(a+1)x_i^{2c}}{[(\alpha x_i+\lambda x_i^c)^2]} + 2a(-a-1)\sum_{i=1}^{n}\frac{x_i^{2c}(\alpha x_i+\lambda x_i^c)^{-a-2}}{[1+(\alpha x_i+\lambda x_i^c)^{-a}]} \\
&+2a^2\sum_{i=1}^{n}\frac{x_i^{2c}(\alpha x_i+\lambda x_i^c)^{-2a-2}}{[1+(\alpha x_i+\lambda x_i^c)^{-a}]^2}, \\
\frac{\partial^2 lnL}{\partial \lambda c} &= \sum_{i=1}^{n}\frac{x_i^{c-1}(1+cln[x_i])}{(\alpha+c\lambda x_i^{c-1})} - \lambda c\sum_{i=1}^{n}\frac{x_i^{2c-2}(1+cln[x_i])}{(\alpha+c\lambda x_i^{c-1})^2} - \sum_{i=1}^{n}\frac{(a+1)x_i^c ln[x_i]}{[\alpha x_i+\lambda x_i^c]} \\
&+\sum_{i=1}^{n}\frac{\lambda(a+1)x_i^{2c}ln[x_i]}{[\alpha x_i+\lambda x_i^c]^2} + 2a(-a-1)\lambda\sum_{i=1}^{n}\frac{x_i^{2c}(\alpha x_i+\lambda x_i^c)^{-a-2}ln[x_i]}{[1+(\alpha x_i+\lambda x_i^c)^{-a}]} \\
&+2a^2\lambda\sum_{i=1}^{n}\frac{x_i^{2c}(\alpha x_i+\lambda x_i^c)^{-2a-2}}{[1+(\alpha x_i+\lambda x_i^c)^{-a}]^2}, \\
\frac{\partial^2 lnL}{\partial \lambda \alpha} &= -\sum_{i=1}^{n}\frac{cx_i^{c-1}}{(\alpha+c\lambda x_i^{c-1})^2} + \sum_{i=1}^{n}\frac{(a+1)x_i^{c+1}}{[\alpha x_i+\lambda x_i^c]^2} + 2a(-a-1)\lambda\sum_{i=1}^{n}\frac{x_i^{c+1}(\alpha x_i+\lambda x_i^c)^{-2a-2}}{[1+(\alpha x_i+\lambda x_i^c)^{-a}]^2} \\
&+2a^2\sum_{i=1}^{n}\frac{x_i(\alpha x_i+\lambda x_i^c)^{-2a-2}}{[1+(\alpha x_i+\lambda x_i^c)^{-a}]^2},
\end{aligned}$$

$$\begin{aligned}
\frac{\partial^2 lnL}{\partial c^2} &= \sum_{i=1}^{n}\frac{\lambda x_i^{c-1}ln[x_i][2+cln[x_i]]}{(\alpha+c\lambda x_i^{c-1})} - \sum_{i=1}^{n}\frac{\lambda x_i^{c-1}[1+cln[x_i]]}{(\alpha+c\lambda x_i^{c-1})^2} - \sum_{i=1}^{n}\frac{(a+1)\lambda x_i^c ln[x_i]^2}{[\alpha x_i+\lambda x_i^c]} \\
&+\sum_{i=1}^{n}\frac{(a+1)\lambda^2 x_i^{2c}ln[x_i]^2}{[\alpha x_i+\lambda x_i^c]^2} \\
&+2a\lambda\sum_{i=1}^{n}\frac{x_i^{c-1}ln(x_i)^2[(\alpha x_i+\lambda x_i^c)^{-a-1}][1+(-a-1)\lambda x_i^c((\alpha x_i+\lambda x_i^c)^{-1})}{[1+(\alpha x_i+\lambda x_i^c)^{-a}]} \\
&+2a^2\lambda\sum_{i=1}^{n}\frac{x_i^{c+1}(\alpha x_i+\lambda x_i^c)^{-2a-2}ln[x_i]}{[1+(\alpha x_i+\lambda x_i^c)^{-a}]^2},
\end{aligned}$$

$$\begin{aligned}
\frac{\partial^2 lnL}{\partial c\alpha} &= -\sum_{i=1}^{n}\frac{\lambda x_i^{c-1}[1+cln[x_i]]}{(\alpha+c\lambda x_i^{c-1})^2}+\sum_{i=1}^{n}\frac{(a+1)[\lambda x_i^{c+1}ln[x_i]]^2}{(\alpha x_i+c\lambda x_i^{c})^2} \\
&+2a(-a-1)\lambda\sum_{i=1}^{n}\frac{x_i^{c+1}ln[x_i][\alpha x_i+\lambda x_i^{c}]^{-a-2}}{1+[\alpha x_i+\lambda x_i^{c}]^{-}a}+2a^2\sum_{i=1}^{n}\frac{x_i^{c+1}ln[x_i][\alpha x_i+\lambda x_i^{c}]^{-2a-2}}{(1+[\alpha x_i+\lambda x_i^{c}]^{-a})^2}, \\
\frac{\partial^2 lnL}{\partial \alpha^2} &= -\sum_{i=1}^{n}\frac{1}{(\alpha+c\lambda x_i^{c-1})^2}+\sum_{i=1}^{n}\frac{(a+1)x_i^2}{(\alpha x_i+c\lambda x_i^{c})^2} \\
&+2a(-a-1)\sum_{i=1}^{n}\frac{x_i^2[\alpha x_i+\lambda x_i^{c}]^{-a-2}}{1+[\alpha x_i+\lambda x_i^{c}]^{-}a} \\
&+2a^2\sum_{i=1}^{n}\frac{x_i^2[\alpha x_i+\lambda x_i^{c}]^{-2a-2}}{(1+[\alpha x_i+\lambda x_i^{c}]^{-a})^2}.
\end{aligned}$$

and the observed information matrix is given by

$$J(\theta)=-\begin{bmatrix} I_{a,a} & I_{a,c} & I_{a,\lambda} & I_{a,\alpha} \\ I_{c,a} & I_{c,c} & I_{c,\lambda} & I_{c,\alpha} \\ I_{\lambda,a} & I_{\lambda,c} & I_{\lambda,\lambda} & I_{\lambda,\alpha} \\ I_{\alpha,a} & I_{\alpha,c} & I_{\alpha,\lambda} & I_{\alpha,\alpha} \end{bmatrix} \tag{16}$$

Inverting the information matrix and replacing the unknown parameters by their mles to obtain the asymptotic variance-covariance matrix of $(\hat{c},\hat{a},\hat{\lambda},\hat{\alpha})$ $100(1-\gamma)\%$ approximate confidence intervals for the parameters a, c , λ andα are respectively, $(\hat{a}-z_{\frac{\gamma}{2}}\sqrt{var(\hat{a})},\hat{a}+z_{\frac{\gamma}{2}}\sqrt{var(\hat{a})})$, $(\hat{c}-z_{\frac{\gamma}{2}}\sqrt{var(\hat{c})},\hat{c}+z_{\frac{\gamma}{2}}\sqrt{var(\hat{c})})$, $(\hat{\lambda}-z_{\frac{\gamma}{2}}\sqrt{var(\hat{\lambda})},\hat{\lambda}+z_{\frac{\gamma}{2}}\sqrt{var(\hat{\lambda})})$,and $(\hat{\alpha}-z_{\frac{\gamma}{2}}\sqrt{var(\hat{\alpha})},\hat{\alpha}+z_{\frac{\gamma}{2}}\sqrt{var(\hat{\alpha})})$

3.2 Bayesian Estimation

Here, we assume that prior distribution is non-informative, i.e. $\pi_0(a,\lambda,c,\alpha)\propto\frac{1}{a\lambda c\alpha}$ where a,λ,c and $\alpha>0$.To compute approximate Bayes estimates, one can use the Gibbs sampling procedure which used to generate samples from posterior distributions. The joint posterior distribution is
$\pi(a,\lambda,c,\alpha|x)\propto\pi_0(a,\lambda,c,\alpha)\exp l(x,a,\lambda,c,\alpha)$,
where $l(x,a,\lambda,c,\alpha)$ is the logarithm of the likelihood function is given in (14).When we set $\rho_1=\log(a),\rho_2=\log(\lambda),\rho_3=\log(c)$ and $\rho_4=\log(\alpha)$, the joint prior distribution will be $\pi(\rho_1,\rho_2,\rho_3,\rho_4)\propto$ constant,$-\infty<\rho_1,\rho_2,\rho_3$,and

$\rho_4 < \infty$ and the joint posterior distribution take the form

$$\pi(\rho_1, \rho_2, \rho_3, \rho_4|x) \propto \pi(\rho_1, \rho_2, \rho_3, \rho_4)\; n\exp(\rho_1) + \sum_{i=1}^{n} \log[exp(\rho_4) + exp(\rho_2 + \rho_3)(x_i^{exp(\rho_3)-1})]$$
$$-(\exp(\rho_1) + 1) \sum_{i=1}^{n} log[exp(\rho_4)x_i + exp(\rho_2)x_i^{exp(\rho_3)})]$$
$$-2 \sum_{i=1}^{n} log[1 + (\exp(\rho_4)x_i + \exp(\rho_2)x_i^{exp(\rho_3)})^{-exp(\rho_1)}] \quad (17)$$

Using the WinBUGS software, posterior summaries of interest can be obtained such as: the mean; the standard deviation; credible interval and others.

3.3 A Numerical Example

Now, we generate a sample of size $n = 100$ from LMW distribution to estimate the four unknown parameters a, λ, c,and α. For our sample, let $a_0 = 1.5, \lambda_0 = 1.5, c_0 = 0.5$,and $\alpha_0 = 0.5$. The MLEs and 95 % confidence intervals for the four parameters a,λ, c and α are listed in The following table:

Table 1: the MLEs and confidence intervals for the four parameters a,λ, c and α of the LMW model

Parameter	Estimate	Standard Deviation	95% Confidence Interval
a	1.59339	2.14579	(0.00000, 4.46450)
λ	1.56919	0.15846	(0.78898, 2.34940)
c	0.465364	0.138631	(0.0000, 1.195134)
α	0.227414	0.696207	(0.0000, 1.862819)

Consider the sample mentioned above and consider the reparameterization $\rho_1 = \log(a), \rho_2 = \log(\lambda), \rho_3 = \log(c) and \rho_4 = \log(\alpha)$ for the LMW distribution with non-informative uniform priors U(-0.001, 0.001), U(-8.0, 6.5), U(-0.001, -0.001)and U(-0.001, -0.001) respectively. Using WinBUGS software, a set of 70000 Gibbs samples was generated after a "burn-in-sample" of size 1000 to eliminate the initial values considered for the Gibbs sampling algorithm. To asses the accuracy of the posterior estimates, the Monte Carlo error (MC error) for each parameter is calculated. As table (2) shows the MC error is less than 5 % of the sample standard deviation, indicating convergence of the algorithm.

Table 2: Summary results for the posterior parameters in the case of the LMW model

Parameter	Estimate	Standard Deviation	MC error	95 % Credible Interval
a	1.000	5.775E-4	2.371E-6	(0.991, 1.001)
λ	658.6	6.49300	0.04353	(641.2, 665.0)
α	1.000	5.777E-4	2.226E-6	(0.991, 1.001)
c	1.000	5.781E-4	1.275E-6	(0.999, 1.001)

3.4 Application of real data

Data Set:"This data set is generated data to simulate the strengths of glass fibers which reported in Smith and Naylor (1987). The data set is: 1.014- 1.082- 1.081- 1.185- 1.223- 1.248- 1.267- 1.272- 1.271- 1.276- 1.275- 1.278- 1.288- 1.286- 1.292- 1.306- 1.304- 1.355- 1.364- 1.361- 1.379- 1.409- 1.426- 1.46- 1.459- 1.476- 1.481- 1.484- 1.501- 1.506- 1.524- 1.526- 1.535- 1.541- 1.568- 1.579- 1.581- 1.591- 1.593- 1.602- 1.67- 1.666- 1.684- 1.691- 1.704- 1.735- 1.731- 1.747- 1.748- 1.757- 1.806- 1.800- 1.867- 1.878- 1.876- 1.91- 1.916- 1.972- 2.012- 2.456- 2.592- 3.197- 4.121".
Here, we put $\alpha = 0$. The mles for the LW distribution are
$\hat{a} = 3.55$,$\hat{\lambda} = 0.5$, and $\hat{c} = 1.55$
Using Kolmogorov-Smirnov goodness of fit test, one can note that the LW distribution fit to this data with Kolmogorov-Smirnov goodness test statistic D=0.1513 and p-value=0.112. For our data set, the MLEs and 95 % confidence intervals for the three parameters a,λ and c are listed in The following table:

Table 1: the MLEs and confidence intervals for the three parameters a,λ and c of the LW model

Parameter	Estimate	Standard Deviation	95% Confidence Interval
a	3.55	0.348	(2.855, 4.245)
λ	0.50	0.129	(0.243, 0.758)
c	1.55	0.0633	(1.424, 1.677)

To obtain the MLE of the survival function of LW distribution, replace the parameters a, λ and c by their MLEs $\hat{a}, \hat{\lambda}$ and $\hat{c}$ in (8). Also, one can obtain Kaplan - Meier estimator of survival function for the data set. The estimates are displayed graphically in figure (3).

Figure (4) displays TTT plot which has concave shape and so it has increasing shaped failure rate which agrees with the plot of the MLE of failure rate.

To compute approximate Bayes estimates in the case of the LW distribution, one can use the Gibbs sampling procedure which used to generate samples from posterior distributions. Here, we assume that prior distribution is non-informative, i.e. $\pi_0(a, \lambda, c) \propto \frac{1}{a\lambda c}$ where $a, \lambda, c > 0$.Then the joint posterior distribution is
$\pi(a, \lambda, c|x) \propto \pi_0(a, \lambda, c) \exp l(x, a, \lambda, c)$
where $l(x, a, \lambda, c)$ is the logarithm of the likelihood function is given in (14).When we set $\rho_1 = \log(a), \rho_2 = \log(\lambda)$ and $\rho_3 = \log(c)$, the joint prior distribution will be $\pi(\rho_1, \rho_2, \rho_3) \propto constant, -\infty < \rho_1, \rho_2,$and $\rho_3 < \infty$ and the joint posterior distribution take the form

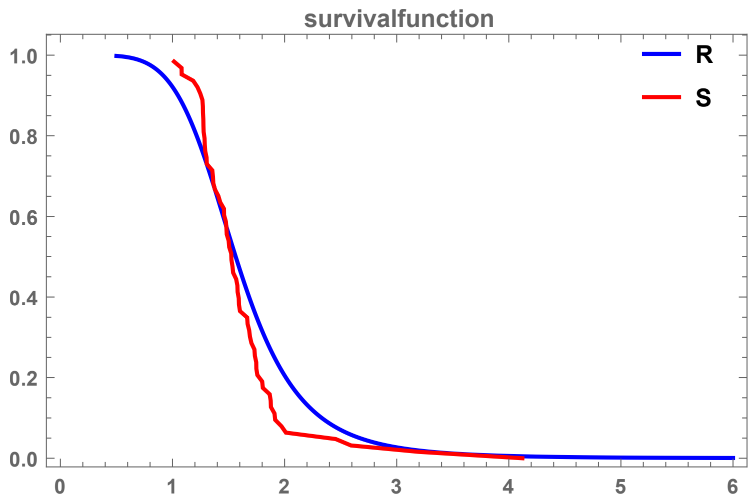

Figure 3: The MLE (R) and Kaplan - Meier (S) estimates of survival function

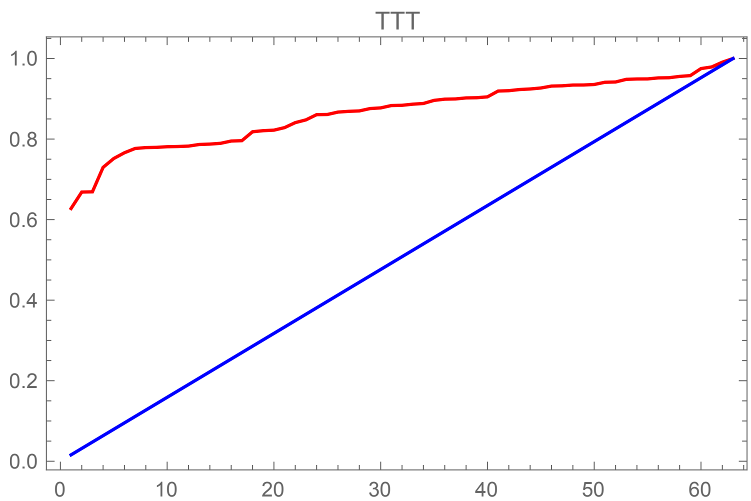

Figure 4: TTT plot for the data set

$$\pi(\rho_1,\rho_2,\rho_3|x) \propto \pi(\rho_1,\rho_2,\rho_3)\ \exp(\rho_1+\rho_3-\exp(\rho_1)\rho_2-(\exp(\rho_1)\exp(\rho_2)+1)* \sum_{i=1}^{n}\log x_i - 2\sum_{i=1}^{n}\log[1+(\exp(\rho_2)(x_i)^{\exp(\rho_3)})^{-\exp(\rho_1)}]) \qquad (18)$$

Posterior summaries such as: the mean; the standard deviation; credible interval and others can be obtained using the WinBUGS software.

Approximate Bayes estimates

Here, we assume that prior distribution is non-informative, i.e. $\pi_0(a,\lambda,c) \propto \frac{1}{a\lambda c}$ where $a,\lambda,c>0$. Consider the sample data mentioned above and consider the reparameterization $\rho_1 = \log(a), \rho_2 = \log(\lambda)$ and$\rho_3 = \log(c)$ for the LW distribution with non-informative uniform priors U(0,1), U(0,0.01) and U(-4,-3) respectively. Using WinBUGS software, a set of 10000 Gibbs samples was generated after a "burn-in-sample" of size 1000 to eliminate the initial values considered for the Gibbs sampling algorithm. Table (2) shows that the MC error is less than 5 % of the sample standard deviation.

Table 2: Summary results for the posterior parameters in the case of the LW model

Parameter	Estimate	Standard Deviation	MC error	95 % Credible Interval
a	1.9070	0.23630	0.002813	(1.4690, 2.3900)
λ	1.0040	0.002873	3.243E-5	(1.0000, 1.0100)
c	0.03654	0.009015	1.246E-4	(0.0195, 0.0493)

From Tables 1-2, one can note that the confidence intervals based on maximum likelihood inference is larger than the credible intervals based on the posterior summaries.

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