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Algorithms for Generating All the Maximal Independent Sets of Some Graphs

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Abstract

An independent set is a set of vertices in a graph, no two of which are adjacent. A maximal independent set is an independent set that is not a proper subset of any other independent set. Note that in general counting the number of maximal independent sets in a graph is NP-complete [5]. In this paper, we give two linear-time algorithms to characterize all the maximal independent sets of the path P_n and the cycle C_n.

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1 Introduction

Let $G = (V, E)$ be a simple undirected graph. An *independent set* is a subset S of V such that no two vertices in S are adjacent. The set of all the maximal independent sets of a graph G is denoted by $I(G)$. A *maximal independent set* is an independent set that is not a proper subset of any other independent set. The set of all the maximal independent sets of a graph G is denoted by $\mathrm{MI}(G)$ and its cardinality by $mi(G)$. Denote P_n a path of order n and C_n a cycle of order n. For notation and terminology in graphs we follow [1] in general.

The problem of determining the largest value of $mi(G)$ in a general graph of order n and those graphs achieving the largest number was proposed by

Erdős and Moser, and solved by Moon and Moser [4]. It was then extensively studied for various classes of graphs in the literature, including trees, forests, (connected) graphs with at most one cycle, bipartite graphs, connected graphs, k-connected graphs, (connected) triangle-free graphs; for a survey see [2]. Note that in general counting the number of maximal independent sets in a graph is NP-complete [5, 6]. Lin and Su [3] proved this problem to remain NP-complete when restricted to directed path graphs but a further restriction to rooted directed path graphs admits a solution in polynomial time. In this paper, we give two linear-time algorithms to characterize all the maximal independent sets of the path P_n and the cycle C_n. The following properties are needed.

Lemma 1.1. *Suppose that S is an independent set in a graph G. Let v_1 and v_2 be two distinct vertices in S. Then $v_i \notin N[v_j]$ for $i \neq j$.*

Proof. Suppose there exists an independent set S in G such that $v_i \in N[v_j]$, where $v_i, v_j \in S$. Then v_i and v_j are adjacent in G, this is a contradiction. We complete the proof. □

Lemma 1.2. *Let S be an independent set of a graph G. If $N[S] = V(G)$, then $S \in MI(G)$.*

Proof. Let S be an independent set of a graph G and $N[S] = V(G)$. If $S \notin \mathrm{MI}(G)$, then there exists a set $S^* \in \mathrm{MI}(G)$ and $S \subset S^*$. Note that $N[S^*] - N[S] \neq \emptyset$. This means that $N[S] \neq V(G)$, it is a contradiction. We complete the proof. □

2 The maximal independent sets of a path P_n

In this section, we provide a constructive characterization of the path P_n, where $n \geq 1$ and $P_n : 1, 2, \ldots, n$. In order to give a constructive characterization of $MI(P_n)$, we introduce two operations.

Operation O1. Assume $S' \in \mathcal{T}_{k-2}$. Add a new number k and let $S = S' \cup \{k\}$.
Operation O2. Assume $S' \in \mathcal{T}_{k-3}$. Add a new number $k-1$ and let $S = S' \cup \{k-1\}$.

Let $\mathcal{T}_1 = \{\{1\}\}, \mathcal{T}_2 = \{\{1\}, \{2\}\}, \mathcal{T}_3 = \{\{1,3\}, \{2\}\}, \mathcal{T}_4, \ldots, \mathcal{T}_{k-2}, \mathcal{T}_{k-1}, \mathcal{T}_k \ldots$ be a sequence of sets, where $\mathcal{T}_k$ be the collection of the sets S which can be obtained from some $S' \in \mathcal{T}_{k-i-1}$ by the Operation Oi, where $i = 1$ and 2.

Lemma 2.1. *For $n \geq 1$, $\mathcal{T}_n \subseteq MI(P_n)$.*

Proof. We prove this lemma by induction on n, where $n \geq 1$. It's true for $n = 1, 2$ and 3. Assume that it's true for all $n' < n$ and let $S \in \mathcal{T}_n$, where $n \geq 4$. Suppose a is the largest number in S, by the operation O1 and operation

O2, then $a = n$ or $n-1$. Let $S' = S - \{a\}$. Then $S' \in \mathcal{T}_{n-2}$ or $S' \in \mathcal{T}_{n-3}$. By the operation O1 and operation O2, we can see that $a \notin N[S']$. We consider two cases.

Case 1. $a = n$. Then S is obtained from S' by the Operation O1 and $S' \in \mathcal{T}_{n-2}$. By the induction hypothesis, $S' \in MI(P_{n-2})$. Hence S is an independent set of P_n and $N[S] = N[S'] \cup N[a] = V(P_{n-2}) \cup N[n] = V(P_n)$. By Lemma 1.2, $S \in MI(P_n)$.

Case 2. $a = n-1$. Then S is obtained from S' by the Operation O2 and $S' \in \mathcal{T}_{n-3}$. By the induction hypothesis, $S' \in MI(P_{n-3})$. Hence S is an independent set of P_n and $N[S] = N[S'] \cup N[a] = V(P_{n-3}) \cup N[n-1] = V(P_n)$. By Lemma 1.2, $S \in MI(P_n)$.

By Case 1 and Case 2, $S \in MI(P_n)$. Thus it's also true for n, and we complete the proof. □

In the following theorem, we will show that $\mathcal{T}_n$ is the characterization of P_n.

Theorem 2.2. *For $n \geq 1$, $MI(P_n) = \mathcal{T}_n$.*

Proof. By Lemma 2.1, we obtain that $\mathcal{T}_n \subseteq MI(P_n)$. Now we want to show that $MI(P_n) \subseteq \mathcal{T}_n$ and prove it by induction on n, where $n \geq 1$. It's true for $n = 1, 2$ and 3. Assume that it's true for all $n' < n$, where $n \geq 4$. Suppose $S \in MI(P_n)$, we have either $n \in S$ or $n-1 \in S$.

Case 1. $n \in S$. Let $S' = S - \{n\}$. Then $S' \in MI(P_{n-2})$, by the induction hypothesis, $S' \in \mathcal{T}_{n-2}$. We can see that S can be obtained from S' by Operation O1. This means that $S \in \mathcal{T}_n$.

Case 2. $n-1 \in S$. Let $S' = S - \{n-1\}$. Then $S' \in MI(P_{n-3})$, by the induction hypothesis, $S' \in \mathcal{T}_{n-3}$. We can see that S can be obtained from S' by Operation O2. This means that $S \in \mathcal{T}_n$.

By Case 1 and Case 2, we obtain that $S \in \mathcal{T}_n$, thus it's also true for n. Hence $MI(P_n) \subseteq \mathcal{T}_n$. We complete the proof. □

By Theorem 2.2, we can see that MI$(P_n) = \mathcal{T}_n$. The following algorithm is designed to determine the set $\mathcal{T}_n$. Hence we provide a constructive characterization of $MI(P_n)$.

For $n \geq 4$ and $1 \leq i \leq 2$, let $\mathcal{T}_n^{(i)}$ be the collection of all maximal independent sets S of P_n which are obtained from some $S' \in \mathcal{T}_{n-i-1}$ by the Operation Oi. In the following lemma, we calculate the cardinality $mi(P_n)$.

Lemma 2.3. *For $n \geq 4$, we have the following results.*
(i) $\mathcal{T}_n^{(1)} \cap \mathcal{T}_n^{(2)} = \emptyset$.
(ii) $mi(P_n) = mi(P_{n-2}) + mi(P_{n-3})$, *where* $mi(P_1) = 1$, $mi(P_2) = 2$ *and* $mi(P_3) = 2$.

Algorithm 1 The collection $\mathcal{T}_n$ of all the maximal independent sets of P_n

Input: A positive integer $n \geq 4$.
Output: $\mathcal{T}_n$.

```
initialize T_1 = {{1}}, T_2 = {{1},{2}} and T_3 = {{1,3},{2}}
for i = 4; i < n+1; i++ do
    if S ∈ T_{i-2} then S = S ∪ {i};
    if S ∈ T_{i-3} then S = S ∪ {i-1};
    T_i = ∪S;
end for
Print T_n
```

Proof. (i) Suppose that $\mathcal{T}_n^{(1)} \cap \mathcal{T}_n^{(2)} \neq \emptyset$, and let $S \in \mathcal{T}_n^{(i)}$ for $i = 1$ and 2. Since $S \in \mathcal{T}_n^{(1)}$, we obtain that $n \in S$. Similarly, $S \in \mathcal{T}_n^{(2)}$, then we obtain that $n-1 \in S$. Note that $\mathcal{T}_n = \mathrm{MI}(P_n)$, then $S \in \mathrm{MI}(P_n)$ and $|S \cap \{n-1, n\}| = 1$. This is a contradiction, hence $\mathcal{T}_n^{(1)} \cap \mathcal{T}_n^{(2)} = \emptyset$. (ii)By Theorem 2.2 and (i), we have that $mi(P_n) = |\mathcal{T}_n| = |\mathcal{T}_n^{(1)} \cup \mathcal{T}_n^{(2)}| = |\mathcal{T}_n^{(1)}| + |\mathcal{T}_n^{(2)}| = |\mathcal{T}_{n-2}| + |\mathcal{T}_{n-3}| = mi(P_{n-2}) + mi(P_{n-2})$. □

3 The maximal independent sets of a cycle C_n

In this section, we provide a constructive characterization of the path C_n, where $n \geq 3$ and $C_n : 1, 2, \ldots, n, 1$. Since we want to calculate the recursive formula for $mi(C_n)$ and provide a constructive characterization of the path C_n, we don't continue the results in section 2. Hence we give a new constructive characterization of $MI(C_n)$. The following operations are needed.

Operation Q1. Assume $S' \in \Theta_{k-2}$ and m is the largest value in S'. Add a new number $m+2$ and let $S = S' \cup \{m+2\}$.
Operation Q2. Assume $S' \in \Theta_{k-3}$ and m is the largest value in S'. Add a new number $m+3$ and let $S = S' \cup \{m+3\}$.

Let $\Theta_3 = \{\{1\},\{2\},\{3\}\}, \Theta_4 = \{\{1,3\},\{2,4\}\}, \Theta_5 = \{\{1,3\},\{1,4\},\{2,4\}, \{2,5\},\{3,5\}\}, \Theta_6, \ldots, \Theta_{k-3}, \Theta_{k-2}, \Theta_k, \ldots$ be a sequence of sets, where Θ_k be the collection of the sets S which can be obtained from some $S' \in \Theta_{k-i-1}$ by the Operation Qi, where $i = 1$ and 2.

Lemma 3.1. *For $n \geq 3$, $\Theta_n \subseteq MI(C_n)$.*

Proof. We prove this lemma by induction on n, where $n \geq 3$. It's true for $n = 3, 4$ and 5. Assume that it's true for all $n' < n$ and let $S \in \Theta_n$, where $n \geq 6$. Suppose that S is obtained from S' by the Operation Q1 or Q2. Then $S' \in \Theta_{n-i}$, where $i = 2$ or 3. By the induction hypothesis, $S' \in MI(C_{n-i})$.

Let a and m be the largest and the second largest numbers in S, respectively. So $S - S' = \{a\}$ and $a = m + i$.

Claim 1. $S \in I(C_n)$.

Note that $S' \in MI(C_{n-i})$, where $i = 2$ or 3. Suppose that $S \notin I(C_n)$, then $1 \in S'$ and $a = n$. So $m = n - i$ and $n - i \in S'$. Thus $\{1, n - i\} \subset S'$, this contradicts that $S' \in I(C_{n-i})$. Hence $S \in I(C_n)$.

Claim 2. $N[S] = V(C_n)$.

Suppose there exists a vertex u in C_n such that $u \notin N[S]$. Note that $S' \in MI(C_{n-i})$, where $i = 2$ or 3. So $m \geq n - i - 2$ and $a = m + i \geq n - 2$, hence $u = 1$ or $u = n$. If $u = 1$, then $2 \notin S'$ and $n \notin S$. So $a \leq n - 1$ and $m = a - i \leq n - i - 1$, thus $n - i \notin S'$. These mean that $1 \notin N[S']$, this contradicts that $S' \in MI(C_{n-i})$. Hence $u = n$. Then $1 \notin S'$ and $n - 1 \notin S$. So $a \leq n - 2$ and $m = a - i \leq n - i - 2$, thus $n - i - 1 \notin S'$. These mean that $n - i \notin N[S']$, this contradicts that $S' \in MI(C_{n-i})$. Hence we obtain that $N[S] = V(C_n)$.

By Claim 1, Claim 2 and Lemma 1.2, $S \in MI(C_n)$. Thus it's also true for n. We complete the proof. □

Theorem 3.2. *For $n \geq 3$, MI(C_n) $= \Theta_n$.*

Proof. By Lemma 3.1, we obtain that $\Theta_n \subseteq MI(C_n)$. Now we want to prove MI(C_n) $\subseteq \Theta_n$ and prove it by induction on n, where $n \geq 3$. Suppose there exists a maximal independent set $S \in MI(C_n)$ and $S \notin \Theta_n$ such that n is as small as possible. Then $n \geq 6$. Let a and m be the largest and the second largest numbers in S, respectively. Since $S \in MI(C_n)$, we have that $n - 2 \leq a \leq n$ and $a = m + k$, where $k = 2$ or 3. Let $S' = S - \{a\}$. Note that $a \notin S'$. We consider three cases.

Case 1. $a = n$. Then $1 \notin S$ and $m = n - k$, where $k = 2$ or 3. Thus $S' \in MI(C_m)$, by the hypothesis, $S' \in \Theta_m$. Hence S can be obtained from S' by Operation Q1 or Q2, it means that $S \in \Theta_n$. This is a contradiction.

Case 2. $a = n - 1$. Then $1 \in N[S']$ and $m = n - 3$ or $n - 4$. If $m = n - 3$, then $S' \in MI(C_{n-2})$. By the hypothesis, $S' \in \Theta_{n-2}$. Hence S can be obtained from S' by Operation Q1, it means that $S \in \Theta_n$. This is a contradiction, so $m = n - 4$. Then $S' \in MI(C_{n-3})$. By the hypothesis, $S' \in \Theta_{n-3}$. Hence S can be obtained from S' by Operation Q2, it means that $S \in \Theta_n$. This is a contradiction.

Case 3. $a = n - 2$. Then $n \notin S$ and $1 \in S$. So $m = n - 4$ or $n - 5$. Note that $a \notin S'$. If $m = n - 4$, then $n - 2 \notin S'$ and $S' \in MI(C_{n-2})$. By the hypothesis, $S' \in \Theta_{n-2}$. Hence S can be obtained from S' by Operation Q1, it means that $S \in \Theta_n$. This is a contradiction, so $m = n - 5$. Then $n - 3 \notin S'$ and $S' \in MI(C_{n-3})$. By the hypothesis, $S' \in \Theta_{n-3}$. Hence S can be obtained from S' by Operation Q2, it means that $S \in \Theta_n$. This is a contradiction.

By Case 1, Case 2 and Case 3, $\mathrm{MI}(C_n) \subseteq \Theta_n$. Hence $\mathrm{MI}(C_n) = \Theta_n$. We complete the proof. $\square$

By Theorem 3.2, we can see that $\mathrm{MI}(C_n) = \Theta_n$. The following algorithm is designed to determine the set Θ_n. Hence we provide a constructive characterization of $MI(C_n)$.

Algorithm 2 The collection Θ_n of all the maximal independent sets of C_n

Input: A positive integer $n \geq 6$.
Output: Θ_n.

```
1: initialize Θ_3 = {{1},{2},{3}}, Θ_4 = {{1,3},{2,4}}, Θ_5 =
   {{1,3},{1,4},{2,4},{2,5},{3,5}}
2: for i = 6; i < n + 1; i + +  do
3:     if S ∈ Θ_{i−2} and m is the maximal number in S, then S = S ∪ {m + 2};
4:     if S ∈ Θ_{i−3} and m is the maximal number in S, then S = S ∪ {m + 3};
5:     Θ_i = ∪S;
6: end for
7: Print Θ_n
```

For $n \geq 6$ and $1 \leq i \leq 2$, let $\Theta_n^{(i)}$ be the collection of all the maximal independent sets S of C_n which are obtained from some $S' \in \Theta_{k-i-1}$ by the Operation Qi. In the following lemma, we calculate the cardinality $mi(C_n)$.

Lemma 3.3. *For $n \geq 6$, we have the following results.*
(i) $\Theta_n^{(1)} \cap \Theta_n^{(2)} = \emptyset$.
(ii) $mi(C_n) = mi(C_{n-2}) + mi(C_{n-3})$, *where* $mi(C_3) = 3$, $mi(C_4) = 2$ *and* $mi(C_5) = 5$..

Proof. (i) Suppose that $\Theta_m^{(1)} \cap \Theta_m^{(2)} \neq \emptyset$ for some $m \geq 6$. Assume $S \in \Theta_m^{(i)}$ for $i = 1, 2$. By Theorem 3.2, $S \in MI(C_n)$. Let a be the largest number in S. By the Operation Q1 and Operation Q2, then $\{a-2, a-3\} \subset S$. This contradicts that $S \in \mathrm{MI}(C_n)$, hence $\Theta_n^{(1)} \cap \Theta_n^{(2)} = \emptyset$..

(ii)By Theorem 3.2 and (i), we have that $mi(C_n) = |\Theta_n| = |\Theta_n^{(1)} \cup \Theta_n^{(2)}| = |\Theta_n^{(1)}| + |\Theta_n^{(2)}| = |\Theta_{n-2}| + |\Theta_{n-3}| = mi(C_{n-2}) + mi(C_{n-3})$. $\square$

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