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An Improved Jacobian Elliptic Function Expansion Method to Find New Traveling Wave Solutions for a Class of Nonlinear Wave Equations

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Abstract

In this paper, abundant new doubly-periodic solutions for a class of nonlinear wave equations are obtained by using the generalized Jacobi elliptic function method and the Wu-elimination method.The solitary wave solutions and triangular periodic solutions can be obtained at their limit conditions.

Keywords: Jacobi elliptic function method, doubly-periodic solutions, nonlinear wave equation

1 Introduction

Exact solutions may describe not only the propagation of nonlinear waves but also spatial localized structures of permanent shape that may be of interest to experiments. Since the inverse scattering method[1]has been increasing interesting in

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searching for new solition equations and the related issue of the construction of exact solutions to a wide class of nonlinear solition equations in solition theory. Up to now, many powerful methods have been developed such as [1], Darboux transformation method [2-4], algebro-geometric method [3], tanh function method [4],Backlund transformation method[5]. One of most effectively straight forward method to constructing exact solutions of PDEs is Jacobin elliptic function method [6]. When the modulus $k \to 1$ or 0, the Jacobi elliptic functions solutions degenerate to soliton solutions or trigonometric function solutions. Therefore, seeking the Jacobi elliptic function solutions of nonlinear wave equations is more significant.

As we know, when applying the direct methods, it is important to obtain more new solutions of auxiliary equation and choose an appropriate ansätz. Along this way, in this paper, by introducing more new solutions in terms of rational formal Jacobi elliptic function of an elliptic equation and a appropriate anätz, we further improve the method, such that it can be used to obtain more new exact doubly-periodic solutions to nonlinear partial equations.

2 Summary of the improved method

The main idea of our method is to take full of advantages of the elliptic equation that Jacobi elliptic functions satisfy and use its new solutions to obtain more new doubly periodic solutions of a class of nonlinear wave equations. The desired elliptic equation [7] reads:

$$\phi'^2 = a + b\phi^2 + c\phi^4, \tag{1}$$

where $\phi' = \frac{d\phi}{d\xi}$ and a, b, c are constants. Fortunately, we find the following nineteen solutions in terms of rational formal Jacobi elliptic function solutions to Eq.(1).

case 1. If $\begin{cases} a = 1 \\ b = 2 - 4\,k^2, \\ c = 1 \end{cases}$ then (1) has solution $\phi = \frac{sn(\xi,k)dn(\xi,k)}{cn(\xi,k)}$.

case 2. If $\begin{cases} a = 1 \\ b = 2, \\ c = k^4 \end{cases}$ then (1) has solution $\phi = \frac{sn(\xi,k)cn(\xi,k)}{dn(\xi,k)}$

case 3. If $\begin{cases} a = -2\,k^2 + k^4 + 1 \\ b = k^2 + 2, \\ c = 1, \end{cases}$ then (1) has solution $\phi = \frac{dn(\xi,k)cn(\xi,k)}{sn(\xi,k)}$.

case 4. If $\begin{cases} a = \frac{k^2 - 2\,k + 1}{4A^2} \\ b = \frac{1+k^2}{2} + 3\,k, \\ c = \frac{A^2(k-1)^2}{4} \end{cases}$ then (1) has solution $\phi = \frac{dn(\xi,k)cn(\xi,k)}{A(1+sn(\xi,k))(1+ksn(\xi,k))}$.

case 5. If $\begin{cases} a = \frac{(k+1)^2}{4A^2} \\ b = \frac{1+k^2}{2} - 3\,k, \\ c = \frac{A^2(1+k)^2}{4} \end{cases}$ then (1) has solution $\phi = \frac{dn(\xi,k)cn(\xi,k)}{A(1+sn(\xi,k))(1-ksn(\xi,k))}$.

case 6. If $\begin{cases} a=-2\,k^3+k^4+k^2 \\ b=6\,k-k^2-1, \\ c=-\frac{4}{k} \end{cases}$ then (1) has solution $\phi=\frac{k\,cn(\xi,k)\,dn(\xi,k)}{k\,sn^2(\xi,k)+1}$.

case 7. If $\begin{cases} a=2\,k^3+k^4+k^2 \\ b=-6\,k-k^2-1, \\ c=\frac{4}{k} \end{cases}$ then (1) has solution $\phi=\frac{k\,cn(\xi,k)\,dn(\xi,k)}{k\,sn^2(\xi,k)-1}$.

case 8. If $\begin{cases} a=1 \\ b=2+6\,k_1-k^2, \\ c=8+8\,k_1-4\,k^2+4 \end{cases}$ then (1) has solution $\phi=\frac{sn(\xi,k)\,cn(\xi,k)}{cn^2(\xi,k)-k_1\,sn^2(\xi,k)}$.

case 9. If $\begin{cases} a=1 \\ b=2-6\,k_1-k^2, \\ c=8-8\,k_1-4\,k^2+4 \end{cases}$ then (1) has solution $\phi=\frac{sn(\xi,k)\,cn(\xi,k)}{cn^2(\xi,k)+k_1\,sn^2(\xi,k)}$.

case 10. If $\begin{cases} a=2+2k_1-k^2, \\ b=2+6k_1-k^2, \\ c=4\,k_1 \end{cases}$ then (1) has solution $\phi=\frac{k^2\,sn(\xi,k)\,cn(\xi,k)}{k_1-dn^2(\xi,k)}$.

case 11. If $\begin{cases} a=2-2k_1-k^2, \\ b=2-6k_1-k^2, \\ c=-4k_1 \end{cases}$ then (1) has solution $\phi=-\frac{k^2\,sn(\xi,k)\,cn(\xi,k)}{dn^2(\xi,k)+k_1}$.

case 12. If $\begin{cases} a=\frac{2-k^2-2k_1}{4}, \\ b=\frac{k^2}{2}-1-3k_1, \\ c=\frac{2-k^2-2\,k_1}{4}, \end{cases}$ then (1) has solution $\phi=\frac{k^2\,cn(\xi,k)sn(\xi,k)}{sn^2(\xi,k)+(1+k_1)\,dn(\xi,k)-1-k_1}$.

case 13. If $\begin{cases} a=\frac{2-k^2+2k_1}{4} \\ b=\frac{k^2}{2}-1+3k_1, \\ c=\frac{2-k^2+2k_1}{4} \end{cases}$ then (1) has solution $\phi=\frac{k^2\,cn(\xi,k)sn(\xi,k)}{sn^2(\xi,k)+(k_1-1)\,dn(\xi,k)-1+k_1}$.

case 14. If $\begin{cases} a=\frac{k^2-1}{4(C^2k^2-B^2)} \\ b=\frac{k^2+1}{2}, \\ c=\frac{C^2k^4-B^2k^2-C^2k^2+B^2}{4} \end{cases}$ then (1) has solution $\phi=\frac{\left(\sqrt{\frac{B^2-C^2}{B^2-C^2k^2}}+sn(\xi,k)\right)}{B\,cn(\xi,k)+C\,dn(\xi,k)}$.

case 15. If $\begin{cases} a=\frac{1}{4\,(B^2+C^2k^2)} \\ b=\frac{1}{2}-k^2, \\ c=\frac{B^2+C^2k^2}{4} \end{cases}$ then (1) has solution $\phi=\frac{\sqrt{\frac{C^2k^2+B^2-C^2}{B^2+C^2k^2}}+cn(\xi,k)}{B\,sn(\xi,k)+C\,dn(\xi,k)}$.

case 16. If $\begin{cases} a=\frac{k^4}{4(B^2+C^2)} \\ b=\frac{k^2}{2}-1, \\ c=\frac{B^2+C^2}{4} \end{cases}$ then (1) has solution $\phi=\frac{\sqrt{\frac{B^2+C^2-k^2C^2}{B^2+C^2}}+dn(\xi,k)}{B\,sn(\xi,k)+C\,cn(\xi,k)}$.

case 17. If $\begin{cases} a=\frac{2\,k-k^2-1}{B^2} \\ b=2\,k^2+2, \\ c=-B^2k^2-B^2-2\,B^2k \end{cases}$ then (1) has solution $\phi=\frac{k\,sn^2(\xi,k)-1}{B(k\,sn^2(\xi,k)+1)}$.

case 18. If $\begin{cases} a=-\frac{2\,k+k^2+1}{B^2} \\ b=2\,k^2+2, \\ c=-B^2k^2-B^2+2\,B^2k \end{cases}$ then (1) has solution $\phi=\frac{k\,sn^2(\xi,k)+1}{B(k\,sn^2(\xi,k)-1)}$.

case 19. If $\begin{cases} a = \frac{2-k^2+2k_1}{B^2} \\ b = -4 + 2\,k^2, \\ c = B^2\left(2 - k^2 - 2k_1\right) \end{cases}$ then (1) has solution $\phi = \frac{k_1 + dn^2(\xi,k)}{B\left(dn^2(\xi,k) - k_1\right)}$.

case 20. If $\begin{cases} a = B^2\left(2 - k^2 - 2k_1\right), \\ b = -4 + 2\,k^2, \\ c = \frac{2-k^2+2\,k_1}{B^2}, \end{cases}$ then (1) has solution $\phi = \frac{B\left(dn^2(\xi,k) - k_1\right)}{dn^2(\xi,k) + k_1}$.

where $k_1 = \sqrt{1-k^2}$, and A, B, C are arbitrary constants. $sn\,(\xi,k)\,,\,cn\,(\xi,k)\,,\,dn\,(\xi,k)$ are Jacobi elliptic functions, and k $(0 < k < 1)$ denote the modulus of the Jacobi elliptic functions. These Jacobi elliptic functions are double periodic and possess the following properties:

1. Properties of triangular function

$sn^2\xi + cn^2\xi = dn^2\xi + k^2 sn^2\xi = 1$

2.Derivatives of the Jacobi elliptic functions

$sn'\xi = cn\xi\, dn\xi, \qquad cn'\xi = -sn\xi\, dn\xi, \qquad dn'\xi = -k^2 sn\xi\, cn\xi,$

When $k \to 0$, the Jacobi elliptic functions degenerate to the triangle functions

$sn\xi \to sin\xi, \qquad cn\xi \to cos\xi, \qquad dn\xi \to 1.$

When $k \to 1$, the Jacobi elliptic functions degenerate to the hyperbolic functions

$sn\xi \to tanh\xi, \qquad cn\xi \to sech\xi, \qquad dn\xi \to sech\xi.$

In the following we would like to outline the main steps of our general method.

step 1: For a given nonlinear evolution equations in two independent variables x, t,

$$F(u, u_t, u_x, u_{xt}, u_{xx}, u_{tt}...) = 0, \tag{2}$$

by using the travelling wave transformation

$$u(x,t) = U(\xi), \qquad \xi = l(x + \lambda t), \tag{3}$$

where l, λ are constants to be determined later. Then the nonlinear partial differential Eq.(2) reduces to a nonlinear ordinary differential equation(ODE):

$$G(U, U', U'', U'''...) = 0, \tag{4}$$

We assume the solutions of Eq.(4) can be expressed in the form

$$U(\xi) = a_0 + \sum_{i=1}^{n}(a_i\phi^i + \frac{b_i}{\phi^i} + \frac{c_i\phi'^i}{\phi^i}) \tag{5}$$

where ϕ satisfies Eq.(1), n can be determined by balancing the highest order partial derivative term and nonlinear term in Eq.(4) or Eq.(2).

step 2 : With the aid of *Maple*, substituting (5) into (4) along with Eq.(1), and collecting the coefficients of the same power $\phi^i(\sqrt{a + b\phi^2 + c\phi^4})^j$ $(j = 0, 1, i = 0, 1, 2...)$ and setting each of the obtained coefficients to be zero, we get a set of over-determined algebraic equations.

step 3: with the aid of *Maple*, solve the above over-determined algebraic equations for $\lambda, a_0, a_i, b_i, c_i$.

step 4: Substituting the obtained conclusions in step 3 into Eq.(5), and using the solutions of Eq.(1) gives the explicit and exact solutions of Eq.(2)

In the following we illustrate the method by considering a class of nonlinear wave equations.

3 Exact solutions of a class of nonlinear wave equations

Consider the nonlinear wave equation in Ref [8,9]:

$$u_{tt} + \alpha u_{xx} + \beta u + \gamma u^3 = 0 \tag{6}$$

where α, β, γ are constants. Some exact solutions were given in [8,9]. Eq.(6) contains the following famous NPDEs, namely
(i) ϕ^4 equation

$$u_{tt} - u_{xx} + u - u^3 = 0, \tag{7a}$$

(ii) Klein-Gordon equation

$$u_{tt} - u_{xx} + m^2 u + n u^3 = 0, \tag{7b}$$

(iii)Duffing equation

$$u_{tt} + bu + cu^3 = 0, \tag{7c}$$

(iv) Landau-Ginburg-Higgs equation

$$u_{tt} - u_{xx} - m^2 u + n^2 u^3 = 0, \tag{7d}$$

(v) Sine-Gordon equation

$$u_{tt} - u_{xx} + u - \frac{1}{6} u^3 = 0. \tag{7e}$$

By the travelling transformation

$$u = U(\xi), \qquad \xi = m\left(x + \lambda t\right). \tag{8}$$

Then Eq.(6) becomes

$$\left(\lambda^2 + \alpha\right) m^2 U'' + \beta U + \gamma U^3 = 0. \tag{9}$$

By homogeneous balancing process, we can suppose that Eq.(9) has the following formal solutions,

$$U = a_0 + a_1 \phi + \frac{b_1}{\phi} + \frac{c_1 \phi'}{\phi}. \tag{10}$$

where $\phi(\xi)$ satisfies Eq.(1), and $a_0, a_1, b_1, c_1, m, \lambda$ are constants to be determined later. Substituting (10) into (9) along with Eq.(1), collecting coefficients of $\phi^i(\sqrt{a + b\phi^2 + c\phi^4})^j$

$(j=0,1,i=0,1,2..)$, and setting them to be zero, we get a set of over-determined algebraic equations with respect to $a_0, a_1, b_1, c_1, m, \lambda$. With the aid of maple solving the above over-determined algebraic equations, we find the solutions as follow:

Case 1.

$$m=m, c_1=a_0=0, \lambda=\frac{1}{m}\sqrt{\frac{36\,m^2\alpha\,ca-b\beta-6\,\beta\,\sqrt{ac}-b^2m^2\alpha}{b^2-36\,ac}}, a_1=\frac{\sqrt{2\beta c}}{\sqrt{\gamma\,(b-6\,\sqrt{ac})}},$$

$$b_1=\sqrt{\frac{2\beta\,a\,(b+6\,\sqrt{ac})}{\gamma\,(b^2-36\,ac)}},$$

Case 2.

$$m=m, a_0=0, b_1=\frac{\beta\,\sqrt{a}}{\sqrt{\gamma\beta\,(-b+6\,\sqrt{ac})}}, c_1=\sqrt{\frac{\beta\,(b+6\,\sqrt{ac})}{\gamma\,(36\,ac-b^2)}}, a_1=\sqrt{\frac{\beta\,cb+6\,\beta\,c\sqrt{ac}}{\gamma\,(36\,ac-b^2)}},$$

$$\lambda=\frac{1}{m}\sqrt{\frac{b^2m^2\alpha-36\,m^2\alpha\,ca-2\,b\beta-12\,\beta\,\sqrt{ac}}{36\,ac-b^2}},$$

Substituting the case 1 and case 2 into (10) and using the solutions of Eq.(1), we obtain the following double periodic solutions of Eq.(6). For simplicity, we set $Q=\sqrt{\frac{\beta\,(b+6\,\sqrt{ac})}{36\,ac-b^2}}$.

$$u_1=\frac{Q\sqrt{c}sn\,(\xi)\,dn\,(\xi)}{\sqrt{\gamma}cn\,(\xi)}-\frac{\beta\,\left(b\sqrt{ac}+6\,ac\right)cn\,(\xi)}{Q\sqrt{\gamma c}\,(36\,ac-b^2)\,sn\,(\xi)\,dn\,(\xi)}+\frac{Q\,\left(k^2sn^4\,(\xi)-2\,sn^2\,(\xi)\,k^2+1\right)}{\sqrt{\gamma}cn\,(\xi)\,sn\,(\xi)\,dn\,(\xi)},$$

$$u_2=\frac{Q\sqrt{c}sn\,(\xi)\,cn\,(\xi)}{\sqrt{\gamma}dn\,(\xi)}-\frac{\beta\,\left(b\sqrt{ac}+6\,ac\right)dn\,(\xi)}{Q\sqrt{c\gamma}\,(36\,ac-b^2)\,sn\,(\xi)\,cn\,(\xi)}+\frac{Q\,\left(k^2sn^4\,(\xi)-2\,sn^2\,(\xi)+1\right)}{\sqrt{\gamma}cn\,(\xi)\,sn\,(\xi)\,dn\,(\xi)},$$

$$u_3=\frac{Q\sqrt{c}cn\,(\xi)\,dn\,(\xi)}{\sqrt{\gamma}sn\,(\xi)}-\frac{\beta\,\left(b\sqrt{ac}+6\,ac\right)sn\,(\xi)}{Q\sqrt{c\gamma}\,(36\,ac-b^2)\,cn\,(\xi)\,dn\,(\xi)}+\frac{Q\,\left(k^2sn^4\,(\xi)-1\right)}{\sqrt{\gamma}cn\,(\xi)\,sn\,(\xi)\,dn\,(\xi)},$$

$$u_4=\frac{Q\sqrt{c}cn\,(\xi)\,dn\,(\xi)}{\sqrt{\gamma}A\,(1+sn\,(\xi))\,(1+ksn\,(\xi))}+\frac{Q\,(k+1)\left(ksn^2\,(\xi)+1\right)}{\sqrt{\gamma}cn\,(\xi)\,dn\,(\xi)}$$
$$-\frac{A\beta\,\left(b\sqrt{ac}+6\,ac\right)(1+sn\,(\xi))\,(1+ksn\,(\xi))}{Q\sqrt{c\gamma}\,(36\,ac-b^2)\,cn\,(\xi)\,dn\,(\xi)},$$

$$u_5=\frac{Q\sqrt{c}cn\,(\xi)\,dn\,(\xi)}{\sqrt{\gamma}A\,(1+sn\,(\xi))\,(1-ksn\,(\xi))}+\frac{Q\,(k-1)\left(ksn^2\,(\xi)+1\right)}{\sqrt{\gamma}cn\,(\xi)\,dn\,(\xi)}$$
$$-\frac{A\beta\,\left(b\sqrt{ac}+6\,ac\right)(1+sn\,(\xi))\,(1-ksn\,(\xi))}{Q\sqrt{c\gamma}\,(-b^2+36\,ac)\,cn\,(\xi)\,dn\,(\xi)},$$

$$u_6=\frac{Q\sqrt{c}kcn\,(\xi)\,dn\,(\xi)}{\sqrt{\gamma}\left(k\,(sn\,(\xi))^2+1\right)}-\frac{\beta\,\left(b\sqrt{ac}+6\,ac\right)\left(k\,(sn\,(\xi))^2+1\right)}{Qk\sqrt{c\gamma}\,(-b^2+36\,ac)\,cn\,(\xi)\,dn\,(\xi)}$$
$$+\frac{Qsn\,(\xi)\,(k+1)^2\left(k\,(sn\,(\xi))^2-1\right)}{\sqrt{\gamma}\left(k\,(sn\,(\xi))^2+1\right)cn\,(\xi)\,dn\,(\xi)},$$

$$u_7=\frac{Q\sqrt{c}kcn\,(\xi)\,dn\,(\xi)}{\sqrt{\gamma}\left(k\,(sn\,(\xi))^2-1\right)}-\frac{\beta\,\left(b\sqrt{ac}+6\,ac\right)\left(k\,(sn\,(\xi))^2-1\right)}{kQ\sqrt{c\gamma}\,(36\,ac-b^2)\,cn\,(\xi)\,dn\,(\xi)}+\frac{Qsn\,(\xi)\,(k-1)^2\left(k\,(sn\,(\xi))^2+1\right)}{\sqrt{\gamma}\left(k\,(sn\,(\xi))^2-1\right)cn\,(\xi)\,dn\,(\xi)},$$

$$u_8 = \frac{Q\sqrt{c}sn\,(\xi)\,cn\,(\xi)}{\sqrt{\gamma}\,(cn^2\,(\xi)-k_1 sn^2\,(\xi))} - \frac{\beta\,\left(b\sqrt{ac}+6\,ac\right)\left(cn^2\,(\xi)-k_1 sn^2\,(\xi)\right)}{Q\sqrt{c\gamma}\sqrt{c}\,(36\,ac-b^2)\,sn\,(\xi)\,cn\,(\xi)}$$
$$-\frac{Q dn\,(\xi)\left(cn^2\,(\xi)+k_1 sn^2\,(\xi)\right)}{\sqrt{\gamma}cn\,(\xi)\,sn\,(\xi)\,(k_1 sn^2\,(\xi)-cn^2\,(\xi))},$$

$$u_9 = \frac{Q\sqrt{c}sn\,(\xi)\,cn\,(\xi)}{\sqrt{\gamma}\,(cn^2\,(\xi)+k_1 sn^2\,(\xi))} - \frac{\beta\,\left(b\sqrt{ac}+6\,ac\right)\left(cn^2\,(\xi)+k_1 sn^2\,(\xi)\right)}{Q\sqrt{c\gamma}\,(36\,ac-B^2)\,sn\,(\xi)\,cn\,(\xi)}$$
$$-\frac{Q dn\,(\xi)\left((1+k_1)\,sn^2\,(\xi)-1\right)}{\sqrt{\gamma}\,(cn^2\,(\xi)+k_1 sn^2\,(\xi))\,sn\,(\xi)\,cn\,(\xi)},$$

$$u_{10} = \frac{Q\sqrt{c}k^2 sn\,(\xi)\,cn\,(\xi)}{\sqrt{\gamma}\left(k_1-(dn\,(\xi))^2\right)} - \frac{\beta\,\left(b\sqrt{ac}+6\,ac\right)\left(k_1-dn^2\,(\xi)\right)}{k^2\sqrt{c\gamma}Q\sqrt{c}\,(36\,ac-b^2)\,sn\,(\xi)\,cn\,(\xi)}$$
$$+\frac{Q dn\,(\xi)\left(1-k_1+\left(k^2-2+2k_1\right)sn^2\,(\xi)\right)}{\sqrt{\gamma}cn\,(\xi)\,sn\,(\xi)\left(dn^2\,(\xi)-k_1\right)},$$

$$u_{11} = -\frac{Q\sqrt{c}k^2 sn\,(\xi)\,cn\,(\xi)}{\sqrt{\gamma}\left(dn^2\,(\xi)+k_1\right)} + \frac{\beta\,\left(b\sqrt{ac}+6\,ac\right)\left(dn^2\,(\xi)+k_1\right)}{Qk^2\sqrt{c\gamma}\,(36\,ac-b^2)\,sn\,(\xi)\,cn\,(\xi)}$$
$$+\frac{Q dn\,(\xi)\left(\left(2-k^2+2k_1\right)sn^2\,(\xi)-1-k_1\right)}{\sqrt{\gamma}cn\,(\xi)\,sn\,(\xi)\left(dn^2\,(\xi)+k_1\right)},$$

$$u_{12} = \frac{Q\sqrt{c}k^2 sn\,(\xi)\,cn\,(\xi)}{\sqrt{\gamma}\,(sn^2\,(\xi)+(1+k_1)\,dn\,(\xi)-1-k_1)} - \frac{\beta\,\left(b\sqrt{ac}+6\,ac\right)\left[sn^2\,(\xi)+(k_1+1)\,(dn\,(\xi)-1)\right]}{Qk^2\sqrt{c\gamma}\,(36\,ac-b^2)\,sn\,(\xi)\,cn\,(\xi)}$$
$$+\frac{Q\left\{k^2\,(1+k_1)\,(sn\,(\xi))^4+[(2\,k_1+1)\,dn\,(\xi)-2\,k_1-2]\,(sn\,(\xi))^2+(1+k_1)\,(1-dn\,(\xi))\right\}}{cn\,(\xi)\,sn\,(\xi)\,[(sn\,(\xi))^2+(1+k_1)\,(dn\,(\xi)-1)]},$$

$$u_{13} = \frac{Q\sqrt{c}k^2 sn\,(\xi)\,cn\,(\xi)}{\sqrt{\gamma}\,(sn^2\,(\xi)+(k_1-1)\,dn\,(\xi)-1+k_1)} - \frac{\beta\,\left(b\sqrt{ac}+6\,ac\right)\left[sn^2\,(\xi)+(k_1-1)\,(dn\,(\xi)+1)\right]}{Qk^2\sqrt{c\gamma}\,(36\,ac-b^2)\,sn\,(\xi)\,cn\,(\xi)}$$
$$-\frac{Q[(sn\,(\xi))^4\,k^2+(2\,k_1-2+(-1+2\,k_1)\,dn\,(\xi))\,(sn\,(\xi))^2-(k_1-1)\,(1+dn\,(\xi))]}{\gamma\,cn\,(\xi)\,sn\,(\xi)\,[(sn\,(\xi))^2+(k_1-1)\,(1+dn\,(\xi))]}$$

$$u_{14} = \frac{Q\sqrt{c}\left(\sqrt{B^2-C^2}+\sqrt{B^2-k^2C^2}sn\,(\xi)\right)}{\sqrt{\gamma\,(B^2-k^2C^2)}[B\,cn\,(\xi)+C\,dn\,(\xi)]} - \frac{\beta\,\left(b\sqrt{ac}+6\,ac\right)\sqrt{\gamma\,c\,(B^2-k^2C^2)}[B\,cn\,(\xi)+C\,dn\,(\xi)]}{Q\,(-b^2+36\,ac)\,[\sqrt{B^2-C^2}+\sqrt{B^2-k^2C^2}sn\,(\xi)]}$$
$$+\frac{Q\left\{\sqrt{B^2-k^2C^2}[B\,dn\,(\xi)+C\,cn\,(\xi)]+\sqrt{B^2-C^2}sn\,(\xi)\,[B\,dn\,(\xi)+Ck^2\,cn\,(\xi)]\right\}}{\sqrt{\gamma}[B\,cn\,(\xi)+C\,dn\,(\xi)][\sqrt{B^2-k^2C^2}sn\,(\xi)+\sqrt{B^2-C^2}]}$$

$$u_{15} = \frac{Q\sqrt{c}\left(\sqrt{B^2-C^2+C^2k^2}+\sqrt{B^2+k^2C^2}cn\,(\xi)\right)}{\sqrt{\gamma\,(B^2+k^2C^2)}[B\,sn\,(\xi)+C\,dn\,(\xi)]} - \frac{\beta\,\left(b\sqrt{ac}+6\,ac\right)\sqrt{\gamma\,c\,(B^2+k^2C^2)}[B\,sn\,(\xi)+C\,dn\,(\xi)]}{Q\,(-b^2+36\,ac)\,[\sqrt{B^2-C^2+C^2k^2}+\sqrt{B^2+k^2C^2}cn\,(\xi)]}$$
$$-\frac{Q\left\{\sqrt{B^2+k^2C^2}[C\,sn\,(\xi)+B\,dn\,(\xi)-Ck^2\,sn\,(\xi)]+\sqrt{B^2-C^2+k^2C^2}cn\,(\xi)\,[B\,dn\,(\xi)-Ck^2\,sn\,(\xi)]\right\}}{\sqrt{\gamma}[\sqrt{B^2+k^2C^2}cn\,(\xi)+\sqrt{B^2-C^2+k^2C^2}][B\,sn\,(\xi)+C\,dn\,(\xi)]}$$

$$u_{16} = \frac{Q\sqrt{c}\left(\sqrt{B^2+C^2-C^2k^2}+\sqrt{B^2+C^2}dn\,(\xi)\right)}{\sqrt{\gamma\,(B^2+C^2)}[B\,sn\,(\xi)+C\,cn\,(\xi)]} - \frac{\beta\,\left(b\sqrt{ac}+6\,ac\right)\sqrt{\gamma\,c\,(B^2+C^2)}[B\,sn\,(\xi)+C\,cn\,(\xi)]}{Q\,(-b^2+36\,ac)\,[\sqrt{B^2+C^2-C^2k^2}+\sqrt{B^2+C^2}dn\,(\xi)]}$$
$$-\frac{Q\left\{\sqrt{B^2+C^2}[C\,sn\,(\xi)\left(-1+k^2\right)+B\,cn\,(\xi)]+\sqrt{B^2+C^2-k^2C^2}dn\,(\xi)\,[B\,cn\,(\xi)-C\,sn\,(\xi)]\right\}}{\sqrt{\gamma}[B\,sn\,(\xi)+C\,cn\,(\xi)][\sqrt{B^2+C^2-k^2C^2}+dn\,(\xi)]},$$

$$u_{17} = \frac{Q\sqrt{c}\left(ksn^2\,(\xi)-1\right)}{\sqrt{\gamma}B\,(ksn^2\,(\xi)+1)} - \frac{\beta\,\left(b\sqrt{ac}+6\,ac\right)B\left(ksn^2\,(\xi)+1\right)}{Q\sqrt{c\gamma}\,(36\,ac-b^2)\,(ksn^2\,(\xi)-1)} + \frac{4Qksn\,(\xi)\,cn\,(\xi)\,dn\,(\xi)}{\sqrt{\gamma}\,(k^2sn^4\,(\xi)-1)},$$

$$u_{18}=\frac{Q\sqrt{c}\left(ksn^{2}\left(\xi\right)+1\right)}{\sqrt{\gamma}B\left(ksn^{2}\left(\xi\right)-1\right)}-\frac{\beta\left(b\sqrt{ac}+6\,ac\right)B\left(ksn^{2}\left(\xi\right)-1\right)}{Q\sqrt{c\gamma}\left(36\,ac-b^{2}\right)\left(ksn^{2}\left(\xi\right)+1\right)}-\frac{4Qksn\left(\xi\right)cn\left(\xi\right)dn\left(\xi\right)}{\sqrt{\gamma}\left(k^{2}sn^{4}\left(\xi\right)-1\right)},$$

$$u_{19}=\frac{Q\sqrt{c}\left(dn^{2}\left(\xi\right)+k_{1}\right)}{\sqrt{\gamma}B\left(dn^{2}\left(\xi\right)-k_{1}\right)}-\frac{\beta\left(b\sqrt{ac}+6\,ac\right)B\left(dn^{2}\left(\xi\right)-k_{1}\right)}{Q\sqrt{c\gamma}\left(36\,ac-b^{2}\right)\left(dn^{2}\left(\xi\right)+k_{1}\right)}+\frac{4Qk^{2}k_{1}\,dn\left(\xi\right)cn\left(\xi\right)sn\left(\xi\right)}{\sqrt{\gamma}\left(dn^{4}\left(\xi\right)-1+k^{2}\right)},$$

$$u_{20}=\frac{Q\sqrt{c}B\left(dn^{2}\left(\xi\right)-k_{1}\right)}{\sqrt{\gamma}\left(dn^{2}\left(\xi\right)+k_{1}\right)}-\frac{\beta\left(b\sqrt{ac}+6\,ac\right)\left(dn^{2}\left(\xi\right)+k_{1}\right)}{QB\sqrt{c\gamma}\left(36\,ac-b^{2}\right)\left(dn^{2}\left(\xi\right)-k_{1}\right)}-\frac{4Qk_{1}k^{2}\,dn\left(\xi\right)cn\left(\xi\right)sn\left(\xi\right)}{\sqrt{\gamma}\left(dn^{4}\left(\xi\right)-1+k^{2}\right)},$$

where $\xi=mx+\sqrt{\frac{b^{2}m^{2}\alpha-36\,m^{2}\alpha\,ca-2\,b\beta-12\,\beta\,\sqrt{ac}}{36\,ac-b^{2}}}t$, and a,b,c are arbitrary constants in $u_1\sim u_{20}$. All the above solutions are quite new and different from the ones reported in [8,9].

Remark 1. These solutions are obtained by case 2, so they are only some solutions of Eq.(6). Other solutions are omitted from verbosity.

Remark 2. The solutions of Eqs. $(7a)\sim(7e)$ can also be obtained if we set α and β to be the special values respectively.

4 Conclusions

In this paper, we have applied an improved generalized Jacobi elliptic function method to find abundant new solutions for a class of nonlinear wave equations which include ϕ^4 equations. When $k\to 1$, some of these obtained solutions degenerate as solitary wave solutions, when $k\to 0$, some of these obtained solutions degenerate as triangular periodic solutions of Eq.(6). These solutions are new to our knowledge. The algorithm can be also applied to many other nonlinear differential equations in mathematical physics.

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