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Fully Uncertain Linear Systems

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Abstract

In this paper, a fully uncertain linear system is defined. The conditions will be presented for the existence of solutions to a fully uncertain linear system. An application to separating metals from mixed ores is investigated by the proposed method.

Mathematics Subject Classification: 15A06

Keywords: Linear system; Uncertainty distribution; Linear uncertain variable; Solution

1 Introduction

Linear systems play an important role in various areas such as economics, finance, mathematics, physics, engineering and social sciences. Unlike differential equation, linear equation is easy to be modeled. The most important thing is that it can be employed to approximate a nonlinear equation. So it is crucial to find a way solving linear equations.

As we know, there are uncertainties involved in engineering design process. Thus, it is immensely important to find a tool to study linear systems with uncertain parameters.

The first tool coming into people's eyes is the fuzzy set theory. Fuzzy linear systems emerged in the 1990s which was first proposed by Friedman et al. [1]. From then on, such systems have been studied broadly in academia, such as Ma et al [2], Wang et al [3], Allahviranloo et al [4], Muzzioli et al [5], Reynaerts et al [5], Dehghan et al [6] and so on. Great developments [7] have been achieved in the field of linear fuzzy systems [8]. However, fuzzy set theory has not been evolved as a mathematical system because of its inconsistence. Sometimes, counterintuitive results may happen when fuzzy concept was treated as belief. In order to study linear systems with indeterminant parameters more thorough, Li and Zhu [9] proposed the uncertain linear systems based on uncertainty theory [10] in 2014. A general model [9] for solving uncertain linear systems whose coefficient matrix is crisp and its right column is an uncertain vector was first introduced in uncertainty theory [11]. In this paper, we will consider a new class of uncertain linear systems that all parameters in the system are uncertain variables, which can be called fully uncertain linear system (FULS), and present the solution of such system.

In Section 2, some fundamental concepts in uncertainty theory will be reviewed. Next, in Section 3, we will introduce a model of fully uncertain linear system and define the solution in distribution for the model. We will explore the solutions of fully uncertain linear systems with some special uncertain variables in Section 4. And Section 5 contains an application.

2 Preliminary Notes

In this section, we will introduce some fundamental concepts in uncertainty theory [10]. Let Γ be a nonempty set, and $\mathcal{L}$ a σ-algebra over Γ. Each element $\Lambda \in \mathcal{L}$ is called an event. For any event A, $\mathcal{M}\{A\}$ is a number in $[0,1]$. The set function $\mathcal{M}$ is called an uncertain measure if it satisfies normality, duality, and subadditivity axioms. And the triplet$(\Gamma, \mathcal{L}, \mathcal{M})$ is called an uncertainty space. An uncertain variable is a measurable function from an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to the set of real numbers. And the uncertain variable ξ is said to be positive if $\mathcal{M}\{\xi \leq 0\} = 0$. The uncertainty distribution $\Phi : \Re \to [0,1]$ of an uncertain variable ξ is defined by $\Phi(x) = \mathcal{M}\{\xi(\gamma) \leq x\}$ for any real number x.

Example 2.1. *An uncertain variable ξ is called linear if it has a linear uncertainty distribution $\Phi(x) = 0$ if $x \leq a$, $(x-a)/(b-a)$ if $a \leq x \leq b$, and 1 if $x \geq b$. Denote an uncertain linear variable by $\xi \sim \mathcal{L}(a,b)$ where a and b are real numbers with $a < b$.*

The uncertain variables $\xi_1, \xi_2, \cdots, \xi_m$ are said to be independent if $\mathcal{M}\{\bigcap_{i=1}^m \{\xi_i \in B_i\}\} = \min_{1 \leq i \leq m} \mathcal{M}\{\xi_i \in B_i\}$ for any Borel sets $B_1, B_2, \cdots, B_n$. An uncertainty distribution Φ is said to be regular if its inverse function $\Phi^{-1}(\alpha)$ exists and is unique for each $\alpha \in (0,1)$. Then the inverse function $\Phi^{-1}(\alpha)$ is called the inverse uncertainty distribution of ξ. A function $\Phi^{-1}(\alpha) : (0,1) \to \mathcal{R}$ is an inverse uncertainty distribution if and only if it is a continuous and strictly increasing function with respect to α. The expected value of ξ is defined by

$E[\xi] = \int_0^{+\infty} \mathcal{M}\{\xi \geq x\}dx - \int_{-\infty}^{0} \mathcal{M}\{\xi \leq x\}dx$ provided that at least one of the two integrals is finite. We have $E[\xi] = \int_0^1 \Phi^{-1}(\alpha)d\alpha$.

Theorem 2.2. *[10] Let $\xi_1, \xi_2, \cdots, \xi_n$ be independent uncertain variables with regular uncertainty distributions $\Phi_1, \Phi_2, \cdots, \Phi_n$, respectively. If the function $f(x_1, x_2, \cdots, x_n)$ is strictly increasing with respect to $x_1, x_2, \cdots, x_m$ and strictly decreasing with respect to $x_{m+1}, x_{m+2}, \cdots, x_n$, then $\xi = f(\xi_1, \xi_2, \cdots, \xi_n)$ is an uncertain variable with inverse uncertainty distribution*

$$\Psi^{-1}(\alpha) = f(\Phi_1^{-1}(\alpha), \cdots, \Phi_m^{-1}(\alpha), \Phi_{m+1}^{-1}(1-\alpha), \cdots, \Phi_n^{-1}(1-\alpha)).$$

Lemma 2.3. *[12] The inverse of a nonnegative matrix A is nonnegative if and only if A is a generalized permutation matrix.*

Lemma 2.4. *[9] Let $S = (s_{ij})_{n\times n}$ with $s_{ii} > 1, 1 \leq i \leq n$. Denote $S = I + W$, where I is a unit matrix. If the vector $q = (q_1, q_2, \cdots, q_n)^\tau$ satisfies the following conditions: (a) $q > 0, i.e, q_i > 0, i = 1, 2, \cdots, n$, (b) $(W - I)q < 0$, (c) $W^m q \to 0$ as $m \to \infty$, then $S^{-1}q$ is larger than zero.*

3 A Model of Fully Uncertain Linear System

We now introduce a kind of fully uncertain linear system. The linear system

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = \xi_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = \xi_2 \\ \quad\vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = \xi_n \end{cases} \tag{1}$$

is called a fully uncertain linear system(FULS), where the elements a_{ij} are uncertain variables for $i, j = 1, 2, \cdots, n$ and $\xi_1, \xi_2, \cdots, \xi_n$ are uncertain variables.

Let $\xi_1, \xi_2, \cdots, \xi_n$ be uncertain variables with regular uncertainty distributions $\Psi_1, \Psi_2 \cdots \Psi_n$, respectively. Assume that $a_{i1}, a_{i2}, \cdots, a_{in}, x_i$ are independent positive uncertain variables with regular uncertainty distributions $\Phi_{i1}, \Phi_{i2}, \cdots, \Phi_{in}, \Theta_i$, respectively, $i = 1, 2, \cdots, n$. Then we will define the concept of a solution to FULS.

Definition 3.1. *An uncertain vector $(x_1, x_2, \cdots, x_n)^\tau$ is called a solution of the fully uncertain linear system (1) in distribution if for any $\alpha \in (0, 1)$, we have*

$$\begin{cases} \Phi_{11}^{-1}(\alpha)\Theta_1^{-1}(\alpha) + \Phi_{12}^{-1}(\alpha)\Theta_2^{-1}(\alpha) + \cdots + \Phi_{1n}^{-1}(\alpha)\Theta_n^{-1}(\alpha) = \Psi_1^{-1}(\alpha) \\ \Phi_{21}^{-1}(\alpha)\Theta_1^{-1}(\alpha) + \Phi_{22}^{-1}(\alpha)\Theta_2^{-1}(\alpha) + \cdots + \Phi_{2n}^{-1}(\alpha)\Theta_n^{-1}(\alpha) = \Psi_2^{-1}(\alpha) \\ \quad\vdots \\ \Phi_{n1}^{-1}(\alpha)\Theta_1^{-1}(\alpha) + \Phi_{n2}^{-1}(\alpha)\Theta_2^{-1}(\alpha) + \cdots + \Phi_{nn}^{-1}(\alpha)\Theta_n^{-1}(\alpha) = \Psi_n^{-1}(\alpha) \end{cases} \tag{2}$$

4 A Special Fully Uncertain Linear System

In general, uncertainty distributions $\Theta_1, \Theta_2, \cdots, \Theta_n$ of the solutions $x_1, x_2, \cdots, x_n$ are related to the uncertainty distributions of $\xi_1, \xi_2, \cdots, \xi_n$. Next, we will discuss a special fully uncertain linear systems, i.e., $\xi_1, \xi_2, \cdots, \xi_n$ are special uncertain variables.

Let $\xi_1, \xi_2, \cdots, \xi_n$ be uncertain variables with inverse uncertain distributions $\Psi_1^{-1}, \Psi_2^{-1}, \cdots, \Psi_n^{-1}$, respectively, where Ψ_i^{-1} are quadratic polynomial of the α, namely, $\Psi_i^{-1}(\alpha) = w_i + m_i\alpha + t_i\alpha^2, t_i \neq 0$, for $i = 1, 2, \cdots, n$. Let $a_{ij} \sim \mathcal{L}(p_{ij}, q_{ij}), 1 \leq i \leq n, 1 \leq j \leq n$, where $q_{ij} \geq p_{ij}$. Especially, when $q_{ij} = p_{ij}$, a_{ij} is a constant and its inverse distribution $\Phi_{ij}^{-1}(\alpha) = 0$ for $\alpha \in (0, 1)$.

Theorem 4.1. *Assume that $x_i \sim \mathcal{L}(r_i, s_i), 1 \leq i \leq n$. Then (2) is equal to*

$$A_p h_r = b_w,\ A_p(h_s - h_r) + (A_q - A_p)h_r = b_m,\ (A_q - A_p)(h_s - h_r) = b_t \quad (3)$$

where

$$\begin{aligned} A_p =&(p_{ij})_{n\times n}, A_q = (q_{ij})_{n\times n}, h_r = (r_1, r_2, \cdots, r_n)^\tau, \\ h_s &= (s_1, s_2, \cdots, s_n)^\tau, b_w = (w_1, w_2, \cdots, w_n)^\tau, \\ b_m &= (m_1, m_2, \cdots, m_n)^\tau, b_t = (t_1, t_2, \cdots, t_n)^\tau. \end{aligned} \quad (4)$$

Proof. The equation (2) can be expressed in the following form of matrix:

$$\begin{bmatrix} \Phi_{11}^{-1}(\alpha) & \Phi_{12}^{-1}(\alpha) & \cdots & \Phi_{1n}^{-1}(\alpha) \\ \Phi_{21}^{-1}(\alpha) & \Phi_{22}^{-1}(\alpha) & \cdots & \Phi_{2n}^{-1}(\alpha) \\ \vdots & & & \\ \Phi_{n1}^{-1}(\alpha) & \Phi_{n2}^{-1}(\alpha) & \cdots & \Phi_{nn}^{-1}(\alpha) \end{bmatrix} \begin{bmatrix} \Theta_1^{-1}(\alpha) \\ \Theta_2^{-1}(\alpha) \\ \vdots \\ \Theta_n^{-1}(\alpha) \end{bmatrix} = \begin{bmatrix} \Psi_1^{-1}(\alpha) \\ \Psi_2^{-1}(\alpha) \\ \vdots \\ \Psi_n^{-1}(\alpha). \end{bmatrix} \quad (5)$$

According to the conditions we know

$$\begin{aligned} \Phi_{ij}^{-1}(\alpha) &= (1-\alpha)p_{ij} + \alpha q_{ij} = p_{ij} + \alpha(q_{ij} - p_{ij}) \\ \Theta_j^{-1}(\alpha) &= (1-\alpha)r_j + \alpha s_j = r_j + \alpha(s_j - r_j) \\ \Psi_i^{-1}(\alpha) &= w_i + m_i\alpha + t_i\alpha^2,\ 1 \leq i \leq n,\ 1 \leq j \leq n. \end{aligned}$$

So Eq. (5) can be represented by

$$[A_p + \alpha(A_q - A_p)][h_r + \alpha(h_s - h_r)] = b_w + b_m\alpha + b_t\alpha^2, \quad (6)$$

where the parameters are showed by (4). Since the Eq. (6) holds for any $\alpha \in (0, 1)$, the coefficients of α and α^2 at the left side equal to the coefficients of α and α^2 at the right side in the Eq. (6), repectively. Then we get

$$A_p h_r = b_w,\ A_p(h_s - h_r) + (A_q - A_p)h_r = b_m,\ (A_q - A_p)(h_s - h_r) = b_t.$$

This completes the proof.

Theorem 4.2. *Assume that $x_i \sim \mathcal{L}(r_i, s_i), 1 \le i \le n$. Use the notations of (4). If A_p is nonsingular matrix and $b_t = (A_qA_p^{-1} - E)(b_m - (A_qA_p^{-1} - E)b_w)$, then we have*

$$h_r = A_p^{-1}b_w,\ h_s = A_p^{-1}b_m - A_p^{-1}(A_qA_p^{-1} - 2E)b_w. \tag{7}$$

Moreover, if $h_r > 0$ and $h_s - h_r > 0$, the solution in distribution of fully uncertain linear system (1) is unique and resonable.

Proof. Since A_p is a nonsingular matrix, it follows from theorem 4.1 that

$$[r_1, r_2, \cdots, r_n]^\tau = h_r = A_p^{-1}b_w = A_p^{-1}[w_1, w_2, \cdots, w_n]^\tau.$$

And we can also get

$$\begin{aligned}[s_1, s_2, \cdots, s_n]^\tau &= h_s = A_p^{-1}b_m - A_p^{-1}(A_qA_p^{-1} - 2E)b_w \\ &= A_p^{-1}[m_1, m_2, \cdots, m_n]^\tau - A_p^{-1}(A_qA_p^{-1} - 2E)[w_1, w_2, \cdots, w_n]^\tau.\end{aligned}$$

Furthermore, A_p is a nonsingular matrix, so we obtain that r_i and $s_i, 1 \le i \le n$ are unique, namely, $x_i \sim \mathcal{L}(r_i, s_i), 1 \le i \le n$, are unique. In addition if $h_r > 0$ and $h_s - h_r > 0$, that is, $r_i > 0$ and $s_i - r_i > 0, 1 \le i \le n$, so $x_i \sim \mathcal{L}(r_i, s_i)$, $1 \le i \le n$, are reasonable positive linear uncertain variables. The theorem is proved.

Theorem 4.3. *Use the notations of (4). Assume that A_p is a nonsingular matrix and $b_t = (A_qA_p^{-1} - E)(b_m - (A_qA_p^{-1} - E)b_w)$.*

(i) If A is a generalized permutation matrix, $b_w > 0$ and $b_m - (A_qA_p^{-1} - E)b_w > 0$, then the fully uncertain linear system (1) has a unique solution in distribution.

(ii) If $p_{ii} > 1, 1 \le i \le n$, b_w and $b_m - (A_qA_p^{-1} - E)b_w$ satisfies the conditions (a),(b),(c) in Lemma 2, then the fully uncertain linear system (1) has a unique solution in distribution.

Proof. (i) Assume that $x_i \sim \mathcal{L}(r_i, s_i),\ 1 \le i \le n$. Since A_p is a generalized permutation matrix, it follows from Lemma 2.3 that $A_p^{-1} \ge 0$. And at least one element of each column in A_p^{-1} is greater than zero. Since $b_w > 0$, we can obtain $A_p^{-1}b_w > 0$. That is $h_r > 0$. In the same way, we are able to get $A_p^{-1}(b_m - (A_qA_p^{-1} - E)b_w) > 0$ That is $h_s - h_r > 0$. So the fully uncertain linear system (1) with linear uncertain variables has a unique solution in distribution.

(ii) Assume that $x_i \sim \mathcal{L}(r_i, s_i),\ 1 \le i \le n$. If $p_{ii} > 1,\ 1 \le i \le n$, then A_p can be written as $A_p = E + W$. By Theorem 4.2, since b_w satisfies the conditions (a) (b) and (c) in the Lemma 2.4, we have $A_p^{-1}b_w > 0$. That is $h_r > 0$. In the same way, we also get $A_p^{-1}(b_m - (A_qA_p^{-1} - E)b_w) > 0$. That is $h_s - h_r > 0$. So the fully uncertain linear system (1) with linear uncertain variables has a unique solution in distribution.

5 Application

In a mixed smelter, we have to separate metals from mixed ores to obtain the metal products what we need. However, mental contents within different ores are dissimilar. For ensuring to get every metal product enough, we should put different ores with different proportion in the production. Assume that there is a mixed smelter producing copper, zinc, plumbum about ξ_1 (kg), ξ_2 (kg), ξ_3 (kg), respectively. Now, there are three sorts of mixed ores (Ore 1, Ore 2, Ore 3) supplied by three pits. Then the relations of the metals (copper, zinc, plumbum) and mixed ores (Ore 1, Ore 2, Ore 3) are showed in Table 1 where a_{ij}, $1 \le i, j \le 3$, are metal contents of ores, and unit is kilogram per ton.

Table 1: Metal content relations of ores

	Ore 1	Ore 2	Ore 3	demands of metals
copper	a_{11}	a_{12}	a_{13}	ξ_1
zinc	a_{21}	a_{22}	a_{23}	ξ_2
plumbum	a_{31}	a_{32}	a_{33}	ξ_3

Since the metals distribute unevenly in an ore, the metal contents of an ore is not very accurate and may be described as "about 10 kg per ton". So these quantities a_{ij} may be regarded as uncertain variables, for $i, j = 1, 2, 3$. Because of the mass production and the production equipment which are not very precise, it is hard to know the exact amount of the goods, even if the amount of the production can be controlled in an interval. Then the demands ξ_i of the metals in the smelter can be regarded as uncertain variables, too. Based on a production plan, we need produce copper, zinc and plumbum about $20kg, 30kg, 36kg$, respectively. The copper contents of Ore 1 (a_{11}), Ore 2 (a_{12}), Ore 3 (a_{13}) are around $2kg/t, 2.5kg/t, 6kg/t$, respectively. The zinc contents of Ore 1 (a_{21}), Ore 2 (a_{22}), Ore 3 (a_{23}) are approximately $2.5kg/t, 9kg/t, 1.5kg/t$, respectively. The plumbum contents of Ore 1 (a_{31}), Ore 2 (a_{32}), Ore 3 (a_{33}) are about $7.5kg/t, 7kg/t, 2kg/t$, respectively. Now, let ξ_1, ξ_2, ξ_3 be uncertain demands for the copper, zinc and plumbum, respectively, where the inverse uncertainty distribution of ξ_1 is $\Phi_1^{-1}(\alpha) = 7\alpha^2 + 15\alpha + 10$, the inverse uncertainty distribution of ξ_2 is $\Phi_2^{-1}(\alpha) = 5\alpha^2 + 19\alpha + 19$, the inverse uncertainty distribution of ξ_3 is $\Phi_3^{-1}(\alpha) = 6\alpha^2 + 28\alpha + 20$. Let a_{ij} be uncertain variables where $a_{11} \sim \mathcal{L}(1,3)$, $a_{12} \sim \mathcal{L}(2,3)$, $a_{13} \sim \mathcal{L}(5,7)$, $a_{21} \sim \mathcal{L}(2,3)$, $a_{22} \sim \mathcal{L}(8,10)$, $a_{23} \sim \mathcal{L}(1,2)$, $a_{31} \sim \mathcal{L}(7,8)$, $a_{32} \sim \mathcal{L}(6,8)$, $a_{33} \sim \mathcal{L}(1,3)$. Let x_1, x_2, x_3 be the amounts of Ore 1, Ore 2, Ore 3, respectively. Based on the Table 1, we establish the following fully uncertain linear system:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + a_{13}x_n = \xi_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_n = \xi_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_n = \xi_3. \end{cases} \tag{8}$$

Assume that $x_1 \sim \mathcal{L}(r_1, s_1)$, $x_2 \sim \mathcal{L}(r_2, s_2)$, $x_3 \sim \mathcal{L}(r_3, s_3)$. Denote

$$A_p = \begin{bmatrix} 1 & 2 & 5 \\ 2 & 8 & 1 \\ 7 & 6 & 1 \end{bmatrix}, A_q = \begin{bmatrix} 3 & 3 & 7 \\ 3 & 10 & 2 \\ 8 & 8 & 3 \end{bmatrix}, b_w = \begin{bmatrix} 10 \\ 19 \\ 20 \end{bmatrix}, b_m = \begin{bmatrix} 15 \\ 19 \\ 28 \end{bmatrix}, b_t = \begin{bmatrix} 7 \\ 5 \\ 6 \end{bmatrix}.$$

According to Theorem 4.1, we have

$$\begin{bmatrix} 1 & 2 & 5 \\ 2 & 8 & 1 \\ 7 & 6 & 1 \end{bmatrix} \begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix} = \begin{bmatrix} 10 \\ 19 \\ 20 \end{bmatrix},$$

$$\begin{bmatrix} 1 & 2 & 5 \\ 2 & 8 & 1 \\ 7 & 6 & 1 \end{bmatrix} \left(\begin{bmatrix} s_1 \\ s_2 \\ s_3 \end{bmatrix} - \begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix} \right) + \left(\begin{bmatrix} 3 & 3 & 7 \\ 3 & 10 & 2 \\ 8 & 8 & 3 \end{bmatrix} - \begin{bmatrix} 1 & 2 & 5 \\ 2 & 8 & 1 \\ 7 & 6 & 1 \end{bmatrix} \right) \begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix} = \begin{bmatrix} 15 \\ 19 \\ 28 \end{bmatrix},$$

$$\left(\begin{bmatrix} 3 & 3 & 7 \\ 3 & 10 & 2 \\ 8 & 8 & 3 \end{bmatrix} - \begin{bmatrix} 1 & 2 & 5 \\ 2 & 8 & 1 \\ 7 & 6 & 1 \end{bmatrix} \right) \left(\begin{bmatrix} s_1 \\ s_2 \\ s_3 \end{bmatrix} - \begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix} \right) = \begin{bmatrix} 7 \\ 5 \\ 6 \end{bmatrix}.$$

After calculation, we know that $b_t = (A_q A_p^{-1} - E)(b_m - (A_q A_p^{-1} - E)b_w)$. Then by Theorem 4.3, we know the fully uncertain linear system has a unique solution in distribution. Therefore, by employing Theorem 4.2, we obtain the solution that $x_1 \sim \mathcal{L}(1, 3)$, $x_2 \sim \mathcal{L}(2, 3)$, $x_3 \sim \mathcal{L}(1, 2)$, and according to $E[\xi] = \int_0^1 \Phi^{-1}(\alpha)d\alpha$, we can get $E[\xi_1] = 19.83$, $E[\xi_2] = 30.17$, $E[\xi_3] = 36$. From the solution, we know that the mixed smelter needs approximately 2 ton of Ore 1, 2.5 ton of Ore 2, and 1.5 ton of Ore 3 so that the smelter can produce about 19.83kg copper, 30.17kg zinc, and 36kg plumbum.

6 Conclusions

In this paper, a model of fully uncertain linear system $Ax = b$ was proposed and investigated, where A is an uncertain matrix and b is an uncertain vector. It is showed that a fully uncertain linear system with linear uncertain variables can be solved by settling corresponding crisp equations. Under some given conditions, the existence and uniqueness of the solution in distribution to the uncertain system were proved. Finally, a mixed smelter problem was modeled by a fully uncertain linear system. The solution gave a result of ore quantity for required metals.

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