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$\beta-$Baire Spaces and $\beta-$Baire Property

Tugba Han Simsekler

Department of Mathematics
Kafkas University
Kars, Turkey

Naime Tozlu

Department of Mathematics
Nigde University
Nigde, Turkey

Saziye Yuksel

Department of Mathematics
Selcuk University
Konya, Turkey

Abstract

The aim of this study is to investigate a property which can be used measure and category named β-Baire property and a space called β-Baire space. For this purpose β-dense, nowhere β-dense and β-first category sets are defined and some results about these new definitions are obtained. Also we give under which conditions β-Baire spaces preserved.

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1 Introduction

Frolik [3] showed that X is a Baire space if and only if Y is a Baire space where f is a bijective feebly open and feebly continuous function from X to Y. In this paper we define new concepts called β-dense sets, nowhere β-dense sets. We investigated some properties of these new concepts. Also we define of β-first category sets which is used to give β-Baire property. We obtain a characterization of β-Baire property. Finally by using β-dense sets we define β-Baire spaces and investigate under which mappings this space preserves.

2 Preliminaries

Throughout the present paper, X and Y denote the topological spaces. Let A be a subset of X. The closure (resp. the interior) of A is denoted by A^-(resp. A°). A subset A is defined to be β-open [1] (or semipreopen [2]) if $A \subset A^{-\circ-}$. The complement of a β-open set is called β-closed. The intersection of all β-closed sets containing A is called the β-closure [1] of A and is denoted by A^-_β. The β-interior of A is defined by the union of all β-open sets contained in A and is denoted by A°_β. The family of all β-open sets of X is denoted by $\beta O(X)$. Andrijevic [2] show that $A^-_\beta = A \cup A^{\circ-\circ}$ and $A^\circ_\beta = A \cap A^{-\circ-}$.

A subset A of X is called dense if $A^- = X$. A subset A of X is called nowhere dense if $A^{-\circ} = \emptyset$.

3 Baire Property and β-Baire Spaces

Definition 3.1 *A subset A of X is defined to be β-dense if $A^-_\beta = X$.*

Remark 3.2 *Every β-dense set is dense but the converse is not true in general.*

Example 3.3 *Let (R, U) be the usual topological space and Q be the rational numbers set. Then Q is a dense set but it is not a β-dense set.*

Lemma 3.4 *If a subset $A \subset X$ is both open and dense then it is β-dense.*

Proof. Let A be an open and dense set. Then, $A^-_\beta = A \cup A^{-\circ} = X$. This shows that A is β-dense.

The converse of Lemma does not hold in general.

Example 3.5 *Let $X = \{a, b, c\}$, $\tau = \{\emptyset, X, \{a\}\}$. Then the set $\{a, b\}$ is a β- dense and so a dense set but it is not an open set.*

Definition 3.6 *A subset $A \subset X$ is defined to be nowhere β-dense set if there exists a β-open and dense set contained in complement of A.*

Lemma 3.7 *A subset $A \subset X$ is nowhere β-dense if and only if (A^-_β) has no interior points.*

Proof. Let A be a nowhere β-dense set. Then there exists a β-open and dense set B such that $B \subset X - A$. It is clear that $B^\circ_\beta \subset (X - A)^\circ_\beta$. Since B is a β-open set $B^\circ_\beta = B \subset (X - A)^\circ_\beta = X - A^-_\beta$. Then $B^- \subset (X - A^-_\beta)^- = X - (A^-_\beta)^\circ$. Since B is a dense set $B^- = X = X - (A^-_\beta)^\circ$. Hence $(A^-_\beta)^\circ = \emptyset$.

Remark 3.8 *Every nowhere dense set is a nowhere β- dense set. The converse is not true in general as shown in example.*

Example 3.9 *Let R be the set of real numbers endowed with usual topology and Q be the set of rational numbers. Then Q is nowhere β-dense but it is not nowhere dense, moreover it is a dense set.*

Proposition 3.10 *Let A, B be subsets of X. The following statements hold:*

1. *If $A \subset B$ and B is nowhere β-dense then A is nowhere β-dense.*
2. *If A is nowhere β-dense then $A - B$ is nowhere β-dense.*
3. *If A or B is nowhere β-dense then $A \cap B$ is nowhere β-dense.*

Proof. 1. Since $A \subset B$, $(A^-_\beta)^\circ \subset (B^-_\beta)^\circ = \emptyset$. Hence A is nowhere β-dense.
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Definition 3.11 *Let A be a subset of X. A is defined to be of β-first category if it can be represented as a countable union of nowhere β-dense sets.*

Remark 3.12 *Every of first category set is of β-first category. But the converse is not true in general as seen in example.*

Example 3.13 *Let R be the set of real numbers endowed with usual topology and Q be the set of rational numbers. Then R is of β-first category since it can be represented as $R = Q \cup (R - Q)$ where Q and $(R - Q)$ are nowhere β-dense sets.*

Theorem 3.14 *The family of all of β-first category sets composes a σ-ideal.*

Proof. Let $A \subset B$ and B is of β-first category. It is clear that A is of β-first category. Now we show that the countable union of β-first category sets is of β-first category. Let A_n be of β-first category set. Then $A_n = \cup_{m \in N} B_m$, where N is the set of natural numbers and B_m is nowhere β-dense. $\cup_{n \in N} A_n = \cup_{m,n \in N} B_{mn}$. Hence $\cup_{n \in N} A_n$ is of β-first category.

Definition 3.15 *A subset $A \subset X$ is defined to have β-Baire property if it can be represented as $A = G\Delta P$, where G is β-open and P is of β-first category.*

Theorem 3.16 *A subset $A \subset X$ has the property of β-Baire if and only if it can be represented as $A = F\Delta Q$, where F is β-closed and Q is of β-first category.*

Proof. Neccesity. Let $A = G\Delta P$, where G is β-open and P is of β-first category. We set $N = G^-_\beta - G$.Then N is a β -closed set since the intersection of two β- closed set is a β-closed set and N is a nowhere β-dense set. Now we show it. $(N^-_\beta)^\circ = ((G^-_\beta \cap (X-G))^-_\beta)^\circ \subset (G^-_\beta \cap (X-G^\circ_\beta))^\circ = (G^-_\beta)^\circ \cap (X-G)^\circ \subset G^-_\beta \cap (X-G)^-) = \emptyset$. $N \cup P$ is of β- first category sets. Since $N\Delta P \subset N \cup P$, $N\Delta P$ is of $\beta-$ first category. We can see that, $G^-_\beta \Delta N = G$. Set $F = G^-_\beta$ and $Q = N\Delta P$. $A = G\Delta P = (G^-_\beta \Delta N)\Delta P = F\Delta Q$, where F is β -closed and Q is of β- first category.

Sufficiency. Let $A = F\Delta Q$, where F is β-closed and Q is of β-first category. We set $G = F^\circ_\beta$, $N = F - G$ and $P = N\Delta Q$. N is nowhere β-dense since $(N^-_\beta)^\circ = ((F \cap (X-G))^-_\beta)^\circ \subset (F^-_\beta \cap (X-G)^-_\beta)^\circ = (F \cap (X-G)^-_\beta)^\circ = (F \cap (X - G^\circ_\beta))^\circ \subset F^\circ \cap (X - F^\circ_\beta) = \emptyset$ and $N \cup Q$ is of β-first category. Hence $N\Delta Q$ is of β-first category. We can easily see that, $G\Delta N = F$. $A = F\Delta Q = (G\Delta N)\Delta Q = G\Delta P$, where G is β-open and P is of β-first category.

Proposition 3.17 *If A has the β-Baire property then $X - A$ has the β-Baire property.*

Proof. Let $A = G\Delta P$, where G is a β-open and P is of β-first category set. $X - A = X - (G\Delta P) = (X - G)\Delta P$, where $X - G$ is β-closed and P is of β-first category set. This shows that $X - A$ has the β-Baire property.

Lemma 3.18 *If a subset $A \subset X$ is β-dense then A is β-open.*

Proof. Let A be a β-dense set. Then $X = A^-_\beta$ and $A^-_\beta \subset A^-$. $A^\circ_\beta = A \cap A^{-\circ-} = A \cap X = A$. Hence A is a β-open set.

Definition 3.19 *A space X is defined to be β-Baire if the countable intersection of β-dense sets of X is β-dense in X.*

Example 3.20 *The usual topological space (R, U) is a β-Baire space.*

Theorem 3.21 *The following properties are equivalent for a space X:*

1. X is a β-Baire space

2. Every countable union of sets with no β- interior point in X has no β-interior point in X.

Proof. (1) $\Rightarrow$ (2) : Let X be a β-Baire space and $(A_n)^\circ_\beta = \emptyset$ for each $n \in N$, where N is the set of the natural numbers. Then $X - ((A_n)^\circ_\beta) = (X - A_n)^-_\beta = X$. Since X is a β-Baire space, $(\cap_{n\in N}(X - A_n))^-_\beta = X$. Therefore $(\cup_{n\in N} A_n)^\circ_\beta = \emptyset$.

(2) $\Rightarrow$ (1) : It is obvious.

Definition 3.22 *A surjective function* $f : X \longrightarrow Y$ *is defined to be*

1. β*-feebly continuous if* $(f^{-1}(V))^\circ_\beta \neq \emptyset$ *whenever* $V^\circ_\beta \neq \emptyset$ *for a subset* V *of* Y*.*

2. β*-feebly open if* $(f(U))^\circ_\beta \neq \emptyset$ *whenever* $U^\circ_\beta \neq \emptyset$ *for a subset* U *of* X*.*

Theorem 3.23 *Let* $f : X \longrightarrow Y$ *be a surjective function. The following statements hold:*

1. If f *is* β*-feebly continuous and* A *is* β*-dense in* X*, then* $f(A)$ *is* β*-dense in* Y*.*

2. If f *is* β*-feebly open and* B *is* β*-dense in* Y*, then* $f^{-1}(B)$ *is* β*-dense in* X*.*

Proof.

1. Let f be a β-feebly continuous function and A be a β-dense set in X. Suppose that $f(A)$ is not β-dense. Then $X \neq (f(A))^-_\beta$ and $\emptyset \neq Y - (f(A))^-_\beta$. Set $T = Y - (f(A))^-_\beta$. Then T is a nonempty β-open set. Since f is β-feebly continuous $(f^{-1}(T))^\circ_\beta \neq \emptyset$. Also $(f^{-1}(T))^\circ_\beta \cap A \subset f^{-1}(T) \cap f^{-1}(f(A)) = f^{-1}(T \cap f(A)) \subset f^{-1}(T \cap (f(A))^-_\beta) = \emptyset$. This is a contradiction since A is β-dense. Hence $f(A)$ is β-dense.

2. Let f be β-feebly open and B be β-dense in Y. Suppose that $f^{-1}(B)$ is not β-dense in X. Then there exists a nonempty β-open set U of X such that $U \cap f^{-1}(B) = \emptyset$. Since f is β-feebly open $(f(U))^\circ_\beta \neq \emptyset$. Moreover, we have $(f(U))^\circ_\beta \cap B \subset f(U) \cap B = \emptyset$. This is a contradiction since B is β-dense. Hence $f^{-1}(B)$ is β-dense.

Theorem 3.24 *Let* $f : X \longrightarrow Y$ *be a* β*-feebly continuous and* β*-feebly open surjection. If* X *is a* β*-Baire space , then* Y *is a* β*-Baire space.*

Proof. Let X be a β-Baire space and $B_n \subset Y$ be a β -dense set for each $n \in N$, where N is the set of natural numbers. Since f is β-feebly open $f^{-1}(B_n)$ is β-dense in X. Since X is a β-Baire space, $\cap_{n\in N} f^{-1}(B_n)$ is β-dense in X. By the β-feebly continuity of f, $f(\cap_{n\in N} f^{-1}(B_n)) = \cap_{n\in N} B_n$ is β-dense in Y. This shows that Y is a β-Baire space.

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