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Common Fixed Point Theorem for Generalized Hybrid Tangential Property

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Abstract

We define a new property as a generalization of the hybrid tangential property [Kamran T. and Cakic N., Hybrid tangential property and coincidence point theorems, Fixed Point Theory, vol. 9, no. 2, (2008), 487-496] and give common fixed point theorems using implicit relation.

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1 Introduction and Preliminary

Let (X, d) be a metric space. Then for $x \in X$ and $A \subseteq X$, $d(x, A) = inf\{d(x, y) : y \in A\}$. We denote by $B(X)$ the class of all bounded subsets of X and

$$\delta(A, B) = sup\{d(a, b) : a \in A,\ b \in B\}$$

for every $A, B \in B(X)$. Also, it follows from definition of $\delta(A, B)$ that

$$\delta(A, B) = \delta(B, A) \geq 0$$

$$\delta(A, B) \leq \delta(A, C) + \delta(C, B)$$

$$\delta(A, B) = 0 iff A = B = a$$

$$\delta(A,A) = diam(A)$$

for all $A, B, C \in B(X)$. Let $f : X \to X$ and $T : X \to B(X)$. The set of coincidence points of f and T is denoted by $C(f,T)$

Definition 1.1 [4] *Let $T : X \to B(X)$. The map $f : X \to X$ is said to be T-weakly commuting at $x \in X$ if $ffx \in Tfx$.*

Definition 1.2 [1] *Maps $f : X \to X$ and $S : X \to B(X)$are said to satisfy property (E.A) if there exists a sequence $\{x_n\}$ in X, some $t \in X$ and $A \in B(X)$ such that $lim_{n\to\infty} fx_n = t \in A = lim_{n\to\infty} Tx_n$.*

Definition 1.3 [6] *Let $f, g : X \to X$ and $S, T : X \to B(X)$. The pairs (f, S) and (g, T) are said to satisfy common property (E.A) if there exist two sequences $\{x_n\}$ and $\{y_n\}$ in X, some $t \in X$ and $A, B \in B(X)$ such that $lim_{n\to\infty} Sx_n = A$, $lim_{n\to\infty} Ty_n = B$, $lim_{n\to\infty} fx_n = lim_{n\to\infty} g_n = t \in A \cap B$.*

Definition 1.4 [5] *Let $f, g : X \to X$ and $S, T : X \to B(X)$. The hybrid pair (f, T) is said to be g-tangential at $t \in X$ if there exist two sequences $\{x_n\}$ and $\{y_n\}$ in X, $A \in B(X)$ such that $lim_{n\to\infty} Sy_n = B(X)$ and $lim_{n\to\infty} fx_n = lim_{n\to\infty} gy_n = t = lim_{n\to\infty} Tx_n = A$.*

Definition 1.5 [2] *A sequence $\{A_n\}$ of subsets of X is said to be convergent to a subset A of X if*
(i) given $a \in A$, there is a sequence $\{a_n\}$ in X such that $a_n \in A_n$ for $n = 1, 2, ...,$ and $\{a_n\}$ converges to a.
(ii) given $\varepsilon > 0$, there exists a positive integer N such that $A_n \subseteq A_\varepsilon$ for $n > N$ where A_ε is the union of all open spheres with centers in A and radius ε.

Lemma 1.6 [2, 3] *If $\{A_n\}$ and $\{B_n\}$ are sequences in $B(X)$ converging to A and B in $B(X)$, respectively, then the sequence $\{\delta(A_n, B_n)\}$ converges to $\delta(A, B)$.*

Lemma 1.7 [3] *Let $\{A_n\}$ be a sequence in $B(X)$ and y a point in X such that $\delta(A_n, y) \to 0$. Then the sequence $\{A_n\}$ converges to the set $\{y\}$ in $B(X)$.*

Hybrid pair contains single valued and multi valued maps. Fixed point theory for such hybrid pairs is a new development in the context of contraction type multi valued theory. Aamri and Moutawakil [1] introduced (E.A) property for single valued maps so that compatible and noncompatible maps may be studied together. Then Kamran [4] extended this to the hybrid pairs of single valued and multi valued mappings, introducing the notion of T-weak commutativity generalizing the notion of (IT)-commutativity for such pairs. Recently, Liu et al [5] defined common property (E.A) for two pairs of hybrid maps and this was again generalized by Kamran and Cakic [5] by introducing hybrid tangential maps. In this paper, we introduce a new property called generalized hybrid tangential property and prove common fixed point theorems for such pairs by using an implicit relation. Our new notion is a generalization of definition 2.1 of Kamran and Cakic [5].

2 Main Results

Now we introduce the following definitions.

Definition 2.1 *Let $f, g : X \to X$ and $S, T : X \to B(X)$. The hybrid pairs (f, S) and (g, T) are said to satisfy generalized hybrid tangential property if there exist sequences $\{x_n\}$ and $\{y_n\}$ in X , $A \in B(X)$ such that $lim_{n\to\infty} Ty_n \in B(X)$ and $lim_{n\to\infty} fx_n = t_1 \in A$, $lim_{n\to\infty} gy_n = t_2 \in A$ and $lim_{n\to\infty} Sx_n = A$.*

Remark 2.2 *Clearly, if the hybrid pairs (f, S) is g-tangential at $t \in X$, then the pairs (f, S) and (g, T) satisfy generalized hybrid tangential property. However, the converse is not true in general.*

Example 2.3 *Let $(X, d) = [1, 5]$ with usual metric. Define $f, g : X \to X$ and $S, T : X \to B(X)$ by $fx = 3 - \frac{1}{4}x$, $gx = 4 - \frac{1}{6}x$, $Sx = [2, 4]$, $Tx = [2, 4]$. Let $\{x_n\} = \{1 + \frac{1}{n}\}$ and $\{y_n\} = \{2 + \frac{1}{n}\}$. Then $fx_n \to t = \frac{11}{4}$, $gy_n = \frac{11}{3}$ and $Sx_n \to A = [2, 4]$ i.e. $lim_{n\to\infty} fx_n = t_1 \in A$, $lim_{n\to\infty} gy_n = t_2 \in A$. But there are no sequences $\{x_n\}$ and $\{y_n\}$ in X, such that $lim_{n\to\infty} fx_n = lim_{n\to\infty} g_n = t \in A = lim_{n\to\infty} Sx_n$ for some $t \in X$.*

Example 2.4 *Let $X = [1, \infty)$ with usual metric. Define $f, g : X \to X$ and $S, T : X \to B(X)$ by $fx = 2 + \frac{1}{2x}$, $gx = 3 + \frac{1}{6x}$, $Sx = [2, 3 + x]$, $Tx = [1, 2]$. Let $\{x_n\} = \{1 + \frac{1}{n}\}_{n\geq 1}$ and $\{y_n\} = \{2 + \frac{1}{n}\}_{n\geq 1}$. Then for $n \to \infty$, $fx_n \to \frac{5}{2}$, $gy_n \to \frac{37}{12}$ and $Sx_n \to A = [2, 4]$ i.e. $lim_{n\to\infty} fx_n \in A$, $lim_{n\to\infty} gy_n \in A$. But there are no sequences $\{x_n\}$ and $\{y_n\}$ in X such that $lim_{n\to\infty} fx_n = lim_{n\to\infty} g_n = t \in A$ for some $t \in X$.*

Definition 2.5 *Let Ψ be the family of all real lower semi-continuous functions $Q : R^6_+ \to R$ satisfying the following conditions*

(Q1): Q is nonincreasing in the variables t_3, t_4, t_5, t_6

(Q2): there exists $h \in (0, 1)$ such that for every $u, v \geq 0$ with

$$(Q2a) : Q(u, w, v, u, u, 0) \leq 0$$
$$(Q2b) : Q(u, w, u, v, 0, u)) \leq 0$$

we have $u \leq hv$.

Example 2.6 *Let* $Q(t_1, ..., t_6) = t_1 - amax\{t_3, t_4, t_2t_5t_6\}, 0 < a < 1.$
$(Q1)$*: It is obvious*
$(Q2)$*: Let* $u, v \geq 0$ *and if* $u \geq v$*, then*

$$Q(u, w, v, u, u, 0) \leq 0$$

$$\Rightarrow u - a \ max\{u, u, 0\}$$

$$u - au \leq 0$$

a contradiction for $0 < a < 1$.
Thus, $u < v$ *and* $q(u, w, v, u, u, 0) \leq 0$

$$\Rightarrow u - av \leq 0 \Rightarrow u \leq av \Rightarrow u \leq hv$$

where $h = a$.
Similarly, let $u, v \geq 0$ *and if* $u \geq v$*, then*

$$Q(u, w, u, v, 0, u) \leq 0$$

$$\Rightarrow u - a \ max\{u, v, 0\} \leq 0$$

$$u - au \leq 0 \ \ a \ \ contradiction$$

Thus $u < v$ *and* $Q(u, w, u, v, 0, u) \leq 0 \Rightarrow u - av \leq 0 \Rightarrow u \leq hv$ *where* $h = a$

Example 2.7 *Let* $Q(t_1, ..., t_6) = t_1 - a\,max\{t_3, t_4, t_5, t_6\}, 0 < a < 1.$

Theorem 2.8 *Let* f, g *be two self mappings of a metric space* (X, d) *and* $S, T : X \to B(X)$ *such that*
(a) the hybrid pairs (f, S) *and* (g, T) *satisfy generalized hybrid tangential property*
(b) for all $x \neq y$ *in* X

$$Q(\delta(Sx, Ty), d(fx, gy), d(fx, Sx), d(gy, Ty), d(fx, Ty), d(gy, Sx)) \leq 0 \quad (1)$$

If $f(X)$ *and* $g(X)$ *are closed subsets of* X*, then*
(i) f *and* S *have a coincidence point*
(ii) g *and* T *have a coincidence point*
(iii) $f(X)$ *and* $g(X)$ *have a common fixed point provided that* f *is S-weakly commuting at* v *and* $ffv = fv$ *where* $v \in C(f, S)$*;*
(iv) g *and* T *have a common fixed point provided that* g *is T-weakly commuting at* u *and* $ggu = gu$ *where* $u \in C(g, T)$
(v) f, g, S *and* T *have a common fixed point provided that (iii) and (iv) are true.*

Proof. Suppose (f, S) and (g, T) satisfy generalized hybrid tangential property. Then there exist sequences $\{x_n\}$ and $\{y_n\}$ in X , $A \in B(X)$ such that $lim_{n\to\infty}Ty_n \in B(X)$, $lim_{n\to\infty}fx_n = t_1 \in A = lim_{n\to\infty}Sx_n$ and $lim_{n\to\infty}gy_n = t_2 \in A$.
Putting $x = x_n$ and $y = y_n$ in (b), we have
$Q(\delta(Sx_n, Ty_n), d(fx_n, gy_n), d(fx_n, Sx_n), d(gy_n, Ty_n), d(fx_n, Ty_n), d(gy_n, Sx_n)) \leq 0$
By lemma 1.6, taking limit as $n \to \infty$ we have
$Q(\delta(A, lim_{n\to\infty}Ty_n), d(t_1, t_2), 0, \delta(A, lim_{n\to\infty}Ty_n), \delta(A, lim_{n\to\infty}Ty_n), 0) \leq 0$
Then by $(Q2a)$ and lemma1.7, we have
$\delta(A, lim_{n\to\infty}Ty_n = 0 \Rightarrow lim_{n\to\infty}Ty_n = A = \{t\}$ and hence $t_1 = t_2 = t$
Since $f(X)$ is closed, there exists $v \in X$ such that $t = fv$. Now we want to show that $t = ft \in Sv$. Putting $s = v, y = y_n$ in (b), we have
$Q(\delta(Sv, Ty_n), d(fv, gy_n), d(fv, Sv), d(gy_n, Ty_n), d(fv, Ty_n), d(gy_n, Sv)) \leq 0$
Taking limit as $n \to \infty$, we have

$$Q(\delta(Sv, A), 0, d(t, Sv), 0, 0, d(t, Sv)) \leq 0$$

$$\Rightarrow Q(\delta(Sv, A), 0, \delta(Sv, A), 0, 0, \delta(Sv, A)) \leq 0$$

Then by $(Q2b)$, $\delta(Sv, A) = 0 \Rightarrow Sv = A = t$. Therefore $t = fv = Sv$. Again, since $g(X)$ is closed, there exists $u \in X$ such that $lim_{n\to\infty}gy_n = t = gu$. Now we want to show that $A = Tu$.
Taking $x = x_n, y = u$ in (b), we have
$Q(\delta(Sx_n, Tu), d(fx_n, gu), d(fx_n, Sx_n), d(gu, Tu), d(fx_n, Tu), d(gu, Sx_n)) \leq 0$
Taking limit as $n \to \infty$, we have

$$Q(\delta(A, Tu), 0, 0, \delta(A, Tu), \delta(A, Tu), 0) \leq 0$$

Then by $(Q2a)$, $\delta(A, Tu) = 0 \Rightarrow t = A = Tu = Sv$.
This ends the proofs of (i) and (ii). Suppose f is S-weakly commuting and $ffv = fv$ for $v \in C(f, S)$. Then $ffv = fv$ and $ffv \in Sfv$ implies that $t = ft \in St$. This proves (iii). Similarly g is T-weakly commuting and $ggu = gu$ for $u \in C(g, T)$. Then $ggu = gu\ ggu \in Tgu$ implies that $t = gt \in Tt$. This proves (iv). Condition (v) holds immediately.

Remark 2.9 *We denote by $CB(X)$ the class of all nonempty bounded closed subsets of X and*

$$H(A, B) = max\{sup_{a\in A}d(a, B), sup_{b\in B}d(b, A)\}$$

for every $A, B \in CB(X)$. *Here* H *is called Hausdorff metric on* $CB(X)$ *induced by d. If we use* $H(Sx, Ty)$ *in place of* $\delta(Sx, Ty)$ *in equation (1) and* $S, T : X \to CB(X)$ *of above theorem 2.8, then we may not have common fixed point of* f, g, S *and* T.

Example 2.10 *Let* $X = [0, 4]$ *with usual metric* $d(x, y) =| x - y |$.
Let $fx = 1$,*if* $x \in [0, 1]$, $fx = 2 - x$, *if* $1 < x \leq 2$, $fx = x - 2$, *if* $2 < x \leq 3$, $fx = 4 - x$, *if* $3 < x \leq 4$. *and* $gx = 3$, *if* $x \in [3, 4]$, $gx = 6 - x$, *if* $2 \leq x < 3$, $gx = x + 2$, *if* $1 \leq x < 2$, $gx = 3$, *if* $0 \leq x < 1$.

$Sx = [1, 3] = Tx$. *Then* $f(X) = [0, 1]$ *and* $g(X) = [3, 4]$ *so that* $f(X)$ *and* $g(X)$ *are closed subsets of* X. *We know that* $C(f, S) = \{1\}$ *and* $C(g, T) = \{3\}$. *Therefore,* $ffx \in Sfx$ *for* $x \in C(f, S)$ *and* $ggx \in Tgx$ *for* $x \in C(g, T)$. *Also* $ffx = fx$ *for* $x \in C(f, S)$ *and* $ggx = gx$ *for* $ggx \in Tgx$. *Let us consider the sequences* $\{x_n\} = \{\frac{1}{n}\}_{n\in N}$ *and* $\{y_n\} = \{3 + \frac{1}{n}\}_{n\in N}$ *in* X. *Then* $lim_{n\to\infty} Sx_n = [1, 3] = A$, $lim_{n\to\infty} Ty_n = [1, 3] \subseteq CB(X)$ *and* $lim_{n\to\infty} fx_n = 1 \in A$, $lim_{n\to\infty} gy_n = 3 \in A$. *Thus* (f, S) *and* (g, T) *satisfy generalized hybrid tangential property. Also, we have* $H(Sx, Ty) = 0$ *so that the implicit relation is satisfied. But* f *and* S *have common fixed point 1. Similarly* g *and* T *have common fixed point 3. But* f, g, S *and* T *have no common fixed point.*

Corollary 2.11 *Let* f, g *be two self mappings of a metric space* (X, d) *and* $S, T : X \to B(X)$ *such that*
(a) the hybrid pairs (f, S) *is g-tangential with respect to* T
(b) for all $x \neq y$ *in* X

$$Q(\delta(Sx, Ty), d(fx, gy), d(fx, Sx), d(gy, Ty), d(fx, Ty), d(gy, Sx)) \leq 0$$

If $f(X)$ *and* $g(X)$ *are closed subsets of* X, *then* $C(f, S) \neq \phi$ *and* $C(g, T) \neq \phi$
(ii) $f(X)$ *and* $g(X)$ *have a common fixed point provided that* f *is S-weakly commuting at* v *and* $ffv = fv$ *where* $v \in C(f, S)$;
(iii) g *and* T *have a common fixed point provided that* g *is T-weakly commuting at* u *and* $ggu = gu$ *where* $u \in C(g, T)$
(iv) f, g, S *and* T *have a common fixed point provided that (ii) and (iii) are true.*

Now we give an example supporting theorem 2.8

Example 2.12 *Let* $X = [0, \infty)$ *with usual metric* $d(x, y) =| x - y |$.
Assume that $Q(t_1, t_2...t_6) = t_1 - \frac{1}{3}max\{t_3, t_4\}$ *for every* $t_1, ...t_6 \in [0, \infty)$. *Let* $fx = 2x$, $gx = 2x^2$, $Sx = [0, \frac{x}{2})$ $Tx = [0, \frac{x^2}{2})$
Now $C(f, S) = \{0\}$, $C(g, T) = \{0\}$. *Here* $ffx \in Sfx$ *for* $x \in C(f, S)$ *and* $ggx \in Tgx$ *for* $x \in C(g, T)$. *Therefore* f *is S-weakly commuting at* $x \in C(f, S)$

and g is T-weakly commuting at $x \in C(g,T)$. Also, $ffx = fx$ for $x \in C(f,S)$ and $ggx = gx$ for $x \in C(g,T)$.
Consider $\{x_n\} = \{\frac{1}{n}\}_n \in N$ and $\{y_n\} = \{\frac{1}{2n}\}_n \in N$ in X. Then $lim_{n\to\infty} fx_n = 0 \in \{0\} = lim_{n\to\infty} Sx_n$ and $lim_{n\to\infty} gy_n = 0 \in \{0\} = lim_{n\to\infty} Sx_n$. Therefore, the hybrid pairs (f,S) and (g,T) satisfy generalized hybrid tangential property. Also the functions satisfy the implicit relation. It is clear that $f(X)$ and $g(X)$ are closed subsets of X. We know that 0 is a unique common fixed point of f, g, S and T.

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