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Method of Successive Approximations for an Unsteady Fluid Structure Interaction Problem

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Abstract

In this paper, we present a method for solving unsteady coupled problem. This work is coming with a view to improve and extend successive approximations method arising from fluid structure interaction context [7]. The external force acting on the structure is replaced by $\lambda(t) = p(x, H + u(x, \lambda(t)))$. Then we have a nonlinear equation of unknown $\lambda(t)$ to solve by successive approximations method. By this method, good results are obtained with only a few iterations.

Mathematics Subject Classification: 74F10, 65N06, 65N30, 65N12

Keywords: wave equation, Navier-Stokes equations, Finite element method, successive approximations method

1 Introduction

Problem involved in fluid structure interactions occur in a wide variety of engineering problem and therefore have attracted the interest of many investigations from different engineering disciplines. As results, much effort has gone into the development of general computational method for fluid structure system by Sow, Mbaye, Murea, Osses, Fernandez, [7],[4],[5],[6],[2]. In this paper, successive approximations method is applied to solve a fluid-structure interaction problem. We replace the external force acting on the interface between fluid and structure by $\lambda(t) = p(x, H + u(x, \lambda(t)))$. Then we introduce a nonlinear equation to solve by successive approximations method at each time step such that the coupled problem is achievable. A former paper [7] had shown that successive approximations method is appropriate for solving fluid structure interaction problem. This paper is coming to improve and extend our previous work [7]. The fluid is modelled by two dimensional Navier-Stokes equations for unsteady flow and the structure is represented by the one dimensional wave equation.

2 Definition

We denote by Ω_u^t the two dimensional domain occupied by the fluid, Γ_u^t the elastic interface between fluid and structure and $\Sigma_1 \bigcup \Sigma_2 \bigcup \Sigma_3$ be the remaining external boundaries of the fluid as depicted in Figure 1.

Definition 2.1 *Domain definition*

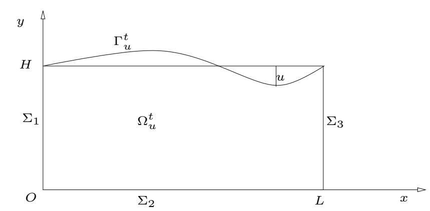

Figure 1: Sets appearing to the fluid structure interaction problem

Definition 2.2 *Structure properties*
We start from the simple equation that governs the structure. The balance equations are:

$$\rho^S h^S \frac{\partial^2 u}{\partial t^2} - a\frac{\partial^2 u}{\partial x^2} + bu = p(x, H + u(x,t)),\ in\,[0,L]\times[0,T] \tag{1}$$

$$u(0,t) = u(L,t) = 0, \quad in\,[0,T] \tag{2}$$

$$u(x, t=0) = u^0(x), \quad in\,[0,L] \tag{3}$$

$$\frac{\partial u}{\partial t}(x, t=0) = \dot{u}^0(x), \quad in\,[0,L] \tag{4}$$

These equations represent the wave equation which is used to describe the motion of the vessel in blood flow.
Where,

- E *is the young modulus,*
- h^S *is the structure thickness,*
- ν *is the poisson coefficient,*
- ρ^S *is the strcuture density,*
- σ^S *is the tangential component of the longitudinal stress,*
- $a = h^S \sigma^S$,
- $b = \frac{Eh^S}{12(1-\nu^2)H^2}$,
- u_0 *is the initial displacement of the structure,*
- $\dot{u}_0$ *is the initial velocity of the strcuture,*

Definition 2.3 *Fluid properties*
We suppose that the fluid is governing by the Navier-Stokes equations for unsteady flow in Ω_u^t *:*

$$\rho^F(\frac{\partial v}{\partial t} + (v.\nabla v)v) - \mu\Delta v + \nabla p = f^F, \quad in\, \Omega_u^t \times [0,T] \tag{5}$$
$$\nabla \cdot v = 0, \quad in\, \Omega_u^t \times [0,T] \tag{6}$$
$$v(t=0) = v_0, \quad in\, \Omega_u^t \tag{7}$$
$$-pI_2 n + \mu\nabla v \cdot n = p_{in} I_2 n, \quad on\, \Sigma_1 \times [0,T] \tag{8}$$
$$-pI_2 n + \mu\nabla v \cdot n = 0, \quad on\, \Sigma_3 \times [0,T] \tag{9}$$
$$v(x, H + u(x,t)) = (0, \frac{\partial u}{\partial t}(x,t)), \quad on\, \Gamma_u^t \times [0,T] \tag{10}$$
$$v_2 = 0, \quad on\, \Sigma_2 \times [0,T] \tag{11}$$
$$\frac{\partial v_1}{\partial y} = 0, \quad on\, \Sigma_2 \times [0,T] \tag{12}$$

Where,

- ρ^F *is the fluid density,*
- I_2 *is the identity matrix,*
- μ *is the fluid viscosity,*
- v_1 *first component of* v,
- v_2 *second component of* v,

- *v_0 is the initial velocity of the fluid,*
- *n is the unit outward normal vector,*
- *Σ_2 is the symmetric axis,*
- *On Σ_2, we have the non penetration condition: $v \cdot n = v_2 = 0$,*
- *On Σ_2, we have the continuity of Cauchy shear stress: $\sigma \cdot n = \frac{\partial v_1}{\partial y} = 0$,*

Definition 2.4 *Formulation of coupled problem*
The problem is to find u, v and p such that:

$$\begin{aligned}
\rho^S h^S \frac{\partial^2 u}{\partial t^2} - a\frac{\partial^2 u}{\partial x^2} + bu &= p(x, H + u(x,t)),\ in\,[0,L]\times[0,T] && (13)\\
u(0,t) &= u(L,t) = 0, \quad in\,[0,T] && (14)\\
u(x,t=0) &= u^0(x), \quad in\,[0,L] && (15)\\
\frac{\partial u}{\partial t}(x,t=0) &= \dot{u}^0(x), \quad in\,[0,L] && (16)\\
\rho^F\left(\frac{\partial v}{\partial t} + (v.\nabla v)v\right) - \mu\Delta v + \nabla p &= f^F, \quad in\,\Omega_u^t\times[0,T] && (17)\\
\nabla\cdot v &= 0, \quad in\,\Omega_u^t\times[0,T] && (18)\\
v(t=0) &= v_0, \quad in\,\Omega_u^t && (19)\\
-pI_2 n + \mu\nabla v\cdot n &= p_{in} I_2 n, \quad on\,\Sigma_1\times[0,T] && (20)\\
-pI_2 n + \mu\nabla v\cdot n &= 0, \quad on\,\Sigma_3\times[0,T] && (21)\\
v(x, H + u(x,t)) &= (0, \frac{\partial u}{\partial t}(x,t)), \quad on\,\Gamma_u^t\times[0,T] && (22)\\
v_2 &= 0, \quad on\,\Sigma_2\times[0,T] && (23)\\
\frac{\partial v_1}{\partial y} &= 0, \quad on\,\Sigma_2\times[0,T] && (24)
\end{aligned}$$

Then, we have a fluid structure interaction problem. The domain of the fluid depends on the displacement and the displacement depends on the velocity and the pressure of the fluid.

Definition 2.5 *Successive approximations method*
We assume that $\lambda(t) = p(x, H + u(x, \lambda(t)))$. Corresponding to each $\lambda(t)$, we consider the coupled problem:

$$\begin{aligned}
\rho^S h^S \frac{\partial^2 u}{\partial t^2} - a\frac{\partial^2 u}{\partial x^2} + bu &= \lambda(t),\ in\,[0,L]\times[0,T] && (25)\\
u(0,t) &= u(L,t) = 0, \quad in\,[0,T] && (26)\\
u(x,t=0) &= u^0(x), \quad in\,[0,L] && (27)\\
\frac{\partial u}{\partial t}(x,t=0) &= \dot{u}^0(x), \quad in\,[0,L] && (28)
\end{aligned}$$

$$
\begin{aligned}
\rho^F \left(\frac{\partial v}{\partial t} + (v.\nabla v)v \right) - \mu \Delta v + \nabla p &= f^F, \quad in\, \Omega_u^t \times [0,T] &(29)\\
\nabla \cdot v &= 0, \quad in\, \Omega_u^t \times [0,T] &(30)\\
v(t=0) &= v_0, \quad in\, \Omega_u^t &(31)\\
-pI_2 n + \mu \nabla v \cdot n &= p_{in} I_2 n, \quad on\, \Sigma_1 \times [0,T] &(32)\\
-pI_2 n + \mu \nabla v \cdot n &= 0, \quad on\, \Sigma_3 \times [0,T] &(33)\\
v(x, H + u(x,t)) &= (0, \frac{\partial u}{\partial t}(x,t)), \quad on\, \Gamma_u^t \times [0,T] &(34)\\
v_2 &= 0, \quad on\, \Sigma_2 \times [0,T] &(35)\\
\frac{\partial v_1}{\partial y} &= 0, \quad on\, \Sigma_2 \times [0,T] &(36)
\end{aligned}
$$

To solve this coupled problem, we need to solve a nonlinear equation of unknown $\lambda(t)$ for all t define as: $\lambda(t) = p(x, H + u(x, \lambda(t)))$ by the successive approximations method. Then we will find the pressure p, the velocity v, and the displacement u.

Definition 2.6 *Description of the method*
We summarize step by step our computational method to find $(\lambda^m(t))_{m \in \mathbf{N}}$ for all t such that:

$$
\begin{aligned}
\lambda^0(t) &= \alpha \quad (\alpha \text{ is done}) &(37)\\
\lambda^{m+1}(t) &= p(x, H + u(x, \lambda^m(t))), \forall t \in [0,T] &(38)
\end{aligned}
$$

Step 1: At each time step, we give the initial value $\lambda^0(t)$ then the initial displacement and the fluid domain are compute.

Step 2: We solve the Navier-Stokes equation by finite element method in the previous domain. We find p_0, v_0.

Step 3: we fine $\lambda^1(t) = p(x, H + u(x, \lambda^0(t)))$.

Step 4: Do:
-$\lambda^0(t) = \lambda^1(t)$,
-compute $u(x, \lambda^1(t))$ and $\Omega_u^t(\lambda^1(t))$,
- solve the Navier-Stokes equation in $\Omega_u^t(\lambda^1(t))$, we find (v, p),
- we compute $\lambda^1(t) = p(x, H + u(x, \lambda^0(t)))$,
While $(|\lambda^1(t) - \lambda^0(t)| > toll\ )$

Step 5: Give $\lambda(t)$, u, v and p and return to Step 1 at each time step.

3 Numerical Results

3.1 Leap-frog scheme

To compute numerical solutions to strcuture equation (1)-(4), we use the leap-frog scheme.
Let $N \in \mathbf{N}$, $\Delta t = \frac{T}{N}$ the time step, and $t_n = n\Delta t$, and we use for space discretization the classic finite difference method, Let $M \in \mathbf{N}$, $\Delta x = \frac{L}{M}$ the spacial step size, and the equidistributed grid points $(x_j)_{1\leq j\leq M}$ given by $x_j = j\Delta x$. We assume that λ_j^n, u_j^{n+1}, u_j^n and u_j^{n-1} are the approximations of $\lambda(t_n)$, $u(x_j,t_{n+1})$, $u(x_j,t_n)$,$u(x_j,t_{n-1})$ respectively and also u_{j+1}^n, u_j^n and u_{j-1}^n are the approximations of $u(x_{j+1},t_n)$, $u(x_j,t_n)$,$u(x_{j-1},t_n)$ respectively. We define the Leap-frog scheme as follow: for all $n \in \mathbf{N}$, knowing the initial data discretized as $u_j^0 = u^0$ and $u_j^1 = u^0 + \Delta t\dot{u}^0$ we find u_j^n for all $j \in \mathbf{N}$ such that:

$$u_j^{n+1} - 2u_j^n + u_j^{n-1} = \frac{a}{\rho^S h^S}\left(\frac{\Delta t}{\Delta x}\right)^2\left(u_{j+1}^n - 2u_j^n + u_{j-1}^n\right) - \frac{b\Delta t^2}{\rho^S h^S}u_j^n + \frac{\Delta t^2}{\rho^S h^S}\lambda^n. \tag{39}$$

3.2 Total discretization in time and space for the fluid equations

The time discretization and the spatial discretization are detailed in [5].

3.3 Boundary conditions and parameters values

The boundary conditions imposed to the pressure [5]:

$$p_{in}(x,y,t) \ = \begin{cases} 20000\left(1-\cos(\frac{\pi t}{0.0025})\right), \ (x,y) \in \Sigma_1\,, 0 \leq t \leq 0.005 \\ 0, \ (x,y) \in \Sigma_1\,, 0.005 \leq t \leq T \end{cases} \tag{40}$$

The parameter values of the fluid and the structure are:

Parameter related to fluid: The fluid viscosity is $\mu = 0.035\frac{g}{cm\cdot s}$, the vessel length is $L = 3cm$, the vessel width is $H = 0.5cm$, the volume force of the fluid $f^F = (0,0)$, the fluid density $\rho^F = 1$. Time step is $\Delta t = 0.1ms$, $toll = 10^{-4}$ and $v_0 = 0$.

Parameter related to structure: The structure thickness $h^S = 0.1cm$, the Young modulus is $E = 3\cdot 10^6\frac{g}{cm\cdot s^2}$, the structure density is $\rho^S = 1.1\frac{g}{cm^3}$, the poisson coefficient is $\nu = 0.3$, $\sigma^S = 25000$, $M = 4$, $\Delta x = \frac{L}{M}$, $u^0 = 0$, $\dot{u}^0 = 0$.

FreeFem++ [3] is used for the numerical tests. At each time step, a few number of iterations in successive approximations method is done. In addition, the stability condition of leap-frog scheme is satisfied at each time step.

n	t_n	Error=$\lvert\lambda^1(t_n) - \lambda^0(t_n)\rvert$	Number of iterations
30	3 ms	4.52884e-05	12
40	4 ms	9.31976e-05	12
75	7.5 ms	8.96308e-05	9
100	10 ms	6.75434e-05	6

Table 1: Optimal values with successive approximations method.

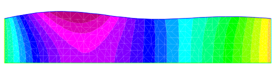

Figure 2: Wave pressure and structure displacement at $t = 3ms$.

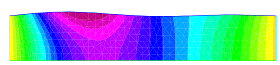

Figure 3: Wave pressure and structure displacement at $t = 4ms$.

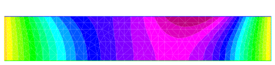

Figure 4: Wave pressure and structure displacement at $t = 7.5ms$.

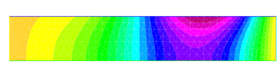

Figure 5: Wave pressure and structure displacement at $t = 10ms$.

The above figures display the modification step by step of the wave pressure and the structure displacement at $t = 3, 4, 7.5, 10ms$.

4 Conclusion

In this work, we applied successive approximations method to solve fluid structure interaction problem. This method gives good results when the displacement is small. We note that at each time step and after a few iterations, we found good approximate solution of the nonlinear equation $\lambda(t) = p(x, H + u(x, \lambda(t)))$ and also we obtain the solution of coupled problem. This paper shown that our computational method used in steady case [7] remains good. In a forthcoming work, we will be showed the theoretical convergence of $\lambda^{m+1}(t) = p(x, H + u(x, \lambda^m(t)))$.

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