International Journal of Contemporary Mathematical Sciences
Vol. 11, 2016, no. 5, 227 - 233
HIKARI Ltd, www.m-hikari.com
http://dx.doi.org/10.12988/ijcms.2016.637

Sun Decompositions of Crowns

Jenq-Jong Lin

Ling Tung University, Taichung 40852, Taiwan

Abstract

In this paper we give some sufficient conditions for the decomposition of crowns into isomorphic suns.

Mathematics Subject Classification: 05C51

Keywords: sun, decomposition, crown

1 Introduction

Suppose that G and H are graphs and the edges of G can be decomposed into subgraphs which are isomorphic to H. Then we say that G has an H-*decomposition*. The *crown* $C_{n,n-1}$ is the graph with vertex set $\{a_1, a_2, \cdots, a_n, b_1, b_2, \cdots, b_n\}$ and edge set $\{a_ib_j : 1 \le i, j \le n, i \ne j\}$. Equivalently, the crown $C_{n,n-1}$ is the graph obtained by deleting a perfect matching from the complete bipartite graph $K_{n,n}$. The crown has been investigated for the star decomposition [10, 11], path decomposition [11] and complete bipartite decomposition [9]. A *k-cycle*, denoted by C_k, is a cycle of length k. Let $(v_1v_2 \ldots v_k)$ denote the k-cycle with vertex set $\{v_1, v_2, \ldots, v_k\}$ and edge set $\{v_1v_2, v_2v_3, \ldots, v_{k-1}v_k, v_kv_1\}$. For an even integer $k \ge 6$, a k-sun $\mathbb{SUN}_k$ is obtained from $C_{k/2}$ by adding a pendant edge to each vertex of $C_{k/2}$. If the pendant vertex set of $\mathbb{SUN}_k$ is $\{w_1, w_2, \ldots, w_{k/2}\}$ and the pendant edge set is $\{v_1w_1, v_2w_2, \ldots, v_{k/2}w_{k/2}\}$, then $\mathbb{SUN}_k$ is denoted by

$$\mathbb{SUN}_k = \begin{pmatrix} v_1 & v_2 & \ldots & v_{k/2} \\ w_1 & w_2 & \ldots & w_{k/2} \end{pmatrix}.$$

The concept of a sun was defined by Harary [6]. Anitha and Lekshmi [1, 2] have decomposed K_{2k} into k-sun, Hamilton cycles, and perfect matchings. A k-sun system of order v is a decomposition of the complete graph K_v into k-suns. Liang and Guo [7, 8] gave the existence spectrum of a k-sun system of order v as $k = 3, 4, 5, 6, 8$ by using a recursive construction. Recently, Fu et al. [3] investigate the problem of the decomposition of complete tripartite graphs into 3-suns and find the necessary and sufficient condition for the existence of a k-sun system of order v in [4, 5]. In this paper we give some sufficient conditions for the decomposition of crowns into isomorphic suns.

2 Preliminaries

We consider only $\mathbb{SUN}_{4r}$-decompositions of crowns because crowns (being bipartite) possess no odd cycles. For positive integers m and n, $B_{m,n}$ denotes the bipartite graph with parts of sizes m and n. A bipartite graph is *balanced* if $m = n$. In a balanced bipartite graph, the label of the edge a_ib_j is defined to be $j - i$ if $i \leq j$ and $n + j - i$ if $i > j$. Let $B_{pr,qr} = \cup_{j=0}^{q-1} \cup_{i=0}^{p-1} G_{i,j}$, where $G_{i,j}$ is a complete bipartite graph (or a crown) with bipartition $(\{a_{ir}, a_{ir+1}, a_{ir+2}, \ldots, a_{ir+r-1}\}, \{b_{jr}, b_{jr+1}, b_{jr+2}, \ldots, b_{jr+r-1}\})$ for $0 \leq i \leq p-1$ and $0 \leq j \leq q-1$. Let $M_{i,j}^{(\ell)}$ (respectively, $C_{i,j}^{(\ell)}$) be the matching (respectively, cycle) with the edges labeled ℓ (respectively, labeled ℓ and $\ell+1$) in $G_{i,j}$. Note that $C_{i,j}^{(\ell_1)}$, $M_{i,j'}^{(\ell_2)}$ and $M_{i',j}^{(\ell_3)}$, for $i \neq i'$, $j \neq j'$, constitute a $4r$-sun. The following lemma for the sun decomposition of the complete bipartite graph is needed for our discussions.

Lemma 2.1. *For an integer $r \geq 2$, then $K_{4r,2r}$ has a $\mathbb{SUN}_{4r}$-decomposition.*

Proof. We distinguish two cases by the values of r.

Case 1. r is even.

First, we have that $K_{2r,2r}$ has a $\mathbb{SUN}_{4r}$-decomposition as follows.

$$\begin{aligned} K_{2r,2r} &= \cup_{j=0}^{1} \cup_{i=0}^{1} G_{i,j} \\ &= \cup_{\ell=0}^{\frac{r}{2}-1} [(C_{0,0}^{(2\ell)} \cup M_{0,1}^{(2\ell)} \cup M_{1,0}^{(2\ell)}) \cup (C_{1,1}^{(2\ell)} \cup M_{1,0}^{(2\ell+1)} \cup M_{0,1}^{(2\ell+1)})]. \end{aligned}$$

Since $K_{4r,2r} = K_{2r,2r} \cup K_{2r,2r}$, it follows that $K_{4r,2r}$ can be decomposed into $2r$ copies of $\mathbb{SUN}_{4r}$.

Case 2. r is odd.

Similar to Case 1, we have that

$$\begin{aligned} K_{4r,2r} &= \cup_{j=0}^{1} \cup_{i=0}^{3} G_{i,j} \\ &= \cup_{\ell=0}^{\frac{r-3}{2}} [(C_{0,0}^{(2\ell)} \cup M_{0,1}^{(2\ell)} \cup M_{1,0}^{(2\ell)}) \cup (C_{1,1}^{(2\ell)} \cup M_{1,0}^{(2\ell+1)} \cup M_{0,1}^{(2\ell+1)}) \cup \\ &\qquad (C_{2,0}^{(2\ell)} \cup M_{2,1}^{(2\ell)} \cup M_{3,0}^{(2\ell)}) \cup (C_{3,1}^{(2\ell)} \cup M_{3,0}^{(2\ell+1)} \cup M_{2,1}^{(2\ell+1)})] \cup \\ &\qquad (\cup_{j=0}^{1} \cup_{i=0}^{3} M_{i,j}^{(r-1)}). \end{aligned}$$

On the other hand, since

$$\begin{aligned} & (C_{0,0}^{(0)} \cup M_{0,1}^{(0)} \cup M_{1,0}^{(0)}) \cup (C_{1,1}^{(r-3)} \cup M_{1,0}^{(r-2)} \cup M_{0,1}^{(r-2)}) \cup (\cup_{j=0}^{1} \cup_{i=0}^{3} M_{i,j}^{(r-1)}) \\ = \; & (C_{0,0}^{(0)} \cup M_{0,1}^{(0)} \cup M_{2,0}^{(r-1)}) \cup (C_{1,1}^{(r-3)} \cup M_{1,0}^{(r-2)} \cup M_{2,1}^{(r-1)}) \cup \\ & \qquad (C_{1,0}^{(r-1)} \cup M_{1,1}^{(r-1)} \cup M_{3,0}^{(r-1)}) \cup (C_{0,1}^{(r-2)} \cup M_{0,0}^{(r-1)} \cup M_{3,1}^{(r-1)}), \end{aligned}$$

it follows that $K_{4r,2r}$ can be decomposed into $2r$ copies of $\mathbb{SUN}_{4r}$. □

3 Main results

Suppose $\mathbb{S}$ is the $4r$-sun

$$\begin{pmatrix} v_1 & v_2 & \dots & v_{2r} \\ w_1 & w_2 & \dots & w_{2r} \end{pmatrix}$$

in $C_{n,n-1}$, and μ is a nonnegative integer. Then we use $\mathbb{S} + \mu$ to be denote the the $4r$-sun

$$\begin{pmatrix} v_1 + \mu & v_2 + \mu & \dots & v_{2r} + \mu \\ w_1 + \mu & w_2 + \mu & \dots & w_{2r} + \mu \end{pmatrix},$$

where the indices are taken modulo n. In this paper we prove the following results.

Theorem A. *Suppose that $n \geq 9$, $r \geq 2$ are integers such that $4r|n-1$. Then $C_{n,n-1}$ has a $\mathbb{SUN}_{4r}$-decomposition.*

Theorem B. *Suppose that $n \geq 8$, $r \geq 2$ are integers such that $4r|n$. Then $C_{n,n-1}$ has a $\mathbb{SUN}_{4r}$-decomposition.*

We now prove Theorem A. Let us begin with Lemma 3.1.

Lemma 3.1. *$C_{4r+1,4r}$ has a $\mathbb{SUN}_{4r}$-decomposition for all $r \geq 2$.*

Proof. We distinguish two cases by the values of r.

Case 1: r is even.

Let $\mathbb{S}$ be the $4r$-sun

$$\mathbb{S}=\begin{pmatrix} b_1 & a_0 & b_2 & a_{4r} & b_3 & a_{4r-1} & b_4 & a_{4r-2} & \cdots \\ a_{r+2} & b_{4r} & a_{r+5} & b_{4r-2} & a_{r+7} & b_{4r-4} & a_{r+9} & b_{4r-6} & \cdots \end{pmatrix.$$
$$\begin{matrix} \cdots & b_{r-2} & a_{3r+4} & b_{r-1} & a_{3r+3} & b_r & a_{2r+2} \\ \cdots & a_{3r-3} & b_{2r+6} & a_{3r-1} & b_{2r+4} & a_{3r+2} & b_{r+2} \end{pmatrix}.$$

It is not difficult to check that all vertices of $\mathbb{S}$ are distinct. In addition, one sees that that $\mathbb{S}$ consists of edges in the cycle with labels $1,2,3,4,5,6,7,\cdots,2r-5,2r-4,2r-3,2r-2,3r-1,2r$ and the pendant edges with labels $3r,4r,3r-2,4r-1,3r-3,4r-2,3r-4,4r-3,\ldots,2r+2,3r+3,2r+1,3r+2,2r-1,3r+1$ consecutively. Thus $C_{4r+1,4r}$ is decomposed into the following $4r$-suns: $\mathbb{S}+\mu$ ($\mu=0,1,2,\cdots,4r$).

Case 2: r is odd.

Let $\mathbb{S}$ be the $4r$-sun

$$\mathbb{S}=\begin{pmatrix} b_1 & a_0 & b_2 & a_{4r} & b_3 & a_{4r-1} & b_4 & a_{4r-2} & \cdots \\ a_{r+2} & b_{4r} & a_{2r+4} & b_{4r-2} & a_{r+6} & b_{4r-4} & a_{r+8} & b_{4r-6} & \cdots \end{matrix}$$
$$\begin{matrix} \cdots & b_{r-2} & a_{3r+4} & b_{r-1} & a_{3r+3} & b_r & a_{2r+2} \\ \cdots & a_{3r-4} & b_{2r+6} & a_{3r-2} & b_{2r+4} & a_{3r} & b_{r+2} \end{pmatrix}.$$

It is not difficult to check that all vertices of $\mathbb{S}$ are distinct. In addition, one sees that that $\mathbb{S}$ consists of edges in the cycle with labels $1,2,3,4,5,6,7,\cdots,2r-5,2r-4,2r-3,2r-2,3r-1,2r$ and the pendant edges with labels $3r,4r,2r-1,4r-1,3r-2,4r-2,3r-3,4r-3,\ldots,2r+3,3r+3,2r+2,3r+2,2r+1,3r+1$ consecutively. Thus $C_{4r+1,4r}$ is decomposed into the following $4r$-suns: $\mathbb{S}+\mu$ ($\mu=0,1,2,\cdots,4r$). □

Proof of Theorem A. Let $n=4rq+1$ where $r\geq 2$, $q\geq 1$. We need to show that $C_{4rq+1,4rq}$ has a $\mathbb{SUN}_{4r}$-decomposition. We prove the result by induction on q. By Lemma 3.1, the case $q=1$ is true. Let $q\geq 2$ and suppose the result holds for values smaller than q. Note that $C_{4rq+1,4rq}$ can be decomposed into subgraphs G_1,G_2,G_3,G_4 where $G_1\cong C_{4r+1,4r}$, $G_2\cong C_{4r(q-1)+1,4r(q-1)}$ and $G_3\cong G_4\cong K_{4r,4r(q-1)}$. By the induction hypothesis, both G_1 and G_2 have $\mathbb{SUN}_{4r}$-decompositions. Since $K_{4r,4r(q-1)}=\underbrace{K_{4r,2r}\cup K_{4r,2r}\cup\cdots\cup K_{4r,2r}}_{2(q-1)\text{ copies of }K_{4r,2r}}$, by Lemma 2.1, both G_3 and G_4 have $\mathbb{SUN}_{4r}$-decompositions. Thus $C_{4rq+1,4rq}$ has a $\mathbb{SUN}_{4r}$-decomposition. □

We next prove Theorem B. Let us begin with Lemma 3.2.

Lemma 3.2. *$C_{4r,4r-1}$ has a $\mathbb{SUN}_{4r}$-decomposition for all $r\geq 2$.*

Proof. We distinguish two cases by the values of r.

Case 1. r is even.

First, we have that $C_{4r,4r-1}$ has a decomposition as follows.

$$\begin{aligned}C_{4r,4r-1} &= \cup_{j=0}^{3}\cup_{i=0}^{3} G_{i,j}\\ &= \cup_{\ell=0}^{\frac{r}{2}-1}[(C_{0,1}^{(2\ell)}\cup M_{0,0}^{(2\ell+1)}\cup M_{1,1}^{(2\ell+1)})\cup(C_{2,3}^{(2\ell)}\cup M_{2,2}^{(2\ell+1)}\cup M_{3,3}^{(2\ell+1)})\cup\\ &\quad (C_{2,0}^{(2\ell)}\cup M_{2,1}^{(2\ell)}\cup M_{3,0}^{(2\ell)})\cup(C_{3,1}^{(2\ell)}\cup M_{3,0}^{(2\ell+1)}\cup M_{2,1}^{(2\ell+1)})\cup\\ &\quad (C_{0,2}^{(2\ell)}\cup M_{0,3}^{(2\ell)}\cup M_{1,2}^{(2\ell)})\cup(C_{1,3}^{(2\ell)}\cup M_{1,2}^{(2\ell+1)}\cup M_{0,3}^{(2\ell+1)})]\cup\\ &\quad \cup_{\ell=1}^{\frac{r}{2}-1}[(C_{1,0}^{(2\ell)}\cup M_{1,1}^{(2\ell)}\cup M_{0,0}^{(2\ell)})\cup(C_{3,2}^{(2\ell)}\cup M_{3,3}^{(2\ell)}\cup M_{2,2}^{(2\ell)})]\cup\\ &\quad (M_{1,0}^{(0)}\cup M_{1,0}^{(1)})\cup(M_{3,2}^{(0)}\cup M_{3,2}^{(1)}).\end{aligned}$$

On the other hand, since

$$\begin{aligned}&(C_{2,0}^{(0)}\cup M_{2,1}^{(0)}\cup M_{3,0}^{(0)})\cup(C_{3,1}^{(0)}\cup M_{3,0}^{(1)}\cup M_{2,1}^{(1)})\cup(M_{1,0}^{(0)}\cup M_{1,0}^{(1)})\cup(M_{3,2}^{(0)}\cup M_{3,2}^{(1)})\\ =\;&(C_{2,0}^{(0)}\cup M_{2,1}^{(0)}\cup M_{1,0}^{(0)})\cup(C_{3,1}^{(0)}\cup M_{3,2}^{(0)}\cup M_{2,1}^{(1)})\cup(C_{3,0}^{(0)}\cup M_{3,2}^{(1)}\cup M_{1,0}^{(1)}),\end{aligned}$$

it follows that $C_{4r,4r-1}$ can be decomposed into $4r-1$ copies of $\mathbb{SUN}_{4r}$.

Case 2. r is odd.

Similar to Case 1, we have that

$$\begin{aligned}C_{4r,4r-1} &= \cup_{j=0}^{3}\cup_{i=0}^{3} G_{i,j}\\ &= \cup_{\ell=0}^{\frac{r-3}{2}}[(C_{2,0}^{(2\ell)}\cup M_{2,1}^{(2\ell)}\cup M_{3,0}^{(2\ell)})\cup(C_{3,1}^{(2\ell)}\cup M_{3,0}^{(2\ell+1)}\cup M_{2,1}^{(2\ell+1)})\cup\\ &\quad (C_{1,2}^{(2\ell)}\cup M_{1,3}^{(2\ell)}\cup M_{0,2}^{(2\ell)})\cup(C_{0,3}^{(2\ell)}\cup M_{0,2}^{(2\ell+1)}\cup M_{1,3}^{(2\ell+1)})\cup\\ &\quad (C_{0,0}^{(2\ell+1)}\cup M_{0,1}^{(2\ell)}\cup M_{1,0}^{(2\ell)})\cup(C_{1,1}^{(2\ell+1)}\cup M_{1,0}^{(2\ell+1)}\cup M_{0,1}^{(2\ell+1)})\cup\\ &\quad (C_{2,2}^{(2\ell+1)}\cup M_{2,3}^{(2\ell)}\cup M_{3,2}^{(2\ell)})\cup(C_{3,3}^{(2\ell+1)}\cup M_{3,2}^{(2\ell+1)}\cup M_{2,3}^{(2\ell+1)})]\cup\\ &\quad (\cup_{i\neq j}M_{i,j}^{(r-1)}).\end{aligned}$$

On the other hand, since

$$\begin{aligned}&(C_{2,0}^{(0)}\cup M_{2,1}^{(0)}\cup M_{3,0}^{(0)})\cup(C_{3,3}^{(r-2)}\cup M_{3,2}^{(r-2)}\cup M_{2,3}^{(r-2)})\\ &\quad\cup(C_{1,2}^{(0)}\cup M_{1,3}^{(0)}\cup M_{0,2}^{(0)})\cup(\cup_{i\neq j}M_{i,j}^{(r-1)})\\ =\;&(C_{2,0}^{(0)}\cup M_{2,1}^{(0)}\cup M_{1,0}^{(r-1)})\cup(C_{3,0}^{(r-1)}\cup M_{3,1}^{(r-1)}\cup M_{2,0}^{(r-1)})\cup\\ &\quad (C_{3,3}^{(r-2)}\cup M_{3,2}^{(r-2)}\cup M_{0,3}^{(r-1)})\cup(C_{2,3}^{(r-2)}\cup M_{2,1}^{(r-1)}\cup M_{1,3}^{(r-1)})\cup\\ &\quad (C_{1,2}^{(0)}\cup M_{1,3}^{(0)}\cup M_{3,2}^{(r-1)})\cup(C_{0,2}^{(r-1)}\cup M_{0,1}^{(r-1)}\cup M_{1,2}^{(r-1)}),\end{aligned}$$

it follows that $C_{4r,4r-1}$ can be decomposed into $4r-1$ copies of $\mathbb{SUN}_{4r}$. □

Proof of Theorem B. Let $n = 4rq$ where $r \geq 2$, $q \geq 1$. We need to show that $C_{4rq,4rq-1}$ has a $\mathbb{SUN}_{4r}$-decomposition. We prove the result by induction on q. By Lemma 3.2, the case $q = 1$ is true. Let $q \geq 2$ and suppose the result holds for values smaller than q. Note that $C_{4rq,4rq-1}$ can be decomposed into subgraphs G_1', G_2', G_3', G_4' where $G_1' \cong C_{4r,4r-1}$, $G_2' \cong C_{4r(q-1),4r(q-1)-1}$ and $G_3' \cong G_4' \cong K_{4r,4r(q-1)}$. By the induction hypothesis, both G_1' and G_2' have $\mathbb{SUN}_{4r}$-decompositions. Since $K_{4r,4r(q-1)} = \underbrace{K_{4r,2r} \cup K_{4r,2r} \cup \cdots \cup K_{4r,2r}}_{2(q-1) \text{ copies of } K_{4r,2r}}$, by Lemma 2.1, both G_3' and G_4' have $\mathbb{SUN}_{4r}$-decompositions. Thus $C_{4rq,4rq-1}$ has a $\mathbb{SUN}_{4r}$-decomposition. □

References

[1] R. Anitha, R.-S. Lekshmi, N-sun decomposition of complete graphs and complete bipartite graphs, *World Acad. Sci. Eng. Tech.*, **27** (2007), 262-266.

[2] R. Anitha, R.-S. Lekshmi, N-sun decomposition of complete, complete bipartite and some Harary graphs, *Int. J. Math. Sci.*, **2** (2008), 33-38.

[3] C.-M. Fu, N.-H. Jhuang, Y.-L. Lin and H.-M. Sung, From Steiner triple systems to 3-sun systems, *Taiwanese J. Math.*, **16** (2012), no. 2, 531-543.

[4] C.-M. Fu, N.-H. Jhuang, Y.-L. Lin and H.-M. Sung, On the existence of k-sun systems, *Discrete Math.*, **312** (2012), 1931-1939. http://dx.doi.org/10.1016/j.disc.2012.03.007

[5] C.-M. Fu, M.-H. Huang and Y.-L. Lin, On the existence of 5-sun systems, *Discrete Math.*, **313** (2013), 2942-2950. http://dx.doi.org/10.1016/j.disc.2013.09.007

[6] F. Harary, *Graph Theory*, Addison-Wesley, New York, 1969.

[7] Z. Liang, J. Guo, J. Wang, On the crown graph decompositions containing odd cycle, *Int. J. Comb. Graph Theory Appl.*, **2** (2008), 125-160.

[8] Z. Liang, J. Guo, Decomposition of complete multigraphs into crown graphs, *J. Appl. Math. Comput.*, **32** (2010), 507-517. http://dx.doi.org/10.1007/s12190-009-0267-0

[9] C. Lin, J.-J. Lin and T.-W. Shyu, Complete Bipartite Decomposition of Crowns, with Applications to Complete Directed Graphs, Chapter in *Combinatorics and Computer Science*, Springer Berlin Heidelberg, 1996, 58-66. http://dx.doi.org/10.1007/3-540-61576-8_74

[10] C. Lin, J.-J. Lin, and T.-W. Shyu, Isomorphic Star Decompositions of Multicrowns and the Power of Cycles, *Ars Combin.*, **53** (1999), 249-256.

[11] T.-W. Shyu, *The Decomposition of Complete Graphs, Complete Bipartite Graphs and Crowns*, Ph.D. Thesis, Department of Mathematics, National Central University, Taiwan, 1997.

Received: March 16, 2016; Published: April 25, 2016