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# $\{C_k, P_k, S_k\}$-Decompositions of Balanced Complete Bipartite Graphs

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**Abstract**

Let $L = \{H_1, H_2, \ldots, H_r\}$ be a family of subgraphs of a graph $G$. An $L$-decomposition of $G$ is an edge-disjoint decomposition of $G$ into positive integer $\alpha_i$ copies of $H_i$, where $i \in \{1, 2, \ldots, r\}$. Let $C_k$, $P_k$ and $S_k$ denote a cycle, a path and a star with $k$ edges, respectively. In this paper, we prove that a balanced complete bipartite graph with $2n$ vertices has a $\{C_k, P_k, S_k\}$-decomposition if and only if $k$ is even, $4 \leq k \leq n$ and $n^2 \equiv 0 \pmod{k}$.

**Mathematics Subject Classification:** 05C51

**Keywords:** cycle, path, star, decomposition, balanced complete bipartite graph

## 1 Introduction

Let $L = \{H_1, H_2, \ldots, H_r\}$ be a family of subgraphs of a graph $G$. An $L$-*decomposition* of $G$ is an edge-disjoint decomposition of $G$ into positive integer $\alpha_i$ copies of $H_i$, where $i \in \{1, 2, \ldots, r\}$. Furthermore, if each $H_i (i \in \{1, 2, \ldots, r\})$ is isomorphic to a graph $H$, then we say that $G$ has an $H$-*decomposition*.

For positive integers $m$ and $n$, $K_{m,n}$ denotes the complete bipartite graph with parts of sizes $m$ and $n$. A complete bipartite graph is *balanced* if $m = n$. A $k$-*cycle*, denoted by $C_k$, is a cycle of length $k$. A $k$-*star*, denoted by $S_k$, is the complete bipartite graph $K_{1,k}$. A *k-path*, denoted by $P_k$, is a path with $k$ edges.

Decompositions of some families of graphs into $k$-cycles has been a popular topic of research in graph theory; see [4, 7] for surveys of this topic. Articles of $P_k$-decompositions of interest include [9, 11]. Decompositions of graphs into $k$-stars have also attracted a fair share of interest; see [16, 17, 18]. The study of $\{G, H\}$-decomposition was introduced by Abueida and Daven in [1]. Abueida and Daven [2] investigated the problem of $\{K_k, S_k\}$-decomposition of the complete graph $K_n$. Abueida and O'Neil [3] settled the existence problem for $\{C_k, S_{k-1}\}$-decomposition of the complete multigraph $\lambda K_n$ for $k \in \{3, 4, 5\}$. In [10], Priyadharsini and Muthusamy gave necessary and sufficient conditions for the existence of a $\{G, H\}$-factorization of $\lambda K_n$ where $G, H \in \{C_n, P_{n-1}, S_{n-1}\}$. Furthermore, Shyu [12] investigated the problem of decomposing $K_n$ into paths and stars with $k$ edges, giving a necessary and sufficient condition for $k = 3$. In [13], Shyu considered the existence of a decomposition of $K_n$ into paths and cycles with $k$ edges, giving a necessary and sufficient condition for $k = 4$. Shyu [14] investigated the problem of decomposing $K_n$ into cycles and stars with $k$ edges, settling the case $k = 4$. Recently, Lee [5, 6] established necessary and sufficient conditions for the existence of a $\{C_k, S_k\}$-decomposition of a complete bipartite graph and $\{P_k, S_k\}$-decomposition of a balanced complete bipartite graph. In this paper, we consider the existence of a $\{C_k, P_k, S_k\}$-decomposition of the balanced complete bipartite graph, giving necessary and sufficient conditions.

## 2 Preliminaries

Let $G$ be a graph. The *degree* of a vertex $x$ of $G$, denoted by $\deg_G x$, is the number of edges incident with $x$. The vertex of degree $k$ in $S_k$ is the *center* of $S_k$. For $A \subseteq V(G)$ and $B \subseteq E(G)$, we use $G[A]$ and $G - B$ to denote the subgraph of $G$ induced by $A$ and the subgraph of $G$ obtained by deleting $B$, respectively. When $G_1, G_2 \ldots, G_m$ are graphs, not necessarily disjoint, we write $G_1 \cup G_2 \cup \cdots \cup G_m$ or $\bigcup_{i=1}^{m} G_i$ for the graph with vertex set $\bigcup_{i=1}^{m} V(G_i)$ and edge set $\bigcup_{i=1}^{m} E(G_i)$. When the edge sets are disjoint, $G = \bigcup_{i=1}^{m} G_i$ expresses the decomposition of $G$ into $G_1, G_2, \ldots, G_m$. $nG$ is the short notation for the union of $n$ copies of disjoint graphs isomorphic to $G$. Let $(v_0, v_1, \ldots, v_{k-1})$ denote the cycle $C_k$ with vertices $v_0, v_1, \ldots, v_{k-1}$ and edges $v_0v_1, v_1v_2, \ldots, v_{k-1}v_0$, let $v_0v_1 \ldots v_k$ denote the path $P_k$ with vertices $v_0, v_1, \ldots, v_k$ and edges $v_0v_1, v_1v_2, \ldots, v_{k-1}v_k$ and let $(v_0; v_1, v_2, \ldots, v_k)$ denote the star $S_k$ with centered at vertex $v_0$ and $v_1, v_2, \ldots, v_k$ are other vertices. For any vertex $x$ of a digraph $G$, the *outdegree* $\deg_G^+ x$ (respectively, *indegree* $\deg_G^- x$) of $x$ is the number of arcs incident from (respectively, to) $x$.

**Proposition 2.1.** (Sotteau [15]) *For positive integers $m$, $n$ and $k$, the graph $K_{m,n}$ has a $C_k$-decomposition if and only if $m, n$ and $k$ are even, $k \geq 4$, $\min\{m, n\} \geq k/2$, and $mn \equiv 0 \pmod{k}$.*

**Proposition 2.2.** (Ma et al. [8]) *For positive integers $n$ and $k$, the graph obtained by deleting a 1-factor from $K_{n,n}$ has a $C_k$-decomposition if and only if $n$ is odd, $k$ is even, $4 \leq k \leq 2n$, and $n(n-1)$ is divisible by $k$.*

**Lemma 2.3.** *If $k$ is an even integer with $k \geq 4$, then there exist $(k/2-1)$ edge-disjoint $k$-cycles in $K_{k/2,k}$.*

*Proof.* If $k \equiv 0 \pmod{4}$, then $k/2$ is even. By Proposition 2.1, there exists a $C_k$-decomposition $\mathscr{H}$ of $K_{k/2,k}$ with $|\mathscr{H}| = k/2$, in which $k$-cycles are edge-disjoint. If $k \equiv 2 \pmod{4}$, then $k/2$ is odd. Proposition 2.2 implies that $K_{k/2,k/2}$ with a 1-factor removed has a $C_k$-decomposition $\mathscr{H}'$ with $|\mathscr{H}'| = (k-2)/4$. Hence there exist $2(k-2)/4 = k/2-1$ edge-disjoint $k$-cycles in $K_{k/2,k}$. This completes the proof. □

**Proposition 2.4.** (Parker [9]) *There exists a $P_k$-decomposition of $K_{m,n}$ if and only if $mn \equiv 0 \pmod{k}$ and one of the following cases holds.*

| Case | $k$ | $m$ | $n$ | Conditions |
|---|---|---|---|---|
| 1 | even | even | even | $k \leq 2m, k \leq 2n$, not both equalities |
| 2 | even | even | odd | $k \leq 2m-2, k \leq 2n$ |
| 3 | even | odd | even | $k \leq 2m, k \leq 2n-2$ |
| 4 | odd | even | even | $k \leq 2m-1, k \leq 2n-1$ |
| 5 | odd | even | odd | $k \leq 2m-1, k \leq n$ |
| 6 | odd | odd | even | $k \leq m, k \leq 2n-1$ |
| 7 | odd | odd | odd | $k \leq m, k \leq n$ |

By Proposition 2.4, the following result can be obtained.

**Lemma 2.5.** *If $k$ is an even integer with $k \geq 4$, then there exist $k/2$ edge-disjoint $k$-paths in $K_{k/2,k}$.*

**Proposition 2.6.** (Yamamoto et al. [18]) *For integers $m$ and $n$ with $m \geq n \geq 1$, the graph $K_{m,n}$ has an $S_k$-decomposition if and only if $m \geq k$ and*

$$\begin{cases} m \equiv 0 \pmod{k} & \text{if } n < k \\ mn \equiv 0 \pmod{k} & \text{if } n \geq k. \end{cases}$$

# 3 Main results

The goal of this paper is to settle the $\{C_k, P_k, S_k\}$-decomposition problem for $K_{n,n}$. We prove the following theorem.

**Main Theorem.** *Let $k$ and $n$ be positive integers. The graph $K_{n,n}$ has a $\{C_k, P_k, S_k\}$-decomposition if and only if $k$ is even, $4 \le k \le n$ and $n^2$ is divisible by $k$.*

We first give necessary conditions for a $\{C_k, P_k, S_k\}$-decomposition of $K_{n,n}$.

**Lemma 3.1.** *If $K_{n,n}$ has a $\{C_k, P_k, S_k\}$-decomposition, then $k$ is even, $4 \le k \le n$ and $n^2 \equiv 0 \pmod{k}$.*

*Proof.* Since bipartite graphs contain no odd cycle, $k$ is even. In addition, the minimum length of a cycle and the maximum size of a star in $K_{n,n}$ are 4 and $n$, respectively, we have $4 \le k \le n$. Finally, the size of each member in the decomposition is $k$ and $|E(K_{n,n})| = n^2$; thus $n^2 \equiv 0 \pmod{k}$. □

Throughout this paper, let $(A, B)$ denote the bipartition of $K_{n,n}$, where $A = \{a_0, a_1, \ldots, a_{n-1}\}$ and $B = \{b_0, b_1, \ldots, b_{n-1}\}$. We begin the discussion with the smallest value of $k$, namely $k = 4$.

**Lemma 3.2.** *For an even integer $n \ge 4$, then $K_{n,n}$ has a $\{C_4, P_4, S_4\}$-decomposition.*

*Proof.* First, $K_{4,4}$ can be decomposed into the following one copy of $C_4$, two copies of $P_4$ and one copy of $S_4$: $(b_0, a_0, b_1, a_1)$, $b_2a_0b_3a_2b_1$, $b_3a_1b_2a_2b_0$ and $(a_3; b_0, b_1, b_2, b_3)$. Note that $K_{n,n} = K_{4,4} \cup K_{n-4,4} \cup K_{n,n-4}$ for $n \ge 6$. In addition, by Proposition 2.1, $K_{n-4,4}$ and $K_{n,n-4}$ have $C_4$-decompositions. Hence there exists a $\{C_4, P_4, S_4\}$-decomposition of $K_{n,n}$ for even $n \ge 4$. □

With Lemma 3.2 in mind, it is assumed that $k \ge 6$ in the sequel. We now show that the necessary conditions are also sufficient. The proof is divided into cases $n = k$, $k < n < 2k$, and $n \ge 2k$, which are treated in Lemmas 3.3, 3.4, and 3.5, respectively.

**Lemma 3.3.** *For an even integer $k \ge 6$, then $K_{k,k}$ has a $\{C_k, P_k, S_k\}$-decomposition.*

*Proof.* We distinguish two cases by the values of $k$.

Case 1. $k \equiv 0 \pmod{4}$.

Then $k/2$ is even and $K_{k,k} = 2K_{k/2,k/2+2} \cup K_{k,k/2-2}$ for $k \ge 6$. By Propositions 2.1 and 2.4, $2K_{k/2,k/2+2}$ has a $C_k$-decomposition and a $P_k$-decomposition. In addition, by Proposition 2.6, $K_{k,k/2-2}$ has an $S_k$-decomposition. Hence, $K_{k,k}$ has a $\{C_k, P_k, S_k\}$-decomposition.

Case 2. $k \equiv 2 \pmod{4}$.

Let $G = K_{k,k}[\{a_0, a_1, \ldots, a_{k/2-1}\} \cup \{b_0, b_1, \ldots, b_{k/2}\}]$, $F = K_{k,k}[\{a_{k/2}, a_{k/2+1}, \ldots, a_{k-1}\} \cup \{b_0, b_1, \ldots, b_{k/2}\}]$ and $H = K_{k,k}[\{a_0, a_1, \ldots, a_{k-1}\} \cup \{b_{k/2+1}, b_{k/2+2} \ldots,$

$b_{k-1}\}]$. Note that $K_{k,k} = G \cup F \cup H$. We will show that $G \cup F$ can be decomposed into two copies of $C_k$ and $(k/2-1)$ copies of $P_k$ as follows.

First, a decomposition of $G \cup F$ into $k$-paths is given by the $(k/2+1)$ following paths:

$$P^{(i,j)} = b_{2j}a_{ik/2}b_{2j+1}a_{ik/2+1}\ldots b_{2j+k/2-1}a_{ik/2+(k/2-1)}b_{2j+k/2}$$

for $i = 0, 1$ and $j = 0, 1, \ldots, (k-2)/4$, where the subscripts of $b$ are taken modulo $(k/2+1)$.

Next, let $P^{(0,1)'}$ and $P^{(1,0)'}$ be two new $k$-paths obtained by

$$P^{(0,1)'} = P^{(0,1)} \cup \{a_{k/2}b_0, a_{k/2}b_1\} - \{a_{k/2-1}b_0, a_{k/2-1}b_1\},$$
$$P^{(1,0)'} = P^{(1,0)} \cup \{a_{k/2-1}b_0, a_{k/2-1}b_1\} - \{a_{k/2}b_0, a_{k/2}b_1\}.$$

Finally, $C^{(1)}$ and $C^{(2)}$ are two $k$-cycles are obtained by

$$C^{(1)} = P^{(0,0)} \cup \{a_{k/2-1}b_0\} - \{a_{k/2-1}b_{k/2}\},$$
$$C^{(2)} = P^{(1,0)'} \cup \{a_{k/2-1}b_{k/2}\} - \{a_{k/2-1}b_0\}.$$

Thus $G \cup F$ can be decomposed into $C^{(1)}$, $C^{(2)}$, $P^{(0,1)'}$, $P^{(1,1)}$ and $P^{(i,j)}$ for $i = 0, 1$, $j = 2, 3, \ldots, (k-2)/4$. On the other hand, by Proposition 2.6, $H$ has an $S_k$-decomposition. Hence $K_{k,k}$ has a $\{C_k, P_k, S_k\}$-decomposition. □

**Lemma 3.4.** *Let $k$ be a positive even integer and let $n$ be a positive integer with $6 \le k < n < 2k$. If $n^2$ is divisible by $k$, then $K_{n,n}$ has a $\{C_k, P_k, S_k\}$-decomposition.*

*Proof.* Let $n = k + r$. From the assumption $k < n < 2k$, we have $0 < r < k$. Let $t = r^2/k$. Since $k \mid n^2$, we have $k \mid r^2$, which implies that $t$ is a positive integer. The proof is divided into two parts according to the value of $t$.

Case 1. $t = 1$.

Then $k = r^2$. This implies that $r \ge 4$ and $k \ge 4r$. Let

$$\begin{aligned}
G_1 &= K_{n,n}[\{a_r, a_{r+1}, \ldots, a_{k-1}\}, \{b_r, b_{r+1}, \ldots, b_{k-1}\}],\\
G_2 &= K_{n,n}[\{a_0, a_1, \ldots, a_{r-1}\}, \{b_0, b_1, \ldots, b_{k-1}\}],\\
G_3 &= K_{n,n}[\{a_r, a_{r+1}, \ldots, a_{k+r-1}\}, \{b_0, b_1, \ldots, b_{r-1}\}],\\
G_4 &= K_{n,n}[\{a_k, a_{k+1}, \ldots, a_{k+r-1}\}, \{b_r, b_{r+1}, \ldots, b_{k+r-1}\}],\\
G_5 &= K_{n,n}[\{a_0, a_1, \ldots, a_{k-1}\}, \{b_k, b_{k+1}, \ldots, b_{k+r-1}\}].
\end{aligned}$$

Note that $G_1 = K_{k-r,k-r}$ and $G_i = K_{k,r}$ (or $K_{r,k}$) for $2 \le i \le 5$. Clearly $K_{n,n} = G_1 + G_2 + G_3 + G_4 + G_5$. By Propositions 2.1 and 2.6 , $G_5$ has a $C_k$-decomposition and $G_i$ has a $S_k$-decomposition for $2 \le i \le 5$ .

By Sotteau ([15], p.77), there are $(r-1)^2$ copies of $k$-cycles in the decomposition $\mathscr{D}$ of $G_1$. We take two $k$-cycles in $\mathscr{D}$:

$$C_{0,0} = (a_1, b_{k/4-1}, a_{k/2-2}, b_{k/4-2}, a_{k/2-4}, b_{k/4-3}, a_{k/2-6}, b_{k/4-4}, \dots, a_2, b_0, a_0, b_{k/2-1}, a_{k/2-1}, b_{k/2-2}, a_{k/2-3}, b_{k/2-3}, a_{k/2-5}, b_{k/2-4}, \dots, a_3, b_{k/4}),$$
$$C_{0,1} = (a_{r+k/2-2}, b_{k/4-1}, a_{r+1}, b_{k/4}, a_{r+3}, b_{k/4+1}, a_{r+5}, b_{k/4+2}, \dots, a_{r+k/2-1}, b_{k/2-1}, a_r, b_0, a_{r+2}, b_1, a_{r+4}, b_2, a_{r+6}, b_3, \dots, a_{r+k/2-4}, b_{k/4-2})$$

and interchange the two edges $a_1 b_{k/4-1}$ and $a_{r+k/2-2} b_{k/4-1}$. In doing so, we obtain two paths: $P_{0,0} = C_{0,0} - \{a_1 b_{k/4-1}\} \cup \{a_{r+k/2-2} b_{k/4-1}\}$ and $P_{0,1} = C_{0,1} - \{a_{r+k/2-2} b_{k/4-1}\} \cup \{a_1 b_{k/4-1}\}$. Thus $K_{n,n}$ can be decomposed into $(r-1)^2 - 2$ copies of $C_k$, two copies of $P_k$ and $4r$ copies of $S_k$. This settles the case 1.

Case 2. $t \geq 2$.

Let $G'_0 = K_{n,n}[\{a_0, a_1, \dots, a_{k/2-1}\} \cup \{b_0, b_1, \dots, b_{k-1}\}]$ and $G'_1 = K_{n,n}[\{a_{k/2}, a_{k/2+1}, \dots, a_{k-1}\} \cup \{b_0, b_1, \dots, b_{k-1}\}]$, $F' = K_{n,n}[\{a_k, a_{k+1}, \dots, a_{k+r-1}\} \cup \{b_0, b_1, \dots, b_{k-1}\}]$, and $H' = K_{n,n}[\{a_0, a_1, \dots, a_{k+r-1}\} \cup \{b_k, b_{k+1}, \dots, b_{k+r-1}\}]$. Clearly $K_{n,n} = G'_0 \cup G'_1 \cup F' \cup H'$. Note that $G'_0$ and $G'_1$ are isomorphic to $K_{k/2,k}$, $H'$ is isomorphic to $K_{n,r}$, and $F'$ is isomorphic to $K_{r,k}$, which can be decomposed into $r$ copies of $S_k$ by Proposition 2.6.

Let $p_0 = \lceil t/2 \rceil$ and $p_1 = \lfloor t/2 \rfloor$. In the following, we will show that $G'_0$ can be decomposed into $p_0$ copies of $C_k$ and $k/2$ copies of $S_{k-2p_0}$, $G'_1$ can be decomposed into $p_1$ copies of $P_k$ and $k/2$ copies of $S_{k-2p_1}$, $H'$ can be decomposed into $k/2$ copies of $S_{2p_0}$, $k/2$ copies of $S_{2p_1}$ and $r$ copies of $S_k$.

We first show the required decomposition of $G'_0$ and $G'_1$. Since $r < k$, we have $t < r$. Thus, $p_0 = \lceil t/2 \rceil \leq (t+1)/2 \leq r/2 < k/2$, which implies $p_i \leq k/2 - 1$ for $i \in \{0, 1\}$. This assures us that there exist $p_0$ edge-disjoint $k$-cycles in $G'_0$ and $p_1$ edge-disjoint $k$-paths in $G'_1$ by Lemmas 2.3 and 2.5, respectively. Suppose that $Q_{0,0}, Q_{0,1}, \dots, Q_{0,p_0-1}$ and $Q_{1,0}, Q_{1,1}, \dots, Q_{1,p_1-1}$ are edge-disjoint $k$-cycles and $k$-paths in $G'_0$ and $G'_1$, respectively. Let $W'_i = G'_i - E(\bigcup_{h=0}^{p_i-1} Q_{i,h})$ and $X_{i,j} = W'_i[\{a_{ik/2+j}\} \cup \{b_0, b_1, \dots, b_{k-1}\}]$ where $i = 0, 1$ and $j = 0, 1, \dots, k/2-1$. Since $\deg_{G'_i} a_{ik/2+j} = k$ and each $Q_{i,h}$ uses two edges incident with $a_{ik/2+j}$ for each $i$ and $j$, we have $\deg_{W'_i} a_{ik/2+j} = k - 2p_i$. Hence $X_{i,j}$ is a $(k-2p_i)$-star with center $a_{ik/2+j}$ for $i = 0, 1$ and $j = 0, 1, \dots, k/2-1$.

Next we show the required star-decompositions of $H'$. Equivalently we need show that there exists an orientation of $H'$ such that, for $i = 0, 1$, $j = 0, 1, \dots, k/2-1$, and $w = k, k+1, \dots, k+r-1$,

$$\deg^+_{H'} a_{ik/2+j} = 2p_i \quad (1)$$
$$\deg^+_{H'} b_w = k. \quad (2)$$

We begin the orientation. For $j = 0, 1, \dots, k/2-1$ the edges $a_j b_{k+(2p_0)j}$, $a_j b_{k+(2p_0)j+1}, \dots, a_j b_{k+(2p_0)j+2p_0-1}$ and $a_{k/2+j} b_{(p_0+1)k+(2p_1)j}, a_{k/2+j} b_{(p_0+1)k+(2p_1)j+1}, \dots, a_{k/2+j} b_{(p_0+1)k+(2p_1)j+2p_1-1}$ are oriented from $a_{ik/2+j}$ where the subscripts of

$b$'s are taken modulo $r$ in the set of numbers $\{k, k+1, \ldots, k+r-1\}$. Note that from each $a_{ik/2+j}$, we orient $2p_i$ edges. Since $2p_1 \le 2p_0 \le t+1 \le r$, this assures us that there are enough edges for the above orientation. Finally, the edges which are not oriented yet are all oriented from $\{b_k, b_{k+1}, \ldots, b_{k+r-1}\}$ to $\{a_0, a_1, \ldots, a_{k+r-1}\}$.

It is easy to see that (1) is satisfied. Thus we only need to check (2). From the construction of the orientation, for all $w, w' \in \{k, k+1, \ldots, k+r-1\}$, we have

$$|\deg^-_{H'} b_w - \deg^-_{H'} b_{w'}| \le 1. \tag{3}$$

Since $\deg^+_{H'} b_w + \deg^-_{H'} b_w = k+r$ for $w \in \{k, k+1, \ldots, k+r-1\}$, it follows from (3) that $|\deg^+_{H'} b_w - \deg^+_{H'} b_{w'}| \le 1$ for $w, w' \in \{k, k+1, \ldots, k+r-1\}$. Furthermore,

$$\begin{aligned}\sum_{w=k}^{k+r-1} \deg^+_{H'} b_w &= |E(K_{n,r})| - \sum_{i=0}^{1}\sum_{j=0}^{k/2-1} \deg^+_{H'} a_{ik/2+j} \\ &= (k+r)r - (2p_0 + 2p_1)(k/2) \\ &= (k+r)r - tk \\ &= (k+r)r - r^2 \\ &= kr\end{aligned}$$

Thus $\deg^+_{H'} b_w = k$ for $w \in \{k, k+1, \ldots, k+r-1\}$. This proves (2). Hence there exists a decomposition $\mathscr{D}'$ of $H'$ into $k/2$ copies of $S_{2p_0}$ and $k/2$ copies of $S_{2p_1}$ with centers in $\{a_0, a_1, \ldots, a_{k/2-1}\}$ and $\{a_{k/2}, a_{k/2+1}, \ldots, a_{k-1}\}$ as well as $r$ copies of $S_k$ with centers in $\{b_k, b_{k+1}, \ldots, b_{k+r-1}\}$. Let $X'_{i,j}$ be the $(2p_i)$-star with center $a_{ik/2+j}$ in $\mathscr{D}'$. Note that $X_{i,j} \cup X'_{i,j}$ is a $k$-star for $i = 0, 1$ and $j = 0, 1, \ldots, k/2-1$. Thus $K_{n,n}$ can be decomposed into $p_0$ copies of $C_k$, $p_1$ copies of $P_k$ and $k+2r$ copies of $S_k$. This completes the proof. □

**Lemma 3.5.** *Let $k$ be a positive even integer and let $n$ be a positive integer with $6 \le k \le n/2$. If $n^2$ is divisible by $k$, then $K_{n,n}$ has a $\{C_k, P_k, S_k\}$-decomposition.*

*Proof.* Let $n = qk + r$ where $q$ and $r$ are integers with $0 \le r < k$. From the assumption of $k \le n/2$, we have $q \ge 2$. Note that

$$K_{n,n} = K_{qk+r,qk+r} = K_{(q-1)k,(q-1)k} \cup K_{k+r,(q-1)k} \cup K_{(q-1)k,k+r} \cup K_{k+r,k+r}.$$

Trivially, $|E(K_{(q-1)k,(q-1)k})|$, $|E(K_{k+r,(q-1)k})|$ and $|E(K_{(q-1)k,k+r})|$ are multiples of $k$. Thus $(k+r)^2 \equiv 0 \pmod{k}$ from the assumption that $n^2$ is divisible by $k$. By Proposition 2.6, $K_{(q-1)k,(q-1)k}$, $K_{k+r,(q-1)k}$ and $K_{(q-1)k,k+r}$ have $S_k$-decomposition.

The case of $r = 0$, by Lemma 3.3, we obtain that $K_{k,k}$ has a $\{C_k, P_k, S_k\}$-decomposition. In addition, by Lemma 3.4, $K_{k+r,k+r}$ has a $\{C_k, P_k, S_k\}$-decomposition for $0 < r < k$. Hence there exists a $\{C_k, P_k, S_k\}$-decomposition of $K_{n,n}$. □

Now Lemmas 3.1, 3.2, 3.3, 3.4 and 3.5 serve to prove the Main Theorem.

# References


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