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Singularities of Asymptotic Lines on Surfaces in R^4

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Abstract

We give the local topological classification of configurations of asymptotic lines of generic locally convex surfaces around isolated inflection points in $\mathbb{R}^4$.

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1 Introduction

In this paper we give the local topological classification of configurations of asymptotic lines defined around isolated inflection points of generic locally convex surfaces in $\mathbb{R}^4$. Such local configurations are given by solution curves of differential equations of asymptotic lines (DEAL) which is a certain class of binary differential equation of the form

$$a(x,y)dy^2 + b(x,y)dxdy + c(x,y)dx^2 = 0 \tag{1}$$

with $b^2 - 4ac \geq 0$, where a, b, c are smooth functions which vanish at a point taking $b^2 - 4ac = 0$. If a point in the surface in $\mathbb{R}^4$ is an isolated inflection point, it takes $b^2 - 4ac = 0$ (hence $a = b = c = 0$) and otherwise satisfies $b^2 - 4ac > 0$. The local topological classification of integral curves of binary differential equation of this type is well-known ([6], [1], [2], [10]). Especially such differential equations also appear as differential equations of principal curvature lines around umbilic points of surfaces in $\mathbb{R}^3$ ([4], [10], [1]), and the stable configurations of its integral curves are well-known as Darbouxian which are determined by the 1-jet of this differential equation. Moreover Gutierrez, Sotomayor and Garcia ([11], [16]) established umbilic bifurcations. In [3] Bruce and Tari pointed out the 1-jet of DEAL is precisely the 1-jet of (1) giving the principal curvature lines of a surface in $\mathbb{R}^3$ at the umbilic point. Hence its configuration follows the same pattern.

While Guíñez and Gutierrez([6],[7],[8],[9]) studied singularities of positive quadratic differential forms which are the left-hand side of (1), and these classification were established.

Applying results of Guíñez and Gutierrez([6],[7],[8],[9]), we first investigate singularities depending on the 1-jet of DEAL. Then we give the local topological classification and conditions of possible rank-2 singular point (Theorem 3.1). In the classification, Darbouxian (D_1, D_2, D_3), D_{12} and $\widetilde{D_1}$ appear, which is actually the same as the case of the principal curvature line of a surface in $\mathbb{R}^3$ at the umbilic point. Next we investigate a singularity depending on the 2-jet of DEAL and describes a condition for type D_{23}(Theorem 4.2) which is rank-1 codimension one singularities studied in [9]. In a particular case, the condition for D_{23} corresponds to that for type D_{23}^1 which describes a codimension one umbilic bifurcation studied in [11]. Recently the configurations near umbilic points for surfaces in $\mathbb{R}^4$ were studied in [15], and it was shown that the same configurations as our classification of DEAL appears.

The configurations of the integral curves of DEAL at each singular point above are illustrated in Figure 1.

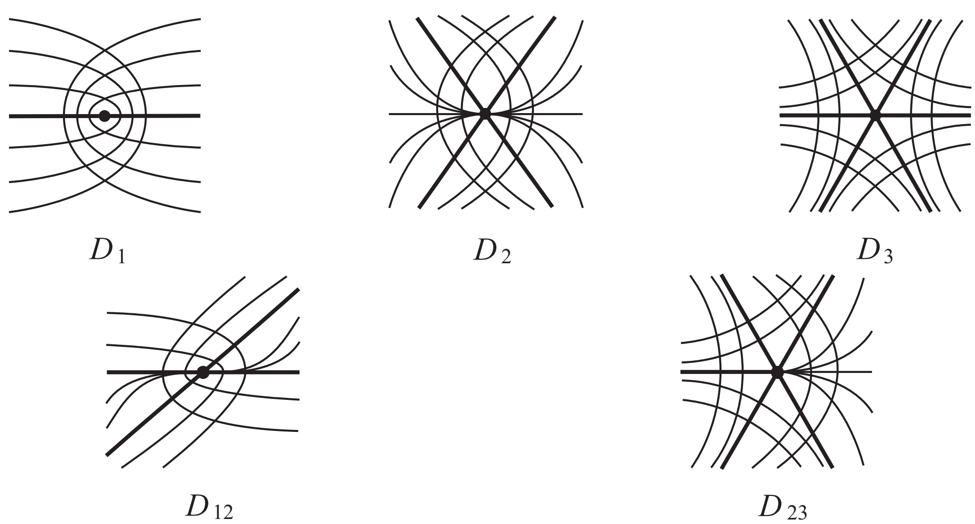

Figure 1: Asymptotic lines near the inflection points D_i, D_{12}, D_{23}, and their separatrices.

We shall suppose that all mappings, map germs and manifolds are of class C^∞, unless otherwise stated.

2 Singularities of positive quadratic differential forms and geometry of surfaces in $\mathbb{R}^4$

According to Guíñez ([6],[7]), we first review elements of the singularities of quadratic differential form

$$\omega = a(x,y)dy^2 + b(x,y)dxdy + c(x,y)dx^2$$

on $\mathbb{R}^2 = \{(x,y)\}$. Let $\Delta_\omega = b^2 - 4ac : \mathbb{R}^2 \to \mathbb{R}$, and let $g_\omega = (a,b,c) : \mathbb{R}^2 \to \mathbb{R}^3$. Then ω is called *positive* if $\Delta_\omega \geq 0$ and $\Delta_\omega^{-1}(0) = g_\omega^{-1}(0)$. Hereafter ω is assumed to be positive quadratic differential forms (PQD). A point in $\mathbb{R}^2$ is said to be *singular point* of ω if Δ_ω vanishes (so is g_ω). Two PQDs ω_1, ω_2 are called *equivalent* if there exists homeomorphism h on $\mathbb{R}^2$ such that h maps the singular set of ω_1 to that of ω_2, and maps the integral curves of $\omega_1 = 0$ to that of $\omega_2 = 0$. A singular point p is said to be of *rank-k* ($k = 0, 1, 2$) if $\mathrm{rank} Dg_\omega(p) = k$, where $Dg_\omega(p)$ is the Jacobian matrix of g_ω at p. Among rank-2 singular points, a point is said to be *simple* if Δ_ω is non-degenerate minimum, and a non-simple point is said to be *semi-simple.* We remark that a singular point is isolated if it is of rank-2.

Let $S(\omega,p)(x,y) := da_p(x,y)y^2 + db_p(x,y)xy + dc_p(x,y)x^2$, where da_p, db_p, dc_p are the linear parts of a, b, c at p respectively. This polynomial is called the *separatrix polynomial.* Then among rank-2 simple singular points, a point p is said to be

(a) *hyperbolic* if $S(\omega,p)$ has only simple roots,

(b) of type D_{12} if $S(\omega,p)$ has one simple and one double roots,

(c) of type $\widetilde{D_1}$ if $S(\omega,p)$ has a triple root.

Moreover the hyperbolic singular points has the three cases (so-called Darbouxian) D_1, D_2, D_3 in Proposition 2.1 below. Semi-simple singularities are also classified in [7].

We now assume that the origin is a rank-2 simple singular point of ω. Let $a_1 = \frac{\partial a}{\partial x}(0), a_2 = \frac{\partial a}{\partial y}(0)$, and also denote by b_i, c_i ($i = 1, 2$) the differential of b, c at 0 respectively. Then by a linear change of coordinate we may suppose that

$$Dg_\omega(0) = \begin{pmatrix} 0 & 1 \\ b_1 & b_2 \\ 0 & -1 \end{pmatrix} \tag{2}$$

with $b_1 \neq 0$. Then the separatrix polynomial is given by $S(\omega, 0) = y(y^2 + b_2xy + (b_1 - 1)x^2)$.

Proposition 2.1. ([6],[7]) *Let ω be a PQD having the linear part (2) above. Suppose that the origin is a rank-2 simple singular point. Then the singular point is one of the following cases:*

$D_1\ :\ \frac{b_2^2}{4}+1-b_1<0\,,$
$D_2\ :\ \frac{b_2^2}{4}+1-b_1>0,\, b_1>0\quad and\quad b_1\neq 1\,,$
$D_3\ :\ b_1<0\,,$
$D_{12}\ :\ b_2\neq 0\quad and\ either\ \frac{b_2^2}{4}+1-b_1=0\quad or\ b_1=1\,,$
$\widetilde{D_1}\ :\ b_1=1\quad and\quad b_2=0.$

Remark 2.2. *The configuration of* $\widetilde{D_1}$ *is homeomorphic to that of* D_1*([8], [16]).*

Next we recall previous results of geometry of surfaces in $\mathbb{R}^4$ (see [5], [13], [14]). Let $f : M^2 \to \mathbb{R}^4$ be an embedding, where M is a compact surface. For a given point $p \in M$, the "curvature ellipse" is defined as a mapping $\eta : S^1 \to N_{f(p)}f(M)$, where S^1 is the unit circle in $T_{f(p)}f(M)$ parametrized by the angle $\theta \in [0, 2\pi]$, and $N_{f(p)}f(M)$ is the normal plane. The image of the curvature ellipse η is an ellipse in $N_{f(p)}f(M)$. This ellipse may degenerate on to a radial segment of a straight line, in which case $f(p)$ is called an *inflection point* of the surface. This inflection point is of *real type* if $f(p)$ belongs to the curvature ellipse, and of *imaginary type* if it doesn't. A direction θ in $T_{f(p)}f(M)$ is said to be an *asymptotic direction* if $\frac{\partial\eta}{\partial\theta}$ and $\eta(\theta)$ are parallel. There are the following cases for asymptotic directions at $f(p)$: exactly two directions ($f(p)$ is called a hyperbolic point), just one direction ($f(p)$ is called a parabolic point) and all directions (this case occurs at any inflection point). If no direction is asymptotic, then the point is called elliptic.

In [14] it is shown that for a generically embedded surface $f(M)$ the set of hyperbolic points is a non empty open region. The line field of asymptotic directions at hyperbolic points "collapse" at the set of parabolic points and inflection points which are made of the union of curves and isolated points. Their integral lines are called *asymptotic lines* and their singular points coincide with the inflection points. In particular if $f(M)$ is locally convex (i.e. has a locally support hyperplane at each point), then $f(M)$ has no elliptic points, and can only have inflection points of imaginary type. Moreover these inflection points of imaginary type are isolated.

Let f be an embedding of a compact surface M to $\mathbb{R}^4$ such that $f(M)$ is locally convex, and let $f(p)$ be an isolated inflection point of imaginary type. In [12] it is shown that there exist local coordinates at p and $f(p)$, f around p is given by $f : (\mathbb{R}^2, 0) \to (\mathbb{R}^4, 0)$ which is defined by

$$f(x, y) = (x, y, \frac{1}{2}(x^2 + y^2) + F(x, y), G(x, y)),$$

where F and G are function germs such that $j^2F(0) = j^2G(0) = 0$. Conversely for any F, G with $j^2F(0) = j^2G(0) = 0$ the embedding f defined by the above gives a local parametrization around an isolated inflection point of imaginary type.

Moreover in [12] the differential equation of the asymptotic lines (DEAL) of f is obtained as the following form

$$\begin{vmatrix} dy^2 & -dxdy & dx^2 \\ 1+F_{xx} & F_{xy} & 1+F_{yy} \\ G_{xx} & G_{xy} & G_{yy} \end{vmatrix} = 0.$$

Then DEAL is rewritten as

$$(-G_{xy}+\Delta_{23})dy^2 + (G_{yy}-G_{xx}+\Delta_{13})dxdy + (G_{xy}+\Delta_{12})dx^2 = 0 \quad (3)$$

where

$$\Delta_{23} = \begin{vmatrix} F_{xy} & F_{yy} \\ G_{xy} & G_{yy} \end{vmatrix}, \Delta_{13} = \begin{vmatrix} F_{xx} & F_{yy} \\ G_{xx} & G_{yy} \end{vmatrix}, \Delta_{12} = \begin{vmatrix} F_{xx} & F_{xy} \\ G_{xx} & G_{xy} \end{vmatrix}.$$

We remark that the quadratic differential form giving the left-hand side of DEAL (3) is positive around the origin, the isolated singular point.

For DEAL (3) we have

$$Dg_\omega(0) = \begin{pmatrix} -G_{xxy}(0) & -G_{xyy}(0) \\ G_{xyy}(0)-G_{xxx}(0) & G_{yyy}(0)-G_{xxy}(0) \\ G_{xxy}(0) & G_{xyy}(0) \end{pmatrix}.$$

By a rotation in (x, y)-plane we may suppose that $G_{xxy}(0) = 0$. In a particular case that the linear part of ω is such a form, we need the following lemma.

Lemma 2.3. *Let ω be a PQD with $a_1 = -c_1 = 0$ and $a_2 = -c_2$. Then*
(i) All rank-2 singular points are simple, that is, there is no semi-simple singular point.
(ii) The origin is a simple singular point if and only if $a_2 \neq 0$ and $b_1 \neq 0$.
(iii) The singular point of rank-2 is one of the following cases:
D_1 : $(\frac{b_2}{2a_2})^2 + 1 - \frac{b_1}{a_2} < 0$,
D_2 : $(\frac{b_2}{2a_2})^2 + 1 - \frac{b_1}{a_2} > 0$, $\frac{b_1}{a_2} > 0$ *and* $\frac{b_1}{a_2} \neq 1$,
D_3 : $\frac{b_1}{a_2} < 0$,
D_{12} : $b_2 \neq 0$ *and either* $(\frac{b_2}{2a_2})^2 + 1 - \frac{b_1}{a_2} = 0$ *or* $\frac{b_1}{a_2} = 1$,
$\widetilde{D_1}$: $b_2 = 0$ *and* $\frac{b_1}{a_2} = 1$.

Proof. (i), (ii) By direct calculations we have $\frac{\partial^2 \Delta_\omega}{\partial x^2}(0) = b_1^2$, $\frac{\partial^2 \Delta_\omega}{\partial y^2}(0) = b_2^2 + 4a_2^2$, the determinant of Hess $\Delta_\omega(0) = 4a_2^2 b_1^2$, where Hess $\Delta_\omega(0)$ is the Hessian matrix of Δ_ω at 0. Hence it is easily shown that the origin is a singular point of rank-2 if and only if the determinant of Hess $\Delta_\omega(0) \neq 0$, that is, $a_2 \neq 0$ and $b_1 \neq 0$. Therefore we conclude (i), (ii).

(iii) The separatrix polynomial is given by $S(\omega, 0)(x, y) = y(a_2 y^2 + b_2 xy + (b_1 - a_2)x^2)$. The equation $S(\omega, 0)(x, y) = 0$ has always a solution $y = 0$.

Consider the equation $a_2(\frac{y}{x})^2 + b_2(\frac{y}{x}) + (b_1 - a_2) = 0$, and its discriminant $D/4 = (b_2/2)^2 + a_2^2 - b_1a_2$.

The case $b_1 = a_2$: If $b_2 = 0$ (resp. $\neq 0$), then $S(\omega, 0)(x, y) = 0$ has the triple root $y = 0$ (resp. the simple root $y = mx (m \neq 0)$ and the double root $y = 0$).

The case $b_1 \neq a_2$: If $D/4 = 0$ (resp. $\neq 0$), then $S(\omega, 0)(x, y) = 0$ has the simple root $y = 0$ and the double root $y = mx (m \neq 0)$ (resp. only simple root).

Therefore by the definitions of $\widetilde{D_1}, D_{12}$ we get the conditions for these singularities. For the hyperbolic cases D_1, D_2, D_3 we find the conditions as follows.

Let $\varphi : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear change of coordinates such that $x = \alpha u + \beta v$, $y = \gamma u + \delta v$. Let $\widetilde{\omega}$ be the pull-back of ω by φ. Denote by

$$\widetilde{\omega} = \varphi^*\omega = \widetilde{a}\, dv^2 + \widetilde{b}\, dudv + \widetilde{c}\, du^2, \quad Dg_{\widetilde{\omega}}(0) = \begin{pmatrix} \widetilde{a_1} & \widetilde{a_2} \\ \widetilde{b_1} & \widetilde{b_2} \\ \widetilde{c_1} & \widetilde{c_2} \end{pmatrix}.$$

Then we have the following formula

$$\begin{array}{rcl} \widetilde{a} & = & a(\varphi)\delta^2 + b(\varphi)\beta\delta + c(\varphi)\beta^2, \\ \widetilde{b} & = & 2a(\varphi)\gamma\delta + b(\varphi)(\alpha\delta + \beta\gamma) + 2c(\varphi)\alpha\beta, \\ \widetilde{c} & = & a(\varphi)\gamma^2 + b(\varphi)\alpha\gamma + c(\varphi)\alpha^2. \end{array}$$

Now set $\beta = 0$, $\gamma = 0$. Then we have

$$Dg_{\omega}(0) = \begin{pmatrix} 0 & a_2 \\ b_1 & b_2 \\ 0 & -a_2 \end{pmatrix}, \; Dg_{\widetilde{\omega}}(0) = \begin{pmatrix} 0 & \delta^3 a_2 \\ \alpha^2\delta b_1 & \alpha\delta^2 b_2 \\ 0 & -\alpha^2\delta a_2 \end{pmatrix}.$$

By taking α, δ such that $\widetilde{a_2} = \delta^3 a_2 = -1$, $\widetilde{c_2} = -\alpha^2\delta a_2 = -1$, we have

$$Dg_{\widetilde{\omega}}(0) = \begin{pmatrix} 0 & 1 \\ \frac{b_1}{a_2} & \frac{b_2}{a_2} \\ 0 & -1 \end{pmatrix}.$$

Then applying Proposition 2.1 we get the conditions for D_1, D_2, D_3. □

3 Rank-2 singularities

Hereafter the left-hand side of DEAL (3) is denoted by

$$\omega = Ldy^2 + Mdxdy + Ndx^2.$$

Since we may suppose that $G_{xxy}(0) = 0$, write

$$G(x,y) = \frac{a}{6}x^3 + \frac{b}{2}xy^2 + \frac{c}{6}y^3 + \text{h.o.t.}$$

Then the linear parts of L, M, N, namely $Dg_\omega(0) = \begin{pmatrix} 0 & -b \\ b-a & c \\ 0 & b \end{pmatrix}$ is rank-2 if and only if $b(b-a) \neq 0$. Applying Lemma 2.3 we have the following classification for rank-2 singularities depending on the 1-jet of DEAL.

Theorem 3.1. *The rank-2 singular points of PQD ω for DEAL (3) at the origin is one of the following types:*
D_1 : $(\frac{c}{2b})^2 - \frac{a}{b} + 2 < 0$,
D_2 : $(\frac{c}{2b})^2 - \frac{a}{b} + 2 > 0$ *and* $1 < \frac{a}{b} \neq 2$,
D_3 : $\frac{a}{b} < 1$,
D_{12} : $bc(a-b) \neq 0$ *and either* $(\frac{c}{2b})^2 - \frac{a}{b} + 2 = 0$ *or* $\frac{a}{b} = 2$,
$\widetilde{D_1}$: $c = 0$ *and* $\frac{a}{b} = 2$.

Remark 3.2. *Type D_{12} corresponds to type D_2^1 with codimension 1 for umbilic points of surfaces in $\mathbb{R}^3$ studied in [11], also type $\widetilde{D_1}$ corresponds to type D_1^2 with codimension 2 studied in [16].*

4 D_{23}-singular point

In [9] the following type of a singular point of PQD is defined to study a generic 1-parameter bifurcation.

Definition 4.1. *Let $\omega = Ldy^2 + Mdxdy + Ndx^2$ be a PQD. Suppose that the origin is rank-1 singular point of ω. Then the origin is called D_{23}-singular point if*

$$Dg_\omega(0) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \\ 0 & -1 \end{pmatrix} \quad \textit{and} \quad \frac{\partial^2 M}{\partial x^2}(0) \neq 0.$$

We need the 2-jet to give a condition that a singular point of ω for DEAL (3) is of type D_{23}. Now write G in DEAL (3) as

$$G(x,y) = \frac{a}{6}x^3 + \frac{b}{2}xy^2 + \frac{c}{6}y^3 + \frac{A}{24}x^4 + \frac{B}{6}x^3y + \frac{C}{4}x^2y^2 + \frac{D}{6}xy^3 + \frac{E}{24}y^4 + \text{h.o.t.}$$

By direct calculations, we have

$$\begin{aligned} L &= -by - \frac{1}{2}(B - 2bF_{xxy}(0))x^2 - (C - cF_{xxy}(0))xy \\ &\quad - \frac{1}{2}(D - 2cF_{xyy}(0) + 2bF_{yyy}(0))y^2 + \text{h.o.t.} \\ M &= (b-a)x + cy + \frac{1}{2}(C - A + 2bF_{xxx}(0) - 2aF_{xyy}(0))x^2 \\ &\quad + (D - B + cF_{xxx}(0) - aF_{yyy}(0) + bF_{xxy}(0))xy \\ &\quad + \frac{1}{2}(E - C + 2cF_{xxy}(0))y^2 + \text{h.o.t.} \\ N &= by + \frac{1}{2}(B - 2aF_{xxy}(0))x^2 + (C + bF_{xxx}(0) - aF_{xyy}(0))xy \\ &\quad + \frac{1}{2}(D + 2bF_{xxy}(0))y^2 + \text{h.o.t.} \end{aligned}$$

Theorem 4.2. *Suppose that the origin is rank-1 singular point, that is, $a^2 + b^2 + c^2 \neq 0$ and $b(b - a) = 0$. Then the origin is D_{23}–singular point if and only if G, F in DEAL (3) satisfy either (i) or (ii) below:*
(i) $a = b \neq 0$ and $cB - b(C - A) \neq 2acF_{xxy}(0) - 2a^2(F_{xyy}(0) - F_{xxx}(0))$
(ii) $b = 0$ and $ac \neq 0$ and $c^2B + 2acC + a^2D \neq 2ac(cF_{xxy}(0) + aF_{xyy}(0))$

Proof. (i) Suppose that $a = b \neq 0$. Calculations give

$$\begin{aligned} L &= -ay - \frac{1}{2}(B - 2aF_{xxy}(0))x^2 - (C - cF_{xxy}(0))xy \\ &\quad - \frac{1}{2}(D - 2cF_{xyy}(0) + 2aF_{yyy}(0))y^2 + \text{h.o.t.} \\ M &= cy + \frac{1}{2}(C - A + 2aF_{xxx}(0) - 2aF_{xyy}(0))x^2 \\ &\quad + (D - B + cF_{xxx}(0) - aF_{yyy}(0) + aF_{xxy}(0))xy \\ &\quad + \frac{1}{2}(E - C + 2cF_{xxy}(0))y^2 + \text{h.o.t.} \\ N &= ay + \frac{1}{2}(B - 2aF_{xxy}(0))x^2 + (C + aF_{xxx}(0) - aF_{xyy}(0))xy \\ &\quad + \frac{1}{2}(D + 2aF_{xxy}(0))y^2 + \text{h.o.t.} \end{aligned}$$

Let $\widetilde{\omega} = \varphi^*\omega = \widetilde{L}dv^2 + \widetilde{M}dudv + \widetilde{N}du^2$ be the pull-back of ω by a linear change of coordinates φ defined by $x = \alpha u + \beta v$, $y = \gamma u + \delta v$. We have

$$\begin{aligned} \widetilde{L} &= L(\varphi)\delta^2 + M(\varphi)\beta\delta + N(\varphi)\beta^2, \\ \widetilde{M} &= 2L(\varphi)\gamma\delta + M(\varphi)(\alpha\delta + \beta\gamma) + 2N(\varphi)\alpha\beta, \\ \widetilde{N} &= L(\varphi)\gamma^2 + M(\varphi)\alpha\gamma + N(\varphi)\alpha^2. \end{aligned}$$

Set $\alpha = [(4a^2 + c^2)/(4a^4)]^{1/6}$, $\beta = c/(2a^2\alpha^2)$, $\gamma = 0$, $\delta = -1/(a\alpha^2)$. Then it follows that $Dg_{\widetilde{\omega}}(0)$ satisfies the required condition of type D_{23} in Definition 4.1.

Moreover, by direct calculations, we have

$$\frac{\partial^2 \widetilde{M}}{\partial u^2}(0) = a^{-2}\alpha \left(a(A-C) + cB - 2acF_{xxy}(0) + 2a^2(F_{xyy}(0) - F_{xxx}(0))\right).$$

Therefore by Definition 4.1 the singular point satisfying the condition (i) is of type D_{23}.

(ii) Suppose that $b = 0$ and $ac \neq 0$. Calculations give

$$\begin{aligned} L &= -\frac{1}{2}Bx^2 - (C - cF_{xxy}(0))xy - \frac{1}{2}(D - 2cF_{xyy}(0))y^2 + \text{h.o.t.} \\ M &= -ax + cy + \frac{1}{2}(C - A - 2aF_{xyy}(0))x^2 \\ &\quad + (D - B + cF_{xxx}(0) - aF_{yyy}(0))xy \\ &\quad + \frac{1}{2}(E - C + 2cF_{xxy}(0))y^2 + \text{h.o.t.} \\ N &= \frac{1}{2}(B - 2aF_{xxy}(0))x^2 + (C - aF_{xyy}(0))xy + \frac{1}{2}Dy^2 + \text{h.o.t.} \end{aligned}$$

By the same argument as (i), setting $\alpha = \beta = (\frac{c}{2a^2})^{\frac{1}{3}}$, $\gamma = (\frac{a}{2c^2})^{\frac{1}{3}}$, $\delta = -\gamma$, $Dg_{\widetilde{\omega}}(0)$ satisfies the required condition of type D_{23} in Definition 4.1.

Calculations give

$$\frac{\partial^2 \widetilde{M}}{\partial u^2}(0) = 2^{-\frac{1}{3}}(a^2+c^2)(ac)^{-\frac{8}{3}}(c^2B + 2acC + a^2D - 2ac^2F_{xxy}(0) - 2a^2cF_{xyy}(0)).$$

Therefore the singular point satisfying the condition (ii) is of type D_{23}.

Conversely if the conditions (i) or (ii) is not satisfied, then we see that $Dg_{\omega}(0)$ does not satisfies the required condition in Definition 4.1 by Remark 14 in [9]. □

Remark 4.3. *If the 3-jet of F at the origin vanishes, a condition (i) for type D_{23} corresponds to the condition for type D^1_{23} with codimension 1 studied in [11]. The classification in the case not satisfying the condition (i) or (ii) (this case corresponds to a case with codimension 2 studied in [16]) shall be studied in a forthcoming paper.*

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References

[1] J.W. Bruce and D.L. Fidal, On binary differential equations and umbilics, *Proceedings of the Royal Society of Edinburgh*, **111A** (1989), 147–168. http://dx.doi.org/10.1017/s0308210500025087

[2] J.W. Bruce and F. Tari, On binary differential equations, *Nonlinearity*, **8** (1995), 255–271. http://dx.doi.org/10.1088/0951-7715/8/2/008

[3] J.W. Bruce and F. Tari, Families of surfaces in $\mathbb{R}^4$, *Proceedings of the Edinburgh Math. Soci.*, **45** (2002), 181–203. http://dx.doi.org/10.1017/s0013091500000213

[4] G. Darboux, Sur la forme des lignes de courbure dans la voisingage d'un ombilic, Leçons sur la théorie générale des surfaces IV, Note VII, Gauthiers-Villars, 1896.

[5] R.A. Garcia, D.K.H. Mochida, M.D.C. Romero-Fuster and M.A.S. Ruas, Inflection points and topology of surfaces in 4-space, *Trans. Amer. Math. Soc.*, **352** (2000), 3029–3043. http://dx.doi.org/10.1090/s0002-9947-00-02404-1

[6] V. Guíñez, Positive quadratic differential forms and foliations with singularities on surfaces, *Trans. Amer. Math. Soc.*, **309** (1988), 477–502. http://dx.doi.org/10.1090/s0002-9947-1988-0961601-4

[7] V. Guíñez, Local stable singularities for positive quadratic differential forms, *J. Differential Equations*, **110** (1994), 1–37. http://dx.doi.org/10.1006/jdeq.1994.1057

[8] C. Gutierrez and V. Guíñez Positive quadratic differential forms: linearization, finite determinacy and versal unfolding, *Annales de la Faculté des Sciences de Toulouse Mathmatiques*, **5** (1996), 661–690. http://dx.doi.org/10.5802/afst.844

[9] V. Guíñez and C.Gutierrez, Rank-1 codimension one singularities of positive quadratic differential forms, *J. Differential Equations*, **206** (2004), 127–155. http://dx.doi.org/10.1016/j.jde.2004.07.015

[10] C. Gutierrez and J. Sotomayor, Structurally stable configurations of lines of principal curvature, *Astérisque*, **98-99** (1982), 195–215.

[11] C. Gutierrez, J. Sotomayor and R. Garcia, Bifurcations of umbilic points and related principal cycles, *J. Dynamics and Differential Equations*, **16** (2004), no. 2, 321–346. http://dx.doi.org/10.1007/s10884-004-2783-9

[12] C. Gutierrez and M. A. S. Ruas, Indices of Newton non-degenerate vector fields and a conjecture of Loewner for surfaces in $\mathbb{R}^4$, Chapter in *Real and Complex Singularities*, Lecture Notes in Pure and Appl. Math., **232** Dekker, 2003, 245–253. http://dx.doi.org/10.1201/9780203912089.ch12

[13] J.A. Little, On singularities of submanifolds of higher dimensional Euclidean space, *Annal. Mat. Pura et Appl.*, **83** (1969), 261–335. http://dx.doi.org/10.1007/bf02411172

[14] D.K.H. Mochida, M.D.C. Romero-Fuster and M.A.S. Ruas, The geometry of surfaces in 4-space from a contact viewpoint, *Geometriae Dedicata*, **54** (1995), 323–332. http://dx.doi.org/10.1007/bf01265348

[15] M. Navarro and A. Serrano, Codimension one bifurcations of non simple umbilical points for surfaces immersed in $\mathbb{R}^4$, *Abstraction & Applications*, **13** (2015), 15–26.

[16] J. Sotomayor and R. Garcia, Codimension two umbilic points on surfaces immersed in $\mathbb{R}^3$, *Discrete and Continuous Dynamical Systems*, **17** (2007), no. 2, 293–308. http://dx.doi.org/10.3934/dcds.2007.17.293

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