International Journal of Contemporary Mathematical Sciences
Vol. 11, 2016, no. 6, 275 - 279
HIKARI Ltd, www.m-hikari.com
http://dx.doi.org/10.12988/ijcms.2016.6525

Homomorphism on Fuzzy Generalised Lattices

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Abstract

This paper deals with generalised lattice homomorphic images and pre-images of fuzzy subgeneralised lattices, fuzzy ideals (filters) and fuzzy prime ideals (filters) of generalised lattices.

Mathematics Subject Classification: 06xx

Keywords: poset, lattice, generalised lattice, fuzzy set, fuzzy lattice

1 Introduction

Murty and Swamy [2] introduced the concept of generalised lattice. In [4], the author developed the theory of generalised lattices that can play an intermediate role between the theories of lattices and posets. The concepts of fuzzy subgeneralised lattice, fuzzy ideal(filter), fuzzy prime ideal(filter) of a generalised lattice are introduced and studied their properties by the author in [5]. In this paper first discussed about the homomorphic images and pre-images of fuzzy subgeneralised lattices, fuzzy ideals (filters) and fuzzy prime ideals (filters) of generalised lattices. Later proved that under certain conditions, for a generalised lattice homomorphism, there is one-one correspondence between fuzzy prime ideals (filters) of domain generalised lattice and of codomain generalised lattice.

2 Preliminary Notes

Through out this section, we denote by P a generalised lattice.

Definition 2.1 *([5]) A fuzzy set μ in P is said to be a fuzzy subgeneralised lattice if for any finite subset A of P, (i) $\mu(s) \geq min_{a\in A}\{\mu(a)\}$ for all $s \in mu(A)$ and (ii) $\mu(t) \geq min_{a\in A}\{\mu(a)\}$ for all $t \in ML(A)$.*

Definition 2.2 *([5]) Let μ be a fuzzy subgeneralised lattice of P. μ is called a fuzzy ideal if $x \leq y$ in P implies $\mu(x) \geq \mu(y)$. μ is called a fuzzy filter if $x \leq y$ in P implies $\mu(x) \leq \mu(y)$.*

Definition 2.3 *([5]) A fuzzy ideal μ of P is said to be a fuzzy prime ideal if for any finite subset A of P, $\mu(t) \leq Max_{a\in A}\{\mu(a)\}$ for all $t \in ML(A)$.*

Definition 2.4 *([5]) A fuzzy filter μ of P is said to be a fuzzy prime filter if for any finite subset A of P, $\mu(s) \leq Max_{a\in A}\{\mu(a)\}$ for all $s \in mu(A)$.*

Definition 2.5 *([4]) Let P_1, P_2 be generalised lattices. A map $f : P_1 \to P_2$ is said to be a homomorphism if for any finite subset A of P_1, $f(ML(A)) = ML(f(A))$ and $f(mu(A)) = mu(f(A))$. A homomorphism $f : P_1 \to P_2$ is said to be a strong homomorphism if $f(x) = f(y)$ for any $x, y \in ML(A)$ and for any $x, y \in mu(A)$.*

Definition 2.6 *([4]) A homomorphism of generalised lattices $f : P_1 \to P_2$ is said to be a strong homomorphism if $f(x) = f(y)$ for any $x, y \in ML(A)$ and for any $x, y \in mu(A)$.*

Definition 2.7 *([3]) A fuzzy set μ in a set S is said to have subproperty if for each $A \subseteq S$, there exists $x_0 \in A$ such that $Sup\{\mu(x) \mid x \in A\} = \mu(x_0)$.*

3 Homomorphic images and pre-images

Through out this section, P, P' are denoted by generalised lattices. The definitions 3.1 and 3.4 of this section for fuzzy sets in generalised lattices are defined in the similar way of the definitions 2.4 and 2.5 in [3] for fuzzy sets in lattices.

Definition 3.1 *Let $f : P \to P'$ be a map. Let μ and η be fuzzy sets in P and P' respectively. Then the fuzzy sets $f(\mu)$ in P' and $f^{-1}(\eta)$ in P are defined as follows: $f(\mu)(y') = Sup\{\mu(y) \mid y \in f^{-1}(y')\}$, if $f^{-1}(y') \neq \phi$ for all $y' \in P'$, otherwise it is equal to 0. $f^{-1}(\eta)(x) = \eta(f(x))$ for all $x \in P$.*

The aim of this paper is to get one-one correspondence between fuzzy prime ideals(filters) of domain and codomain generalised lattices of a homomorphism. In view of that, the first result started with their homomorphic pre-images.

Theorem 3.2 *Let $f : P \to P'$ be homomorphism. Then the following are true: (i) η is a fuzzy subgeneralised lattice of $P' \Rightarrow f^{-1}(\eta)$ is a fuzzy subgeneralised lattice of P (ii) η is a fuzzy ideal (filter) of $P' \Rightarrow f^{-1}(\eta)$ is a fuzzy ideal (filter) of P (iii) η is a fuzzy prime ideal (filter) of $P' \Rightarrow f^{-1}(\eta)$ is a fuzzy prime ideal (filter) of P.*

Proof: (i) Let $x, y \in P$. For any $s \in mu\{x, y\}$, we have $f(s) \in f(mu\{x, y\}) = mu\{f(x), f(y)\}$ and then $\eta(f(s)) \geq min\{\eta(f(x)), \eta(f(y))\}$. Similarly for any $t \in ML\{x,\ y\}$, we can prove $f^{-1}(\eta)(t) \geq min\{f^{-1}(\eta)(x),\ f^{-1}(\eta)(y)\}$. (ii) Let $x,\ y \in P$ and suppose $x \leq y$. Since $f(x) \leq f(y)$ and η is a fuzzy ideal, we have $\eta(f(x)) \geq \eta(f(y))$. (iii) For any $x,\ y \in P$, we have $\eta(f(t)) = \eta(f(x))\ or\ \eta(f(y))$, since $f(t) \in ML\{f(x),\ f(y)\}$ and η is a fuzzy prime ideal.

For the case of homomorphic images of fuzzy ideals(filters), in the following theorem, strong and onto conditions are imposed on the homomorphism.

Theorem 3.3 *Let $f : P \to P'$ be a strong homomorphism. Then the following are true: (i) μ is a fuzzy subgeneralised lattice of $P \Rightarrow f(\mu)$ is a fuzzy subgeneralised lattice of P' (ii) f is onto and μ is a fuzzy ideal (filter) of $P \Rightarrow f(\mu)$ is a fuzzy ideal (filter) of P'*

Proof: (i) Let $x', y' \in P'$. If either $f^{-1}(x')$ or $f^{-1}(y')$ is empty, then $min\{f(\mu)(x'), f(\mu)(y')\} = 0$. Suppose $f^{-1}(x')$ and $f^{-1}(y')$ are non-empty. Choose $x \in f^{-1}(x')$ and $y \in f^{-1}(y')$. Since f is a strong homomorphism, we have $mu\{x', y'\} = mu\{f(x), f(y)\} = f(mu\{x, y\}) = \{f(s)\}$ for any $s \in mu\{x, y\}$. Now consider $min\{f(\mu)(x'), f(\mu)(y')\} = min\{\ Sup\{\mu(p) \mid p \in f^{-1}(f(x))\},\ Sup\{\mu(q) \mid q \in f^{-1}(f(y))\}\ \} =\ Sup\{min\{\mu(p), \mu(q)\} \mid p \in f^{-1}(x'), q \in f^{-1}(y')\} \leq Sup\{Max \{\mu(r)\}_{r \in mu\{p,q\}} \mid p \in f^{-1}(x'),\ q \in f^{-1}(y')\} \leq Sup\{\mu(z) \mid z \in f^{-1}(f(s))\} = f(\mu)(f(s)) = f(\mu)(s')$ for any $s' \in mu\{x', y'\}$. Similarly we can prove (ii).

Definition 3.4 *Let μ be a fuzzy set on P. Then μ is called $f-$ invariant if $f(x) = f(y)$ implies $\mu(x) = \mu(y)$ for all $x,\ y \in P$.*

Lemma 3.5 *Let $f : P \to P'$ be a map. If μ is an $f-$invariant fuzzy set in P then $f^{-1}(f(\mu)) = \mu$*

In the following theorem, proved that onto homomorphic image of a fuzzy ideal (filter) is again the same if it is the map invariant.

Theorem 3.6 *Let $f : P \to P'$ be homomorphism and μ be an $f-$invarient fuzzy set in P. Then the following are true: (i) μ is a fuzzy subgeneralised lattice of $P \Rightarrow f(\mu)$ is a fuzzy subgeneralised lattice of P'. (ii) f is onto and μ is a fuzzy ideal (filter) of $P \Rightarrow f(\mu)$ is a fuzzy ideal (filter) of P'.*

Proof: (i): Let $x', y' \in P'$ and suppose $f^{-1}(x'), f^{-1}(y')$ are non-empty. Choose $x \in f^{-1}(x')$ and $y \in f^{-1}(y')$. For any $s' \in mu\{x', y'\}$, we have $s' = f(s)$ for some $s \in mu\{x, y\}$ and then $f(\mu)(s') = f(\mu)(f(s)) = \mu(s) \geq min\{\mu(x), \mu(y)\} = min\{f(\mu)(f(x)), f(\mu)(f(y))\} = min\{f(\mu)(x'), f(\mu)(y')\}$. (ii): Let $x', y' \in P'$ and suppose $x' \leq y'$. Since f is onto, there exists $x, y \in P$ such that $f(x) = x'$ and $f(y) = y'$. Clearly $y' = f(s)$ for some $s \in mu\{x, y\}$. Now $x \leq s$ and μ is a fuzzy ideal, gives $f(\mu)(x') = f(\mu)(f(x)) = \mu(x) \geq \mu(s) = f(\mu)(f(s)) = f(\mu)(y')$.

Lemma 3.7 *Let $f : P \to P'$ be an onto map. If μ is a fuzzy subgeneralised lattice of with subproperty in P, then $f(\mu)_t = f(\mu_t)$ for $t \in [0, 1]$.*

For the case of homomorphic images of fuzzy prime ideals (filters), in the following theorem proved that onto strong homomorphic image of a fuzzy prime ideal(filter) is again the same if it has the subproperty. The following theorem can be proved by using theorem 3.17 of [5], theorem 3.3 and above lemma.

Theorem 3.8 *Let $f : P \to P'$ be an onto strong homomorphism. If μ is a fuzzy prime ideal (filter) with subproperty in P, then $f(\mu)$ is a fuzzy prime ideal (filter) of P'*

In following theorem, proved that onto homomorphic image of a fuzzy prime ideal (filter) is again the same if it is the map invariant.

Theorem 3.9 *Let $f : P \to P'$ be an onto homomorphism. If μ is an $f-$invarient fuzzy prime ideal (filter) of P, then $f(\mu)$ is a fuzzy prime ideal (filter) of P'.*

Proof: Assume that $f(\mu)$ is not a fuzzy prime ideal of P'. Then by the theorem 3.9 of [5], there exists $f(a), f(b) \in P'$ and $t' \in ML\{f(a), f(b)\}$ such that $f(\mu)(t') \neq f(\mu)(f(a))$ and $f(\mu)(t') \neq f(\mu)(f(b))$. Since f is a homomorphism and μ is an $f-$invarient fuzzy ideal, there exists $t \in ML\{a, b\}$ such that $\mu(t) = f(\mu)(f(t)) > f(\mu)(f(a)) = \mu(a)$ and $\mu(t) = f(\mu)(f(t)) > f(\mu)(f(b)) = \mu(b)$. This is clearly a contradiction.

4 One-one correspondence

The following two theorems are the summary of all the theorems of section 3, so that aim of the paper is fulfilled under certain conditions.

Theorem 4.1 *If $f : P \to P'$ be an onto strong homomorphism, then the map $\mu \to f(\mu)$ defines one-one correspondence between set of all fuzzy prime ideals (filters) with subproperty in P and the set of all fuzzy prime ideals (filters) of P'.*

Theorem 4.2 *If $f : P \to P'$ be an onto homomorphism, then the map $\mu \to f(\mu)$ defines one-one correspondence between set of all $f-$invarient fuzzy prime ideals (filters) of P and the set of all fuzzy prime ideals (filters) of P'.*

Acknowledgements. The author would like to thank Professor M. Krishna Murty for their valuable suggestions during the preparation of the paper.

References

[1] Bo Yuan and W. Wu, Fuzzy ideals on a distributive lattice, *Fuzzy Sets and Systems,* **35** (1990), 231-240. http://dx.doi.org/10.1016/0165-0114(90)90196-d

[2] M.K. Murty and U.M. Swamy, Distributive partially ordered sets, *The Aligarh Bull. of Math.,* **8** (1978), 1-16.

[3] N. Ajmal and K.V. Thomas, Fuzzy lattices, *Information Sciences,* **79** (1994), 271-291. http://dx.doi.org/10.1016/0020-0255(94)90124-4

[4] P.R.K. Kishore, M.K. Murty, V.S. Ramani and M.D.P. Patnaik, On generalised lattices, *Southeast Asian Bulletin of Mathematics,* **33** (2009), 1091-1104.

[5] P.R.K. Kishore, Fuzzy generalized lattices, *International J. of Math. Sci. and Engg. Appls. (IJMSEA),* **9** (2015), no. 3, 1-9.

Received: May 23, 2016; Published: July 5, 2016