International Journal of Contemporary Mathematical Sciences
Vol. 11, 2016, no. 6, 293 - 299
HIKARI Ltd, www.m-hikari.com
http://dx.doi.org/10.12988/ijcms.2016.6418

The Second Largest Number of Maximal Independent Sets in Graphs with at Most Two Cycles

Min-Jen Jou and Jenq-Jong Lin

Ling Tung University, Taichung 40852, Taiwan

Abstract

A maximal independent set is an independent set that is not a proper subset of any other independent set. Jou and Chang determined the largest number of maximal independent sets among all graphs and connected graphs of order n, which contain at most one cycle. Later B. E. Sagan and V. R. Vatter found the largest number of maximal independent sets among all graphs of order n, which contain at most r cycles. In 2012, Jou settled the second largest number of maximal independent sets in graphs with at most one cycle. In this paper, we study the second largest number of maximal independent sets among all graphs of order $n \geq 5$ with at most two cycles. We also characterize those extremal graphs achieving these values.

Mathematics Subject Classification: 05C51

Keywords: independent set, maximal independent set, graph with at most two cycles

1 Introduction

Let $G = (V, E)$ be a simple undirected graph. An *independent set* is a subset S of V such that no two vertices in S are adjacent. A *maximal independent set* is an independent set that is not a proper subset of any other independent set. The set of all maximal independent sets of a graph G is denoted by $\mathrm{MI}(G)$ and its cardinality by $mi(G)$.

The problem of determining the largest value of $mi(G)$ in a general graph of order n and those graphs achieving the largest number was proposed by Erdős and Moser, and solved by Moon and Moser [10]. It was then extensively studied for various classes of graphs in the literature, including trees, forests, (connected) graphs with at most one cycle, bipartite graphs, connected graphs, k-connected graphs, (connected) triangle-free graphs; for a survey see [6]. Recently, Jin and Li [3] determined the second largest number of maximal independent sets among all graphs of order n.

There are researches on independent sets in graphs from a different point of view. The *Fibonacci number* of a graph is the number of independent vertex subsets. The concept of the Fibonacci number of a graph was introduced in [12] and discussed in several papers [9, 13]. In addition, Jou and Chang [8] showed a linear-time algorithm for counting the number of maximal independent sets in a tree.

Jou and Chang [7] determined the largest number of maximal independent sets among all graphs and connected graphs of order n, which contain at most one cycle. Later B. E. Sagan and V. R. Vatter [11] found the largest number of maximal independent sets among all graphs of order n, which contain at most r cycles. In 2012, Jou [4] settled the second largest number of maximal independent sets in graphs with at most one cycle. The purpose of this paper is to determine the second largest number of maximal independent sets among all graphs of order $n \geq 5$ with at most two cycles. We also characterize those extremal graphs achieving these values.

For a graph $G = (V, E)$, the cardinality of $V(G)$ is called the *order*, and it is denoted by $|G|$. The *neighborhood* $N(x)$ of a vertex $x \in V(G)$ is the set of vertices adjacent to x in G and the *closed neighborhood* $N[x]$ is $\{x\} \cup N(x)$. The *degree* of x is the cardinality of $N(x)$, and it is denoted by $\deg(x)$. For a set $A \subseteq V(G)$, the *deletion* of A from G is the graph $G - A$ obtained from G by removing all vertices in A and their incident edges. Two graphs G_1 and G_2 are *disjoint* if $V(G_1) \cap V(G_2) = \emptyset$. The *union* of two disjoint graphs G_1 and G_2 is the graph $G_1 \cup G_2$ with vertex set $V(G_1 \cup G_2) = V(G_1) \cup V(G_2)$ and edge set $E(G_1 \cup G_2) = E(G_1) \cup E(G_2)$. If a graph G is isomorphic to another graph H, we denote $G = H$. Denote K_n a complete graph of order n and C_n a cycle of order n.

2 Preliminaries

We begin with some useful lemmas which are needed in this paper.

Lemma 2.1. ([2, 5]) *For any vertex x in a graph G, $mi(G) \leq mi(G - x) + mi(G - N[x])$.*

Lemma 2.2. ([7]) *For* $n \geq 5$, $mi(C_n) = mi(C_{n-2}) + mi(C_{n-3})$, *where* $mi(C_2) = 2$ *and* $mi(C_3) = 3$.

Lemma 2.3. ([2, 5]) *If* G *is the union of two disjoint graphs* G_1 *and* G_2, *then* $mi(G) = mi(G_1)mi(G_2)$.

3 The main result

We first construct two types of graphs. The *join* of two disjoint graphs G_1 and G_2 is the graph $G_1 + G_2$ with vertex set $V(G_1 \cup G_2) = V(G_1) \cup V(G_2)$ and edge set $E(G_1 \cup G_2) = E(G_1) \cup E(G_2) \cup \{uv : u \in V(G_1) \text{and} v \in V(G_2)\}$. The *star-product* of two disjoint graphs G_1 and G_2 is the graph $G_1 * G_2$ with vertex set $V(G_1 \cup G_2) = V(G_1) \cup V(G_2)$ and edge set $E(G_1 \cup G_2) = E(G_1) \cup E(G_2) \cup \{v_1 v_2\}$, where v_i is a vertex with maximum degree in G_i for $i = 1, 2$.

Jou and Chang [7] determined the largest number $g(n, 1)$ of maximal independent sets among all graphs of order n, which contain at most one cycle.

Theorem 3.1. ([7]) *If* G *is a graph of order* $n \geq 2$ *vertices with at most one cycles, then* $mi(G) \leq g(n, 1)$, *where*

$$g(n,1) = \begin{cases} 2^{\frac{n}{2}}, & \text{if } n \geq 2 \text{ is even;} \\ 3 \cdot 2^{\frac{n-3}{2}}, & \text{if } n \geq 3 \text{ is odd .} \end{cases}$$

Furthermore, $mi(G) = g(n, 1)$ *if and only if* $G = G(n, 1)$, *where*

$$G(n,1) = \begin{cases} \frac{n}{2} K_2, & \text{if } n \geq 2 \text{ is even;} \\ K_3 \cup \frac{n-3}{2} K_2, & \text{if } n \geq 3 \text{ is odd .} \end{cases}$$

Jou [4] settled the second largest number $g'(n, 1)$ of maximal independent sets in graphs with at most one cycle.

Theorem 3.2. ([4]) *If* G *is a graph of order* $n \geq 4$ *with at most one cycle having* $G \neq G(n, 1)$, *then* $mi(G) \leq g'(n, 1)$, *where*

$$g'(n,1) = \begin{cases} 3 \cdot 2^{\frac{n-4}{2}}, & \text{if } n \geq 4 \text{ is even;} \\ 5 \cdot 2^{\frac{n-5}{2}}, & \text{if } n \geq 5 \text{ is odd.} \end{cases}$$

Furthermore, $mi(G) = g'(n, 1)$ *if and only if* $G \in G'(n, 1)$, *where*

$$G'(n,1) = \begin{cases} P_4 \cup \frac{n-4}{2} P_2,\ (K_1 * sK_2) \cup K_3 \cup \frac{n-4-2s}{2} K_2 & \\ \quad \text{or } (K_1 * (K_3 \cup sK_2)) \cup \frac{n-4-2s}{2} K_2, & \text{if } n \geq 4 \text{ is even;} \\ C_5 \cup \frac{n-5}{2} P_2 \text{ or } (K_3 * K_2) \cup \frac{n-5}{2} K_2, & \text{if } n \geq 5 \text{ is odd.} \end{cases}$$

B. E. Sagan and V. R. Vatter [11] found the largest number $g(n,2)$ of maximal independent sets among all graphs of order n, which contain at most two cycles.

Theorem 3.3. ([14]) *If G is a graph of order $n \geq 5$ vertices with at most two cycles, then $mi(G) \leq g(n,2)$, where*

$$g(n,2) = \begin{cases} 3 \cdot 2^{\frac{n-3}{2}}, & \text{if } n \geq 5 \text{ is odd ;} \\ 9 \cdot 2^{\frac{n-6}{2}}, & \text{if } n \geq 6 \text{ is even .} \end{cases}$$

Furthermore, $mi(G) = g(n,2)$ if and only if $G = G(n,2)$, where

$$G(n,2) = \begin{cases} K_3 \cup \frac{n-3}{2} K_2, & \text{if } n \geq 5 \text{ is odd ;} \\ 2K_3 \cup \frac{n-6}{2} K_2, & \text{if } n \geq 6 \text{ is even .} \end{cases}$$

The following theorem is the main result.

Theorem 3.4. *If G is a graph of order $n \geq 5$ with at most two cycles having $G \neq G(n,2)$, then $mi(G) \leq g'(n,2)$, where*

$$g'(n,2) = \begin{cases} 5 \cdot 2^{\frac{n-5}{2}}, & \text{if } n \geq 5 \text{ is odd;} \\ 2^{\frac{n}{2}}, & \text{if } n \geq 6 \text{ is even.} \end{cases}$$

Furthermore, $mi(G) = g'(n,2)$ if and only if $G \in G'(n,2)$, where

$$G'(n,2) = \begin{cases} C_5 \cup \frac{n-5}{2} K_2,\ (K_3 * K_2) \cup \frac{n-5}{2} K_2, & \\ \quad \text{or } (K_1 + (2K_2)) \cup \frac{n-5}{2} K_2, & \text{if } n \geq 5 \text{ is odd ;} \\ \frac{n}{2} K_2 \text{ or } (K_3 * K_3) \cup \frac{n-6}{2} K_2, & \text{if if } n \geq 6 \text{ is even.} \end{cases}$$

Proof. Let G be a graph of order $n \geq 5$ with at most two cycles having $G \neq G(n,2)$ such that $mi(G)$ as large as possible. Then $mi(G) \geq mi(G'(n,2)) = g'(n,2)$. Suppose that G has at most one cycle. Note that $G \neq K_3 \cup \frac{n-3}{2} K_2$. By Theorem 3.1 and Theorem 3.2, then

$$\begin{aligned} g'(n,2) &\leq mi(G) \\ &= \begin{cases} g'(n,1), & \text{if } n \geq 5 \text{ is odd;} \\ g(n,1), & \text{if } n \geq 6 \text{ is even;} \end{cases} \\ &= \begin{cases} 5 \cdot 2^{\frac{n-5}{2}}, & \text{if } n \geq 5 \text{ is odd;} \\ 2^{\frac{n}{2}}, & \text{if } n \geq 6 \text{ is even;} \end{cases} \\ &= g'(n,2). \end{aligned}$$

The equalities hold. Suppose that G has at most one cycle, by Theorem 3.1 and Theorem 3.2, then $G = C_5 \cup \frac{n-5}{2} K_2$, $(K_3 * K_2) \cup \frac{n-5}{2} K_2$ or $\frac{n}{2} K_2$. Now we assume that G have exactly two cycles. Let v be a vertex lying on some cycle such that $deg(v)$ is as large as possible.

Claim. $G - v = G(n-1, 1)$.

Suppose that $G - v \neq G(n-1,1)$, by Theorem 3.2, $mi(G-v) \leq g'(n-1,1)$. Since $deg(v) \geq 2$ and $G - N[v]$ is a graph of order at most $n-3$ with at most one cycle. By Theorem 3.1, $mi(G - N[v]) \leq g(n-3,1)$. So

$$\begin{aligned} g'(n,2) &\leq mi(G) \\ &\leq mi(G-v) + mi(G-N[v]) \\ &\leq g'(n-1,1) + g(n-3,1) \\ &= \begin{cases} 3 \cdot 2^{\frac{n-5}{2}} + 2^{\frac{n-3}{2}}, & \text{if } n \geq 5 \text{ is odd}; \\ 5 \cdot 2^{\frac{n-6}{2}} + 3 \cdot 2^{\frac{n-6}{2}}, & \text{if } n \geq 6 \text{ is even}; \end{cases} \\ &= \begin{cases} 5 \cdot 2^{\frac{n-5}{2}}, & \text{if } n \geq 5 \text{ is odd}; \\ 2^{\frac{n}{2}}, & \text{if } n \geq 6 \text{ is even}; \end{cases} \\ &= g'(n,2). \end{aligned}$$

The equalities hold, then $deg(v) = 2$ and $G - N[v] = G(n-3,1)$. Note that G have exactly two cycles and $deg(v) = 2$, so $G - N[v]$ has exactly one cycle. Thus n is even. Since the equalities hold, the subgraph $G - v = G'(n-1,1)$. Since $deg(v) = 2$, $G - v = C_5 \cup \frac{n-6}{2}K_2$, thus $G = C_3 \cup C_5 \cup \frac{n-8}{2}K_2$. So $mi(G) = 15 \cdot 2^{\frac{n-8}{2}} < 2^{\frac{n}{2}}$, this is a contradiction. Hence $G - v = G(n-1,1)$.

By Claim, $G - v = G(n-1,1)$ and $mi(G-v) = g(n-1,1)$. We consider two cases.

Case 1. $n \geq 5$ is odd. Then $mi(G - N[v]) \geq mi(G) - mi(G-v) \geq g'(n,2) - g(n-1,1) = 5 \cdot 2^{\frac{n-5}{2}} - 2^{\frac{n-1}{2}} = 2^{\frac{n-5}{2}} = g(n-5,1)$. By Theorem 3.1, $deg(v) \leq 4$. Note that G have exactly two cycles and $G - v = G(n-1,1) = \frac{n-1}{2}K_2$. Hence $(K_1 + (2K_2)) \cup \frac{n-5}{2}K_2$.

Case 2. $n \geq 6$ is even. Then $mi(G - N[v]) \geq mi(G) - mi(G-v) \geq g'(n,2) - g(n-1,1) = 2^{\frac{n}{2}} - 3 \cdot 2^{\frac{n-4}{2}} = 2^{\frac{n-4}{2}} = g(n-4,1)$. By Theorem 3.1, $deg(v) \leq 3$. Note that $G \neq 2K_3 \cup \frac{n-6}{2}K_2$ have exactly two cycles and $G - v = G(n-1,1) = K_3 \cup \frac{n-4}{2}K_2$. Hence $G = (K_3 * K_3) \cup \frac{n-6}{2}K_2$.

Suppose G have two cycles, by Case 1 and Case 2, $G = (K_1 + (2K_2)) \cup \frac{n-5}{2}K_2$ or $(K_3 * K_3) \cup \frac{n-6}{2}K_2$. □

References

[1] J. R. Griggs, C. M. Grinstead and D. R. Guichard, The number of maximal independent sets in a connected graph, *Discrete Math.*, **68** (1988), 211-220. http://dx.doi.org/10.1016/0012-365x(88)90114-8

[2] M. Hujtera and Z. Tuza, The number of maximal independent sets in triangle-free graphs, *SIAM J. Discrete Math.*, **6** (1993), 284–288. http://dx.doi.org/10.1137/0406022

[3] Z. Jin and X. Li, Graphs with the second largest number of maximal independent sets, *Discrete Math.*, **308** (2008), 5864–5870. http://dx.doi.org/10.1016/j.disc.2007.10.032

[4] M. J. Jou, The second largest number of maximal independent sets in connected graphs with at most one cycle, *Journal of Combinatorial Optimization*, **24** (2012), no. 3, 192–201. http://dx.doi.org/10.1007/s10878-011-9376-4

[5] M. J. Jou, *The Number of Maximal Independent Sets in Graphs*, Master Thesis, Department of Mathematics, National Central University, Taiwan, 1991.

[6] M. J. Jou and G. J. Chang, Survey on conunting maximal independent sets, in: *Proceedings of the Second Asian Mathematical Conference*, S. Tangmance and E. Schulz eds., World Scientific, Singapore, (1995), 265–275.

[7] M. J. Jou and G. J. Chang, Maximal independent sets in graphs with at most one cycle, *Dicrete Appl. Math.*, **79** (1997), 67–73. http://dx.doi.org/10.1016/s0166-218x(97)00033-4

[8] M. J. Jou and G. J. Chang, Algorithmic aspects of counting independent sets, *ARS Combin.*, **65** (2002), 265–277.

[9] A. Knopfmacher, R. F. Tichy, S. Wagner and V. Ziegler, Graphs, partitions and Fibonacci numbers, *Discrete Appl. Math.*, **155** (2007), 1175–1187. http://dx.doi.org/10.1016/j.dam.2006.10.010

[10] J. W. Moon and L. Moser, On cliques in graphs, *Israel J. Math.*, **3** (1965), 23–28. http://dx.doi.org/10.1007/bf02760024

[11] B. E. Sagan and V. R. Vatter, Maximal and maximum independent sets in graphs with at most r cycles, *J. Graph Theory*, **53** (2006), 283–314. http://dx.doi.org/10.1002/jgt.20186

[12] H. Prodinger and R. F. Tichy, Fibonacci numbers of graphs, *Fibonacci Quart.*, **20** (1982), 16–21.

[13] S. G. Wagner, The Fibonacci Number of Generalized Petersen Graphs, *Fibonacci Quart.*, **44** (2006), 362–367.

[14] G. C. Ying, Y. Y. Meng, B. E. Sagan and V. R. Vatter, Maximal independent sets in graphs with at most r cycles, *J. Graph Theory*, **53** (2006), 270-282. http://dx.doi.org/10.1002/jgt.20185

Received: May 3, 2016; Published: July 5, 2016