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Formulae for Every Integer

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Abstract

In this note we obtain polynomials such that their images are all integers.

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1 Introduction

The problem of the representation of integers is an old problem. Waring's problem establish that every nonnegative integer is the sum of a fixed number of nonnegative k-th powers. This fixed number depends of k. For example, all nonnegative integer is the sum of four nonnegative squares (Lagrange's Theorem). The Waring's problem was solved by Hilbert. This is a very difficult problem. The history of the Waring's problem is given in [3].

Goldbach conjectured that every even positive integer is the sum of two prime numbers. The Goldbach's conjecture is an open problem.

It is also natural to consider the problem of the representation of an integer as the sum of a fixed number of members of the set

$$0, 1^k, 2^k, 3^k, \ldots, -1^k, -2^k, -3^k, \ldots$$

This fixed number depends of k. This problem has also a positive solution, as the Waring's problem (see [3]).

Also, every integer is the sum of three triangular numbers. This was conjectured by Fermat and proved by Gauss (1801), Art. 293 [2]. A modern proof via q-series has been given by Andrews (1986)[1].

In this note we obtain some polynomials with integer coefficients such that every integer pertains to the image of the polynomial, that is, the polynomial generates all integers. The polynomials that we consider have terms of the form Ax_i^k, where A is an integer, $k \geq 2$ and the variable x_i takes only integer values.

2 Main Results

In [3](Chapter XXI) is proved the formula

$$\sum_{r=0}^{k-1}(-1)^{k-1-r}\binom{k-1}{r}(a+r)^k = k!a + d_k, \tag{1}$$

where d_k is an integer independient of a. In fact $d_k = \frac{1}{2}(k-1)k!$, but we make no use of this.

Therefore every integer of the linear form $k!a + d_k$ is representable by the polynomial of k terms and k variables $x_{0,k}, x_{1,k}, \ldots, x_{k-1,k}$.

$$P(x_{0,k}, x_{1,k}, \ldots, x_{k-1,k}) = \sum_{r=0}^{k-1}(-1)^{k-1-r}\binom{k-1}{r}x_{r,k}^k, \tag{2}$$

and consequently every integer is representable by the polynomial in $k + k! - 1$ variables with the same exponent $k \geq 2$

$$P(x_{0,k}, x_{1,k}, \ldots, x_{k-1,k}, y_1, \ldots, y_{(k!-1)}) = \sum_{r=0}^{k-1}(-1)^{k-1-r}\binom{k-1}{r}x_{r,k}^k + \sum_{r=1}^{k!-1} y_r^k. \tag{3}$$

If we wish that in the polynomial appear only the exponents $k_1 > k_2 > \cdots > k_s \geq 2$ ($s \geq 2$), we have the following polynomial whose image are all integers (see (2)).

$$\begin{aligned} &\sum_{r=1}^{k_s!-1} y_r^{k_1} + P(x_{0,k_1}, x_{1,k_1}, \ldots, x_{k_1-1,k_1}) \\ - &\sum_{i=2}^{s}\left(\frac{k_{i-1}!}{k_i!} - 1\right) P(x_{0,k_i}, x_{1,k_i}, \ldots, x_{k_i-1,k_i}), \end{aligned} \tag{4}$$

since (see (1))

$$k_1!a - \sum_{i=2}^{s}\left(\frac{k_{i-1}!}{k_i!} - 1\right) k_i!a = k_s!a.$$

Note that the part of the polynomial

$$\left(\frac{k_{i-1}!}{k_i!}-1\right)P(x_{0,k_i},x_{1,k_i},\ldots,x_{k_i-1,k_i}) \quad (i=2,\ldots,s)$$

can not generates every integer since only generates integers multiple of

$$\left(\frac{k_{i-1}!}{k_i!}-1\right)>1.$$

We also can consider the following another polynomial (see (4))

$$\begin{aligned}&\sum_{r=1}^{k_s!-1} y_r^{k_0}+P(x_{0,k_1},x_{1,k_1},\ldots,x_{k_1-1,k_1})\\ -&\sum_{i=2}^{s}\left(\frac{k_{i-1}!}{k_i!}-1\right)P(x_{0,k_i},x_{1,k_i},\ldots,x_{k_i-1,k_i}),\end{aligned} \tag{5}$$

where the fixed positive integer $k_0>k_1$ can be arbitrarily large and even.

Let us consider a polynomial P_1, we shall say that the polynomial P_2 is included in the polynomial P_1 if and only if every term of P_2 is in P_1. For example the polynomial $P_2=2x_1^5-3x_2^7+x_3^8$ is included in the polynomial $P_1=x_1^3+x_2^3+2x_3^5+5x_4^6-3x_5^7+x_6^7+x_7^8$. Let us consider a polynomial P_1 such that every integer pertain to its image, that is, the polynomial generates every integer. For example the polynomials (3), (4) and (5). We shall say that the polynomial P_1 is primitive if and only if it does not include a polynomial P_2 that also generates every integer. The author thinks the proof that a certain polynomial is primitive is very difficult in the general case.

For example, if we wish to obtain a polynomial where only appear cubes and squares (4) becomes (see (1))

$$x_1^3+x_2^3-2x_3^3+x_4^3-2x_5^2+2x_6^2.$$

This polynomial has 6 terms. It is not the unique polynomial with cubes and squares that generates every integer, there exist polynomials with this property with less terms.

For example, we have the polynomial with 4 terms

$$x_1^3+x_2^3+2x_3^2+x_4^2,$$

since we have the identity

$$(a-1)^3+(-a)^3+2a^2+(a-1)^2=a.$$

We also have the polynomial with 4 terms

$$x_1^3 + x_2^3 + 4x_3^2 + 2x_4^2,$$

since we have the identities

$$4(a)^2 + 2(2a-2)^2 + (2a-2)^3 + (-2a+1)^3 = 2a+1$$

$$4(a)^2 + 2(a-2)^2 + (-a)^3 + (a-2)^3 = 2(2a)$$

$$4(a+1)^2 + 2(a)^2 + (a-1)^3 + (-a-1)^3 = 2(4a+1)$$

$$4(a)^2 + 2(a-4)^2 + (-a+1)^3 + (a-3)^3 = 2(4a+3)$$

If we wish a polynomial with only quartic and cubic powers then polynomial (4) has $5+4+3=12$ terms. In this case, we also have polynomials with less terms. For example the polynomial with 4 terms

$$x_1^4 + 2x_2^4 - 2x_3^4 - x_4^3,$$

since we have the identity

$$2(a+1)^4 - 2a^4 - (2a+1)^3 = 2a+1.$$

If we wish a polynomial with only quintic and quartic powers then polynomial (4) has $23+5+4=32$ terms. It is well-known (see [3]) that every integer can be written as a sum of 10 quintic powers. Therefore this polynomial is not primitive. In this case, we also have polynomials with less terms. For example the polynomial with 5 terms

$$27x_1^5 + 864x_2^5 + x_3^5 - 540x_4^4 - 20x_5^4,$$

since we have the identity

$$27(2a+1)^5 + 864(-a)^5 - 540a^4 - 20(3a+1)^4 = 30a+7,$$

and $x^5 \equiv x \ mod(30)$ $(x = 0, 1, \ldots, 29)$.

If we wish a polynomial with only 6-th powers and quintic powers then polynomial (4) has $119+6+5=130$ terms. In this case, we also have polynomials with less terms. For example the polynomial with 8 terms

$$x_1^5 + 2x_2^5 + 2x_3^5 + 2x_4^5 + 8x_5^5 + x_6^6 - x_7^6 + 4x_8^6,$$

since we have the identity

$$2(a+1)^5 + 2(a-1)^5 + 8a^5 + (a-1)^6 - (a+1)^6 = 8a$$

and $x^5 \equiv x\ mod(8)$ $(x = 1,3,5,7)$, $2x^5 \equiv 2,6\ mod(8)$ $x = 1,3$ respectively, $4.1^6 \equiv 4\ mod(8)$.

If we wish that appear in the polynomial the exponents $n, n-1, \ldots, 2$, where $n \geq 3$, polynomials (4) and (5) have many terms. However, in this case, the problem is not very difficult. Since (binomial formula)

$$(a+1)^n = \sum_{k=0}^{n} \binom{n}{k} a^k = 1 + na + \sum_{k=2}^{n-2} \binom{n}{k} a^k + na^{n-1} + a^n$$

and

$$(a+1)^{n-1} = \sum_{k=0}^{n-1} \binom{n-1}{k} a^k = 1 + (n-1)a + \sum_{k=2}^{n-2} \binom{n-1}{k} a^k + a^{n-1}$$

Therefore

$$(a+1)^n - (a+1)^{n-1} = a + \sum_{k=2}^{n-2} \left(\binom{n}{k} - \binom{n-1}{k} \right) a^k + (n-1)a^{n-1} + a^n$$

That is, we have the identity

$$\sum_{k=2}^{n-2} \left(\binom{n-1}{k} - \binom{n}{k} \right) a^k - (n-1)a^{n-1} - (a+1)^{n-1} + (a+1)^n - a^n = a,$$

and consequently we have the following polynomial, with $n+1$ terms, that represent every integer

$$\sum_{k=2}^{n-2} \left(\binom{n-1}{k} - \binom{n}{k} \right) x_{k-1}^k - (n-1)x_{n-2}^{n-1} - x_{n-1}^{n-1} + x_n^n - x_{n+1}^n$$

Note that if we wish a polynomial with only k-th powers ($k \geq 3$) and squares then the polynomial (4) becomes

$$y_1^k + P(x_{0,k}, x_{1,k}, \ldots, x_{k-1,k}) - \left(\frac{k!}{2} - 1 \right) x_{0,2}^2 + \left(\frac{k!}{2} - 1 \right) x_{1,2}^2.$$

This polynomial has $k+3$ terms.

In identity (1) all exponents are k. Now, we shall obtain identities where the exponents are different. Besides, we shall choose the exponents that we wish in our identity.

Let us consider the binomial formula.

$$(a+b)^k = \sum_{j=0}^{k} \binom{k}{k-j} a^{k-j} b^j.$$

The binomial formula gives

$$(a+i)^k = \sum_{j=0}^{k} \binom{k}{k-j} a^{k-j} i^j = a^k + \sum_{j=1}^{k-2} \binom{k}{k-j} a^{k-j} i^j + k i^{k-1} a + i^k, \tag{6}$$

where $i = 1, 2, \ldots, k$.

Suppose that we wish an identity where only appear the exponents 4 and 3, and clearly, always the exponent 1. Then we write the following 4 formulae (see (6) with $k = 4$)

$$(a+1)^4 = a^4 + \binom{4}{3} a^3 + \binom{4}{2} a^2 + 4a + 1, \tag{7}$$

$$(a+2)^4 = a^4 + \binom{4}{3} 2a^3 + \binom{4}{2} 2^2 a^2 + 4.2^3 a + 2^4, \tag{8}$$

$$(a+3)^4 = a^4 + \binom{4}{3} 3a^3 + \binom{4}{2} 3^2 a^2 + 4.3^3 a + 3^4, \tag{9}$$

$$(a+4)^4 = a^4 + \binom{4}{3} 4a^3 + \binom{4}{2} 4^2 a^2 + 4.4^3 a + 4^4. \tag{10}$$

Now, we establish the following system of linear equations

$$x_1 + x_2 + x_3 + x_4 = 0 \tag{11}$$

$$x_1 + 2x_2 + 3x_3 + 4x_4 = 1 \tag{12}$$

$$x_1 + 2^2 x_2 + 3^2 x_3 + 4^2 x_4 = 0 \tag{13}$$

$$x_1 + 2^3 x_2 + 3^3 x_3 + 4^3 x_4 = 1 \tag{14}$$

The determinant of the coefficients in this system of linear equations is a Vandermonde's determinant whose value is $1!2!3! \neq 0$. Consequently, it has an unique solution. This unique solution is

$$(x_1, x_2, x_3, x_4) = \left(-\frac{9}{2}, 10, -\frac{15}{2}, 2\right). \tag{15}$$

If we multiply equation (7) by x_1, equation (8) by x_2, equation (9) by x_3, equation (10) by x_4 and sum, then we obtain (see equations (11), (12), (13) and (14))

$$-\frac{9}{2}(a+1)^4 + 10(a+2)^4 - \frac{15}{2}(a+3)^4 + 2(a+4)^4 - \binom{4}{3}a^3 = 4a + \left(-\frac{9}{2}1^4 + 10.2^4 - \frac{15}{2}3^4 + 2.4^4\right). \tag{16}$$

Finally. if we multiply (16) by 2 (the least common multiple of the denominators of x_1, x_2, x_3 and x_4) then we obtain the desired identity

$$-9(a+1)^4 + 20(a+2)^4 - 15(a+3)^4 + 4(a+4)^4 - 8a^3 = 8a + 120.$$

This identity can be used to obtain a polynomial that represent every integer, for example, the following polynomial

$$\sum_{i=1}^{7} x_i^n - 9x_8^4 + 20x_9^4 - 15x_{10}^4 + 4x_{11}^4 - 8x_{12}^3,$$

where $n > 4$ is a fixed but arbitrary integer.

The method used in this example is general.

If we wish that in the identity appear only the exponents $k > d_1 > d_2 > \cdots > d_s \geq 2$ then we consider the square system of linear equations in k unknowns $x_1, x_2, \ldots, x_k$. We put $b_1 = 1$, $b_{d_i} = 1$ $(i = 1, 2, \ldots, s)$ and in the rest of the b_i we put $b_i = 0$ (see (6)).

$$\begin{gathered} x_1 + x_2 + x_3 + \cdots + x_k = b_k \\ x_1 + 2x_2 + 3x_3 + \cdots + kx_k = b_{k-1} \\ x_1 + 2^2x_2 + 3^2x_3 + \cdots + k^2x_k = b_{k-2} \\ \vdots \\ x_1 + 2^{k-1}x_2 + 3^{k-1}x_3 + \cdots + k^{k-1}x_k = b_1 \end{gathered}$$

The determinant of the coefficients of this square system of linear equations is a Vandermonde's determinant whose value is $1!2!\ldots(k-1)! \neq 0$. Therefore the system of linear equations has an unique solution $(x_1, x_2, x_3, \ldots, x_k) = (q_1, q_2, q_3, \ldots, q_k)$, where the q_i $(i = 1, 2, \ldots, k)$ are rational numbers. Consequently we obtain the identity.

$$\left(\sum_{i=1}^{k} q_i(a+i)^k\right) - \left(\sum_{i=1}^{s} \binom{k}{d_i} a^{d_i}\right) = ka + \left(\sum_{i=1}^{k} i^k q_i\right). \tag{17}$$

Let l be the least common multiple of the denominators of the q_i $(i = 1, 2, \ldots, k)$. If we multiply both sides of (17) by l then we obtain the desired identity. Note that lq_i $(i = 1, 2, \ldots, k)$ is an integer.

$$\left(\sum_{i=1}^{k} lq_i(a+i)^k\right) - \left(\sum_{i=1}^{s} l\binom{k}{d_i}a^{d_i}\right) = kla + \left(\sum_{i=1}^{k} i^k lq_i\right).$$

This identity can be used to obtain a polynomial that represent every integer, for example, the following polynomial

$$\sum_{i=1}^{kl-1} y_i^n + \left(\sum_{i=1}^{k} lq_i x_i^k\right) - \left(\sum_{i=1}^{s} l\binom{k}{d_i} x_{k+i}^{d_i}\right),$$

where $n > k$ is a fixed but arbitrary integer.

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