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Successive Derivatives of $\arcsin x$

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Abstract

Let us consider the well-known functions $sin^{-1}x = \arcsin x$ and $sinh^{-1}x$. In this article we examine the successive algebraic derivatives of this functions and the relations between both.

Mathematics Subject Classification: 11A99, 11B99

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1 Introduction

In a previous article [1] we examine the successive derivatives de la function $f(x) = \frac{1}{g(x)}$. In another previous [2] article we examine the special case of the successive derivatives when $g(x) = 1 + x^2$ and $g(x) = 1 - x^2$. In this article we examine the special case of the successive derivatives when $g(x) = \sqrt{1 + x^2}$ and $g(x) = \sqrt{1 - x^2}$. The antiderivatives of these algebraic functions are trascendent.

Let us consider the trascendent inverse function of $\sinh x = \frac{e^x - e^{-x}}{2}$, namely

$$f(x) = \sinh^{-1} x = \log\left(x + \sqrt{x^2 + 1}\right)$$

The successive algebraic derivatives of this function are

$$f^{(1)}(x) = \frac{1}{(1 + x^2)^{1/2}} \tag{1}$$

$$f^{(2)}(x) = \frac{-x}{(1+x^2)^{3/2}}$$

$$f^{(3)}(x) = \frac{2x^2-1}{(1+x^2)^{5/2}}$$

$$f^{(4)}(x) = \frac{-6x^3+9x}{(1+x^2)^{7/2}}$$

$$\vdots$$

In this article we examine these successive derivatives, that is, the polynomials with integers coefficients in the numerators of these derivatives. Analogously we study the successive derivatives of $\arcsin x$.

2 Successive derivatives of $\sinh^{-1} x$

Theorem 2.1 *Let us consider the function $f(x) = \sinh^{-1} x$. We have*

$$f^{(n)}(x) = \frac{P_{n-1}(x)}{(1+x^2)^{\frac{2n-1}{2}}} \qquad (n = 1, 2, 3, \ldots) \tag{2}$$

where $P_{n-1}(x)$ is a polynomial of integer coefficients.

These polynomials can be obtained using the recursive formulae

$$P_0(x) = 1 \tag{3}$$

$$P_n(x) = P'_{n-1}(x)(1+x^2) - (2n-1)xP_{n-1}(x) \qquad (n = 1, 2, 3, \ldots) \tag{4}$$

Proof. Use equation (2), mathematical induction and the quotient's rule for derivatives. The theorem is proved.

Example 2.2 *The first polynomials are (using (3) and (4))*

$$P_0(x) = 1$$

$$P_1(x) = -x$$

$$P_2(x) = 2x^2 - 1$$

$$P_3(x) = -6x^3 + 9x$$

$$P_4(x) = 24x^4 - 72x^2 + 9$$

$$P_5(x) = -120x^5 + 600x^3 - 225x$$

$$P_6(x) = 720x^6 - 5400x^4 + 4050x^2 - 225$$

Theorem 2.3 *The polynomial $P_n(x)$ $(n \geq 0)$ has degree n, integer coefficients and leading coefficient $(-1)^n n!$.*

Proof. We shall apply mathematical induction. Clearly the theorem is true for the first polynomials (see Example 2.2). Suppose the theorem is true for $P_{n-1}(x)$ then from equation (4) we obtain that the greater exponent in $P_n(x)$ is n and its coefficient is

$$(n-1)(-1)^{n-1}(n-1)! - (2n-1)(-1)^{n-1}(n-1)! = (-1)^n n! \neq 0$$

The theorem is proved.

Theorem 2.4 *In the polynomial $P_n(x)$ $(n \geq 0)$ appear only even exponents if n is even and only odd exponents if n is odd.*

It is an immediate consequence of Example 2.2, equation (4) and mathematical induction. The theorem is proved.

Theorem 2.5 *The polynomials $P_n(x)$ are of the form*

$$P_{2n}(x) = \sum_{i=0}^{n}(-1)^i a_{2n-2i,2n} x^{2n-2i} \qquad (n = 0, 1, 2, 3, \ldots) \tag{5}$$

where $a_{2n-2i,2n}$ are positive integers $(i = 0, \ldots, n)$.

$$P_{2n-1}(x) = \sum_{i=0}^{n-1}(-1)^{i+1} a_{2n-1-2i,2n-1} x^{2n-1-2i} \qquad (n = 1, 2, 3, \ldots) \tag{6}$$

where $a_{2n-1-2i,2n-1}$ are positive integers $(i = 0, \ldots, n-1)$.

Proof. For sake of simplicity in the proof we shall write $a_{2n-2i,2n} = a_{2n-2i}$ and $a_{2n-1-2i,2n-1} = a_{2n-1-2i}$. We shall use mathematical induction. Clearly the theorem is true for the first polynomials (see Example 2.2). Suppose that the theorem is true for $P_{2n-1}(x)$, then we shall prove the theorem is also true for $P_{2n}(x)$. We have (see (4) and (6))

$$\begin{aligned}
& P_{2n}(x) = P'_{2n-1}(x)(1+x^2) - (4n-1)xP_{2n-1}(x) \\
= \ & \sum_{i=0}^{n-1}(-1)^{i+1}(2n-1-2i)a_{2n-1-2i}x^{2n-2-2i}(1+x^2) \\
- \ & (4n-1)\sum_{i=0}^{n-1}(-1)^{i+1}a_{2n-1-2i}x^{2n-2i} \\
= \ & \sum_{i=0}^{n-1}(-1)^{i+1}(2n-1-2i)a_{2n-1-2i}x^{2n-2i}
\end{aligned}$$

$$
\begin{aligned}
&- \sum_{i=0}^{n-1}(-1)^{i+1}(4n-1)a_{2n-1-2i}x^{2n-2i} \\
&+ \sum_{i=0}^{n-1}(-1)^{i+1}(2n-1-2i)a_{2n-1-2i}x^{2n-2-2i} \\
&= \sum_{i=0}^{n-1}(-1)^{i}(2n+2i)a_{2n-1-2i}x^{2n-2i} + \sum_{i=1}^{n}(-1)^{i}(2n+1-2i)a_{2n+1-2i}x^{2n-2i} \\
&= 2na_{2n-1}x^{2n} \\
&+ \sum_{i=1}^{n-1}(-1)^{i}\left((2n+2i)a_{2n-1-2i} + (2n+1-2i)a_{2n+1-2i}\right)x^{2n-2i} \\
&+ (-1)^n a_1 \qquad (7)
\end{aligned}
$$

where, from the inductive hypothesis, the integers in the last polynomial in (7), namely

$$2na_{2n-1} \tag{8}$$

$$((2n+2i)a_{2n-1-2i} + (2n+1-2i)a_{2n+1-2i}) \qquad (i = 1, \ldots, n-1) \tag{9}$$

$$a_1 \tag{10}$$

are positive. As we desired.

Now suppose that the theorem is true for $P_{2n}(x)$, then we shall prove the theorem is also true for $P_{2n+1}(x)$. We have (see (4) and (5))

$$
\begin{aligned}
&P_{2n+1}(x) = P'_{2n}(x)(1+x^2) - (4n+1)xP_{2n}(x) \\
&= \sum_{i=0}^{n-1}(-1)^{i}(2n-2i)a_{2n-2i}x^{2n-2i-1}(1+x^2) \\
&- (4n+1)\sum_{i=0}^{n}(-1)^{i}a_{2n-2i}x^{2n-2i+1} \\
&= \sum_{i=0}^{n-1}(-1)^{i}(2n-2i)a_{2n-2i}x^{2n-2i-1} + \sum_{i=0}^{n-1}(-1)^{i}(2n-2i)a_{2n-2i}x^{2n-2i+1} \\
&- (4n+1)\sum_{i=0}^{n}(-1)^{i}a_{2n-2i}x^{2n-2i+1} \\
&= \sum_{i=1}^{n}(-1)^{i-1}(2n-2i+2)a_{2n-2i+2}x^{2n-2i+1} \\
&+ \sum_{i=0}^{n-1}(-1)^{i}(2n-2i)a_{2n-2i}x^{2n-2i+1} - (4n+1)\sum_{i=0}^{n}(-1)^{i}a_{2n-2i}x^{2n-2i+1} \\
&= -(2n+1)a_{2n}x^{2n+1}
\end{aligned}
$$

$$
\begin{aligned}
&+ \quad \sum_{i=1}^{n-1}(-1)^{i+1}\left((2n+2i+1)a_{2n-2i}+(2n-2i+2)a_{2n-2i+2}\right)x^{2n-2i+1} \\
&+ \quad (-1)^{n+1}\left(2a_2+(4n+1)a_0\right)x \qquad (11)
\end{aligned}
$$

where, from the inductive hypothesis, the integer in the last polynomial in (11), namely

$$(2n+1)a_{2n} \qquad (12)$$

$$((2n+2i+1)a_{2n-2i}+(2n-2i+2)a_{2n-2i+2}) \qquad (i=1,\ldots,n-1) \qquad (13)$$

$$(2a_2+(4n+1)a_0) \qquad (14)$$

are positive. As we desired. The theorem is proved.

In the following theorem, we examine the sum of the absolute values of the coefficients in the polynomial $P_n(x)$.

Theorem 2.6 *Let us consider the polynomial* $P_n(x)$ $(n=0,1,2,\ldots)$. *The sum of the absolute values of their coefficients is*

$$A_n = 1.\ 3.\ 5\ldots(2n-1) \qquad (n=1,2,\ldots)$$

Proof. We shall use mathematical induction. The theorem is true for the first values of n (see Example 2.2). From the inductive hypothesis we have (see (6))

$$\sum_{i=0}^{n-1} a_{2n-1-2i} = A_{2n-1} = 1.3.5\ldots(2(2n-1)-1) = 1.3.5\ldots(4n-3) \qquad (15)$$

On the other hand, equations (8), (9), (10) and (15) give

$$
\begin{aligned}
A_{2n} &= 2na_{2n-1}+\sum_{i=1}^{n-1}\left((2n+2i)a_{2n-1-2i}+(2n+1-2i)a_{2n+1-2i}\right) \\
&+ \quad a_1 = (4n-1)\sum_{i=0}^{n-1}a_{2n-1-2i} = (4n-1)A_{2n-1} = 1.3.5\ldots(4n-3)(4n-1)
\end{aligned}
$$

As we desired.

From the inductive hypothesis we have (see (5))

$$\sum_{i=0}^{n} a_{2n-2i} = A_{2n} = 1.3.5\ldots(4n-1) \qquad (16)$$

On the other hand, equations (12), (13), (14) and (16) give

$$\begin{aligned} A_{2n+1} &= (2n+1)a_{2n} + \sum_{i=1}^{n-1} ((2n+2i+1)a_{2n-2i} + (2n-2i+2)a_{2n-2i+2}) \\ &+ ((4n+1)a_0 + 2a_2) = (4n+1)\sum_{i=0}^{n} a_{2n-2i} = (4n+1)A_{2n} \\ &= 1.3.5\ldots(4n-1)(4n+1) \end{aligned}$$

As we desired. The theorem is proved.

Theorem 2.7 *The coefficient of x^0 in the polynomial $P_{2n}(x)$ is*

$$(-1)^n(1.3.5\ldots(2n-1))^2$$

and consequently (see (5)) $a_{0,2n} = (1.3.5\ldots(2n-1))^2$. The coefficient of x in the polynomial $P_{2n-1}(x)$ is $(-1)^n(1.3.5\ldots(2n-1))^2$ and consequently (see (6) and (7)) $a_{1,2n-1} = (1.3.5\ldots(2n-1))^2$.

Proof. The following binomial power series is well-known

$$(1+x)^\alpha = \sum_{n=0}^{\infty} \binom{\alpha}{n} x^n, \qquad |x| < 1 \tag{17}$$

where

$$\binom{\alpha}{n} = \frac{\alpha(\alpha-1)(\alpha-2)\cdots(\alpha-n+1)}{n!} \tag{18}$$

Equation (18) gives

$$g(x) = \frac{1}{\sqrt{1+x^2}} = (1+x^2)^{-1/2} = \sum_{n=0}^{\infty} \binom{-1/2}{n} x^{2n} \tag{19}$$

On the other hand, it is well-known that

$$g(x) = \frac{1}{\sqrt{1+x^2}} = \sum_{n=0}^{\infty} \frac{g(0)^{(n)}}{n!} x^n, \qquad |x| < 1 \tag{20}$$

Hence, equations (19) and (20) give

$$g(0)^{(2n-1)} = 0, \qquad (n = 1, 2, 3, \ldots) \tag{21}$$

$$\begin{aligned} g(0)^{(2n)} &= \binom{-1/2}{n}(2n)! = \frac{-\frac{1}{2}\left(-\frac{1}{2}-1\right)\cdots\left(-\frac{1}{2}-(n-1)\right)}{n!}(2n)! \\ &= \frac{(-1)^n \frac{1}{2}\frac{3}{2}\cdots\frac{2n-1}{2}}{n!}(2n)! = (-1)^n \frac{1.3.5\ldots(2n-1)}{2^n n!}(2n)! \\ &= (-1)^n \frac{(1.3.5\ldots(2n-1))^2}{2.4.6\ldots 2n}(2.4.6\ldots 2n) \\ &= (-1)^n (1.3.5\ldots(2n-1))^2 \qquad (n = 1, 2, \ldots) \end{aligned} \tag{22}$$

Now, see (2), we have

$$g^{(n-1)}(0) = f^{(n)}(0) = \frac{P_{n-1}(0)}{(1+0^2)^{\frac{2n-1}{2}}} = P_{n-1}(0) \qquad (n = 1, 2, 3, \ldots) \tag{23}$$

Equations (21), (22) and (23) give

$$P_{2n-1}(0) = 0 \qquad (n = 1, 2, 3, \ldots) \tag{24}$$

$$P_{2n}(0) = (-1)^n (1.3.5 \ldots (2n-1))^2 \qquad (n = 1, 2, \ldots) \tag{25}$$

Consequently, the coefficient of x^0 in the polynomial $P_{2n}(x)$ is

$$P_{2n}(0) = (-1)^n (1.3.5 \ldots (2n-1))^2 \qquad (n = 1, 2, \ldots)$$

and consequently (see (5)) $a_{0,2n} = (1.3.5 \ldots (2n-1))^2$. The theorem is proved.

Theorem 2.8 *We have*

$$\int_0^\infty f^{(2n+2)}(x)\ dx = \int_0^\infty \frac{P_{2n+1}(x)}{(1+x^2)^{\frac{4n+3}{2}}}\ dx = (-1)^{n+1}(1.3.5 \ldots (2n-1))^2$$

where $(n = 1, 2, \ldots)$.

$$\int_0^\infty f^{(2n+1)}(x)\ dx = \int_0^\infty \frac{P_{2n}(x)}{(1+x^2)^{\frac{4n+1}{2}}}\ dx = 0 \qquad (n = 1, 2, 3, \ldots)$$

Proof. We have (see (2), (24) and (25))

$$\int_0^\infty f^{(2n+2)}(x)\ dx = \lim_{a\to\infty} \left(f^{(2n+1)}(a) - f^{(2n+1)}(0)\right)$$

$$= \lim_{a\to\infty} \left(\frac{P_{2n}(a)}{(1+a^2)^{\frac{4n+1}{2}}} - P_{2n}(0)\right) = 0 - P_{2n}(0) = (-1)^{n+1}(1.3.5 \ldots (2n-1))^2$$

$$\int_0^\infty f^{(2n+1)}(x)\ dx = \lim_{a\to\infty} \left(f^{(2n)}(a) - f^{(2n)}(0)\right)$$

$$= \lim_{a\to\infty} \left(\frac{P_{2n-1}(a)}{(1+a^2)^{\frac{4n-1}{2}}} - P_{2n-1}(0)\right) = 0 - P_{2n-1}(0) = 0$$

The theorem is proved.

Theorem 2.9 *The coefficient of* x^2 *in the polynomial* $P_{2n}(x)$ *is*

$$(-1)^{n-1}(1.3.5 \ldots (2n-1))^2 2n^2$$

and consequently (see (5)) $a_{2,2n} = (1.3.5 \ldots (2n-1))^2 2n^2$.

Proof. We have (see equations (11), (14) and Theorem 2.7)

$$\begin{aligned} & 2a_2 + (4n+1)a_0 = 2a_2 + (4n+1)(1.3.5 \ldots (2n-1))^2 \\ = \ & (1.3.5 \ldots (2n-1)(2n+1))^2 \end{aligned} \tag{26}$$

Equation (26) gives $a_2 = (1.3.5 \ldots (2n-1))^2 2n^2$. The theorem is proved.

3 An Observation

The inverse function of $\sin x$ is the function

$$f(x) = \arcsin x \qquad (-1 < x < 1) \tag{27}$$

The successive algebraic derivatives of this function are

$$f^{(1)}(x) = \frac{1}{\sqrt{1-x^2}}$$

$$f^{(2)}(x) = \frac{x}{(1-x^2)^{3/2}}$$

$$f^{(3)}(x) = \frac{2x^2+1}{(1-x^2)^{5/2}}$$

$$f^{(4)}(x) = \frac{6x^3+9x}{(1-x^2)^{7/2}}$$

$$\vdots$$

Theorem 3.1 *Let us consider the function (27), we have*

$$f^{(n)}(x) = \frac{Q_{n-1}(x)}{(1-x^2)^{\frac{2n-1}{2}}} \qquad (n = 1, 2, 3, \ldots) \tag{28}$$

where $Q_{n-1}(x)$ is a polynomial of integer coefficients.

These polynomials can be obtained using the recursive formulae

$$Q_0(x) = 1 \tag{29}$$

$$Q_n(x) = Q'_{n-1}(x)(1-x^2) + (2n-1)xQ_{n-1}(x) \qquad (n = 1, 2, 3, \ldots) \tag{30}$$

Proof. Use equation (28), mathematical induction and the quotient's rule for derivatives. The theorem is proved.

Example 3.2 *The first polynomials are (using (29) and (30))(compare with Example 2.2)*

$$Q_0(x) = 1$$

$$Q_1(x) = x$$

$$Q_2(x) = 2x^2 + 1$$

$$Q_3(x) = 6x^3 + 9x$$

$$Q_4(x) = 24x^4 + 72x^2 + 9$$

$$Q_5(x) = 120x^5 + 600x^3 + 225x$$

$$Q_6(x) = 720x^6 + 5400x^4 + 4050x^2 + 225$$

Theorem 3.3 *The polynomials $Q_n(x)$ are of the form*

$$Q_{2n}(x) = \sum_{i=0}^{n} a_{2n-2i,2n} x^{2n-2i} \qquad (n = 0, 1, 2, 3, \ldots)$$

where $a_{2n-2i,2n}$ $(i = 0, \ldots, n)$ are the same positive integers that in equation (5)(see Theorem 2.5).

$$Q_{2n-1}(x) = \sum_{i=0}^{n-1} a_{2n-1-2i,2n-1} x^{2n-1-2i} \qquad (n = 1, 2, 3, \ldots)$$

where $a_{2n-1-2i,2n-1}$ $(i = 0, \ldots, n-1)$ are the same positive integers that in equation (6)(see Theorem 2.5).

Proof. The theorem is true for the first polynomials (see Example 2.2 and Example 3.2). Now the proof is the same as Theorem 2.5 using equation (30) and mathematical induction. The theorem is proved.

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References

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