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An Optimal Double Inequality between Geometric, Logarithmic and Arithmetic Means[1]

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Abstract

In this paper, we find the greatest value p and the least value q in $(0, 1/2)$ such that the double inequality $G(pa+(1-p)b, pb+(1-p)a) < \alpha A(a,b) + (1-\alpha)L(a,b) < G(qa+(1-q)b, qb+(1-q)a)$ holds for all $a, b > 0$ with $a \neq b$ and any $\alpha \in (0,1)$, where $G(a,b)$, $L(a,b)$ and $A(a,b)$ are respectively the geometric, logarithmic and arithmetic means of a and b.

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1. Introduction

For $a, b > 0$ with $a \neq b$, the classical geometric mean $G(a,b)$, arithmetic mean $A(a,b)$ and logarithmic mean $L(a,b)$ are defined by

$$G(a,b) = \sqrt{ab}, \quad A(a,b) = \frac{a+b}{2} \tag{1.1}$$

and

$$L(a,b) = \frac{a-b}{\log a - \log b}, \tag{1.2}$$

respectively.

It is well known that these means are very useful in the areas of engineering and technology, such as bridge engineering, structural mechanics, hydraulics, hydrology and geology, and the inequalities

$$\min\{a,b\} < G(a,b) < L(a,b) < A(a,b) < \max\{a,b\}$$

hold for all $a, b > 0$ with $a \neq b$.

Recently, the bounds for the logarithmic, arithmetic and geometric means have attracted the attention of many engineers and mathematicians. In particular, many remarkable inequalities for these means can be found in the literature [1-7].

Let $a, b > 0$ with $a \neq b$, $x \in [0, 1/2]$ and $g(x) = G(xa + (1-x)b, xb + (1-x)a)$. Then $g(x)$ is continuous and strictly increasing from $[0, 1/2]$ onto $[G(a,b), A(a,b)]$.

For $p, q, \alpha, \beta \in (0, 1/2)$, Chu et al. [8, 9] proved that the double inequalities

$$G(pa + (1-p)b, pb + (1-p)a) < I(a,b) < G(qa + (1-q)b, qb + (1-q)a)$$

and

$$G(\alpha a + (1-\alpha)b, \alpha b + (1-\alpha)a) < P(a,b) < G(\beta a + (1-\beta)b, \beta b + (1-\beta)a)$$

hold for all $a, b > 0$ with $a \neq b$ if and only if $p \leq (1 - \sqrt{1 - 4/e^2})/2$, $q \geq (3 - \sqrt{3})/6$, $\alpha \leq (1 - \sqrt{1 - 4/\pi^2})/2$ and $\beta \geq (3 - \sqrt{3})/6$. Here, $I(a,b) = 1/e(b^b/a^a)^{1/(b-a)}$ and $P(a,b) = (a-b)/[4\arctan\sqrt{a/b} - \pi]$ are the identric and Seiffert means of a and b, respectively.

Chu and Wang [10] found the least value $p \in (0, 1/2)$ such that the inequality

$$G(pa + (1-p)b, pb + (1-p)a) > AG(a,b)$$

holds for all $a, b > 0$ with $a \neq b$, where $AG(a,b)$ is the classical arithmetic-geometric mean of a and b, which is defined as the common limit of sequences $\{a_n\}$ and $\{b_n\}$ given by

$$a_0 = a, \quad b_0 = b,$$
$$a_{n+1} = \frac{a_n + b_n}{2} = A(a_n, b_n), \quad b_{n+1} = \sqrt{a_n b_n} = G(a_n, b_n).$$

In [11], the authors proved that if $r \in (0, 1/2)$, then the inequality

$$G(ra + (1-r)b, rb + (1-r)a) > L(a,b)$$

holds for all $a, b > 0$ with $a \neq b$ if and only if $r \geq (3 - \sqrt{6})/6$.

The aim of this paper is to find the greatest value $p = p(\alpha)$ and the least value $q = q(\alpha)$ in $(0, 1/2)$ such that the double inequality

$$\begin{aligned} G(pa + (1-p)b, pb + (1-p)a) &< \alpha A(a,b) + (1-\alpha)L(a,b) \\ &< G(qa + (1-q)b, qb + (1-q)a) \end{aligned}$$

holds for all $a, b > 0$ with $a \neq b$ and any $\alpha \in (0,1)$. Our main result is the following Theorem 1.1.

Theorem 1.1. If $p, q \in (0, 1/2)$ and $\alpha \in (0,1)$, then the double inequality

$$\begin{aligned} G(pa + (1-p)b, pb + (1-p)a) &< \alpha A(a,b) + (1-\alpha)L(a,b) \\ &< G(qa + (1-q)b, qb + (1-q)a) \end{aligned}$$

holds for all $a, b > 0$ with $a \neq b$ if and only if $p \leq (1 - \sqrt{1-\alpha^2})/2$ and $q \geq (3 - \sqrt{6(1-\alpha)})/6$.

2. Proof of Theorem 1.1

In order to prove Theorem 1.1 we need two lemmas, which we present in this section.

Lemma 2.1. If $t > 1$ and $\alpha \in (0,1)$, then $(1+\alpha)t\log^2 t - \alpha(t^2-1)\log t - (1-\alpha)(t-1)^2 < 0$.

Proof. Let $f(t) = (1+\alpha)t\log^2 t - \alpha(t^2-1)\log t - (1-\alpha)(t-1)^2$, $f_1(t) = tf'(t)$, $f_2(t) = tf_1''(t)/2$ and $f_3(t) = tf_2'(t)$, then simple computations lead to

$$f(1) = 0, \tag{2.1}$$

$$f_1(t) = (1+\alpha)t\log^2 t + (-2\alpha t^2 + 2\alpha t + 2t)\log t + (\alpha-2)t^2 + 2(1-\alpha)t + \alpha,$$
$$f_1(1) = 0, \tag{2.2}$$
$$f_1'(t) = (1+\alpha)\log^2 t + 4(-\alpha t + \alpha + 1)\log t + 4(1-t),$$

$$f_1'(1) = 0, \tag{2.3}$$

$$f_2(t) = (-2\alpha t + \alpha + 1)\log t + 2(\alpha + 1)(1 - t),$$

$$f_2(1) = 0, \tag{2.4}$$

$$f_3(t) = -2\alpha t \log t - 2(2\alpha + 1)t + \alpha + 1 < 0. \tag{2.5}$$

Therefore, Lemma 2.1 follows easily from (2.1)-(2.5). □

Lemma 2.2. If $t > 1$ and $\alpha \in (0, 1)$, then $[(3\alpha + 1)t^2 + 2(3\alpha + 5)t + (3\alpha + 1)]\log^2 t - 12\alpha(t^2 - 1)\log t - 12(1 - \alpha)(t - 1)^2 > 0$.

Proof. Let $g(t) = [(3\alpha + 1)t^2 + 2(3\alpha + 5)t + (3\alpha + 1)]\log^2 t - 12\alpha(t^2 - 1)\log t - 12(1-\alpha)(t-1)^2$, $g_1(t) = tg'(t)/2$, $g_2(t) = tg_1'(t)$, $g_3(t) = tg_2''(t)/2$ and $g_4(t) = tg_3'(t)$, then simple computations lead to

$$g(1) = 0, \tag{2.6}$$

$$\begin{aligned} g_1(t) =& [(3\alpha + 1)t^2 + (3\alpha + 5)t]\log^2 t + [(1 - 9\alpha)t^2 + 2(3\alpha + 5)t + (3\alpha + 1)]\log t \\ &+ 6(\alpha - 2)t^2 + 12(1 - \alpha)t + 6\alpha, \end{aligned}$$

$$g_1(1) = 0, \tag{2.7}$$

$$\begin{aligned} g_2(t) =& [2(3\alpha + 1)t^2 + (3\alpha + 5)t]\log^2 t + 4[(1 - 3\alpha)t^2 + (3\alpha + 5)t]\log t \\ &+ (3\alpha - 23)t^2 + 2(11 - 3\alpha)t + 3\alpha + 1, \end{aligned}$$

$$g_2(1) = 0, \tag{2.8}$$

$$\begin{aligned} g_2'(t) =& [4(3\alpha + 1)t + (3\alpha + 5)]\log^2 t + 6[2(1 - \alpha)t + (3\alpha + 5)]\log t \\ &+ 6(\alpha + 7)(1 - t), \end{aligned}$$

$$g_2'(1) = 0, \tag{2.9}$$

$$g_3(t) = 2(3\alpha + 1)t\log^2 t + (3\alpha + 5)(2t + 1)\log t + 3(3\alpha + 5)(1 - t),$$

$$g_3(1) = 0, \tag{2.10}$$

$$g_4(t) = 2(3\alpha + 1)t\log^2 t + 2(9\alpha + 7)t\log t + (3\alpha + 5)(1 - t),$$

$$g_4(1) = 0, \tag{2.11}$$

$$g_4'(t) = 2(3\alpha + 1)\log^2 t + 6(5\alpha + 3)\log t + 3(5\alpha + 3) > 0. \tag{2.12}$$

Therefore, Lemma 2.2 follows easily from (2.6)-(2.12). □

Proof of Theorem 1.1. Let $\lambda = (1 - \sqrt{1 - \alpha^2})/2$ and $\mu = (3 - \sqrt{6(1 - \alpha)})/6$, we first prove that the inequalities

$$\alpha A(a, b) + (1 - \alpha)L(a, b) > G(\lambda a + (1 - \lambda)b, \lambda b + (1 - \lambda)a) \tag{2.13}$$

and

$$\alpha A(a,b)+(1-\alpha)L(a,b)<G(\mu a+(1-\mu)b,\mu b+(1-\mu)a) \tag{2.14}$$

hold for all $a,b>0$ with $a\neq b$ and any $\alpha\in(0,1)$.

Since $A(a,b)$, $L(a,b)$ and $G(\lambda a+(1-\lambda)b,\lambda b+(1-\lambda)a)$ are symmetric and homogeneous of degree 1, without loss of generality, we assume that $a>b$. Let $t=a/b>1$ and $p\in(0,1/2)$, then from (1.1) and (1.2) one has

$$\begin{aligned}G(pa+(1-p)b,&pb+(1-p)a)-[\alpha A(a,b)+(1-\alpha)L(a,b)]\\&=\frac{A(a,b)}{(1+t)\log t}h(t),\end{aligned} \tag{2.15}$$

where

$$\begin{aligned}h(t)=&2\sqrt{p(1-p)t^2+(2p^2-2p+1)t+p(1-p)}\log t\\&-[\alpha(1+t)\log t+2(1-\alpha)(t-1)].\end{aligned} \tag{2.16}$$

Let

$$\begin{aligned}J(t)=&[2\sqrt{p(1-p)t^2+(2p^2-2p+1)t+p(1-p)}\log t]^2\\&-[\alpha(1+t)\log t+2(1-\alpha)(t-1)]^2\\=&[(4p-4p^2-\alpha^2)t^2+(8p^2-8p+4-2\alpha^2)t+(4p-4p^2-\alpha^2)]\log^2 t\\&-4\alpha(1-\alpha)(t^2-1)\log t-4(1-\alpha)^2(t-1)^2.\end{aligned} \tag{2.17}$$

We divide the proof into two cases.

Case 1 $p=\lambda=(1-\sqrt{1-\alpha^2})/2$. Then (2.17) becomes

$$J(t)=4(1-\alpha)[(1+\alpha)t\log^2 t-\alpha(t^2-1)\log t-(1-\alpha)(t-1)^2]. \tag{2.18}$$

Therefore, inequality (2.13) follows from (2.15), (2.16) and (2.18) together with Lemma 2.1.

Case 2 $p=\mu=(3-\sqrt{6(1-\alpha)})/6$. Then (2.17) reduces to

$$\begin{aligned}J(t)=&\frac{1}{3}(1-\alpha)\{[(3\alpha+1)t^2+2(3\alpha+5)t+(3\alpha+1)]\log^2 t\\&-12\alpha(t^2-1)\log t-12(1-\alpha)(t-1)^2\}.\end{aligned} \tag{2.19}$$

Therefore, inequality (2.14) follows from (2.15), (2.16) and (2.19) together with Lemma 2.2.

Next, we prove that $\lambda=(1-\sqrt{1-\alpha^2})/2$ is the best possible parameter in $(0,1/2)$ such that inequality (2.13) holds for all $a,b>0$ with $a\neq b$ and any

$\alpha \in (0,1)$. In fact, if $(1-\sqrt{1-\alpha^2})/2 = \lambda < p < 1/2$, then it follows from (2.15) and (2.16) that

$$\begin{aligned}\lim_{t\to+\infty}&[G(pa+(1-p)b, pb+(1-p)a) - \alpha A(a,b) - (1-\alpha)L(a,b)] \\ &= [2\sqrt{p(1-p)} - \alpha]A(a,b) > 0. \end{aligned}\tag{2.20}$$

Inequality (2.20) implies that for any $(1-\sqrt{1-\alpha^2})/2 = \lambda < p < 1/2$ there exists large enough T such that $G(pa+(1-p)b, pb+(1-p)a) > \alpha A(a,b) + (1-\alpha)L(a,b)$ for all $a/b \in (T, +\infty)$.

Finally, we prove that $\mu = (3-\sqrt{6(1-\alpha)})/6$ is the best possible parameter in $(0,1/2)$ such that inequality (2.14) holds for all $a, b > 0$ with $a \neq b$ and any $\alpha \in (0,1)$. Let $0 < \varepsilon < (3-\sqrt{6(1-\alpha)})/6$, $0 < t < 1$ and $p = (3-\sqrt{6(1-\alpha)})/6 - \varepsilon$, then making use of Taylor expansion we have

$$\begin{aligned}&G(p(1+t)+(1-p), p+(1-p)(1+t)) - [\alpha A(1+t,1) + (1-\alpha)L(1+t,1)] \\ &= 1 + \frac{1}{2}t + \frac{1}{2}(p-p^2-\frac{1}{4})t^2 - [\alpha(1+\frac{1}{2})t + (1-\alpha)(1+\frac{1}{2}t - \frac{1}{12}t^2)] + o(t^2) \\ &= -\frac{1}{6}(\sqrt{6(1-\alpha)} + 3\varepsilon)\varepsilon t^2 + o(t^2). \end{aligned}\tag{2.21}$$

Equation (2.21) implies that for $0 < p < (3-\sqrt{6(1-\alpha)})/6$ there exists $0 < \delta < 1$ such that $G(pa+(1-p)b, pb+(1-p)a) < \alpha A(a,b) + (1-\alpha)L(a,b)$ for all $a/b \in (1, 1+\delta)$. □

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