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Asymptotic Regional Gradient Full-Order Observer in Distributed Parabolic Systems

Raheam A. Al-Saphory

Department of Mathematics
College of Education for Pure Sciences
Tikrit University, Tikrit, Iraq

Naseif J. Al-Jawari

Department of Mathematics
College of Science
Al-Mustansriyah University, Baghdad, Iraq

Asmaa N. Al-Janabi

Department of Mathematics
College of Science
Al-Mustansriyah University, Baghdad, Iraq

Abstract

In this paper, the characterization of regional asymptotic gradient observer have been given for a parabolic system. The approach of this characterization derived from Luenberger observer theory which is enable to estimate asymptotically the gradient state of the original system in a subregion ω of a spatial domain Ω in order that the regional asymptotic gradient observability notion can be achieved. Furthermore, we show that the strategic sensors allows the existence of regional asymptotic gradient observer and a sufficient condition have been given for such regional asymptotic gradient observer in identity case. Thus, the obtained result are applied for different types of measurements, domains and boundary conditions.

Mathematics Subject Classification: 93A30; 93B07; 93B30; 93C20

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1. Introduction

There are many situations in modern technology in which it is necessary to estimate the state of a dynamic system using only the measured input and output data of the system [18]. An observer is a dynamic system $\hat{S}$ the purpose of which is to estimate the state of another dynamic system S using only the measured input and output letter. If the order of $\hat{S}$ is equal to the order of S the observer is called full-order state observer [3, 8, 12]. Thus, asymptotic observer theory explored by Luenberger in [20] for finite dimensional linear systems and extended infinite dimensional distributed parameter systems govern by strongly continuous semi-group in Hibert space by Gressang and Lamont as in [19]. The study of this approach via another variable like sensors and actuators developed by El-Jai et al. as in ref.s [3, 8, 12, 14] in order to achieve asymptotic observability. One of the most important approach in system theory is focused on reconstruction the state of the system from knowledge of dynamic system and the output function on a subregion ω of a spatial domain Ω this problem is called regional observability problem has been received much attention as in [7, 15-17]. An extension of this notion has been given in [4, 21] to the regional gradient case. The regional asymptotic notion has been introduced and developed by Al-Saphory and El- Jai in [2, 11]. Thus, this notion consists in studying the asymptotic behavior of the system in an internal subregion ω of a spatial domain Ω. In this paper, we develop the results of asymptotic regional state reconstruction in [6, 10] to the asymptotic regional gradient full order observer which allows to estimate the state gradient of the original system. The main reason for introducing this notion is that it provides a means to deal with some physical problems concern the model of single room shown in (Figure 1) below.

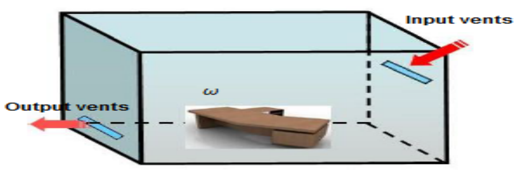

Fig. 1: Room observation problem Ω , workspace ω, and input-output vents.

Now the object to design the room (locate vents, place sensors, etc. …) in order to observe asymptotically the room vents near workspace (for more details see [13]). The outline of this paper is organized as follows: Section 2 is devoted to the problem statement and some basic concept related to the regional gradient stability, regional asymptotic gradient detectability. Section 3 we focus on regional asymptotic gradient observer so we introduced and characterization the existing of identity regional asymptotic gradient observer to provide an identity regional asymptotic gradient estimator of gradient state for the original system in terms of sensors structure. In the last section we have been applied these result to the two dimensional distributed parameter systems for different zone and pointwise sensors case.

2. Problem Formulation and Preliminaries

Let Ω be a regular bounded open subset of R^n, with smooth boundary $\partial\Omega$ and ω be subregion of Ω, $[0,T]$, $T > 0$ be a time measurement interval. We denoted $Q = \Omega \times]0,T[$, $\Sigma = \partial\Omega \times]0,T[$. We considered distributed parabolic systems is described by the following partial deferential equations

$$S\begin{cases} \frac{\partial x}{\partial t}(\xi,t) = Ax(\xi,t) + Bu(t) & Q \\ x(\xi,0) = x_0(\xi) & \Omega \\ x(\eta,t) = 0 & \Sigma \end{cases} \quad (1)$$

Augmented with the output function

$$y(.,t) = Cx(\,,t) \quad (2)$$

where A is a second order linear differential operator, which generator a strongly continuous semi-group $\left(S_A(t)\right)_{t\geq 0}$on the Hilbert space X and is self-adjoin with compact resolvent. The operator $B \in L(R^p, X)$ and $C \in L(R^q, X)$, depend on the structure of actuators and sensors [16]. The space X, U and $\mathcal{O}$ be separable Hilbert spaces where X is the state space, $U = L^2(0,T,R^p)$ is the control space and $\mathcal{O} = L^2(0,T,R^q)$ is the observation space where p and q are the numbers of actuators and sensors (see Figure 2) which is mathematical model more general spatial case in (Figure 1).

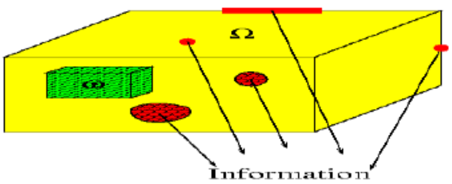

Fig. 2: The domain of $\boldsymbol{\Omega}$, the sub-region $\boldsymbol{\omega}$, various sensors locations.

Under the given assumption, the system (1) has a unique solution [18]:

$$x(\xi\ ,t) = S_A(t)x_0(\xi) + \int_0^t S_A(t-\tau)Bu(\tau)d\tau \quad (3)$$

The measurements are obtained through the output function by using of zone, point wise which may located in Ω (or $\partial\Omega$). [16]

$$y(.,t) = Cx(\xi\ ,t) \quad (4)$$

- We first recall a sensors is defined by any couple *(D, f)*, where *D* is its spatial support represented by a nonempty part of $\overline{\Omega}$ and f represents the distribution of the sensing measurements on *D*.
- Depending on the nature of D and f , we could have various type of sensors. A sensor may be pointwise if *D={b}* with $b \in \overline{\Omega}$ and $f = \delta(.-b)$, where δ is the Dirac mass concentrated at *b*. In this case the operator *C* is unbounded [10] and the output function (2) can be written in the form

$$y(t) = x(b,t) = \int_\Omega x(\xi,t)\delta_b(\xi - b)d\xi$$

It may be zonal when $D \subset \overline{\Omega}$ and $f \in L^2(D)$. The output function (2) can be written in the form

$$y(t) = \int_D x(\xi,t)f(\xi)d\xi$$

In the case of boundary zone sensor, we consider $D_i = \Gamma_i \subset \partial\Omega$ and $f_i \in L^2(\Gamma_i)$, the output function (2) can be written as

$$y(.,t) = Cx(.,t) = \int_{\Gamma_i} x(\eta,t)\, f_i(\eta)d\eta$$

- We define the operator K by the form

$$K: x \in X \to Kx = CS_A(.)x\epsilon\ \mathcal{O}$$

With $K^*: \mathcal{O} \to X$ is the adjoint operator of K defined by

$$K^*y^* = \int_0^t S_A^*(s)C^*y^*(s)ds$$

- Consider the gradient operator $\nabla: H^1(\Omega) \to (H^1(\Omega))^n$

• $\tilde{\chi}_\omega : \begin{cases} H^1(\Omega) \to H^1(\omega) \\ x \to \tilde{\chi}_\omega x = x\,|_\omega \end{cases}$

where $x\,|_\omega$ is the restriction of x to ω and it's adjoint is denoted by $\tilde{\chi}_\omega^*$.

• Finally, we introduced the operator $H = \chi_\omega \nabla K^*$ from $\mathcal{O}$ into $(H^1(\omega))^n$ where its adjoint given by H^*.

The problem is how to build an approach which observe (estimates) regional gradient state in subregion in ω of **Ω** asymptotically by using a dynamic system (an observer) in identity case only may be called full-order observer in region ω. The important of an observer is that to estimates all the gradient state variables, regardless of whether some are available for direct measurements or not [18].

• The systems (1)-(2) are said to be exactly regionally gradient observable on ω (exactly ω_G –observable) if

$$Im\, H = Im\, \chi_\omega \nabla K^* = (H^1(\omega))^n.$$

• The systems (1)–(2) are said to be weakly regionally gradient observable on ω (weakly ω_G –observable) if

$$\overline{Im H} = \overline{Im\, \chi_\omega \nabla K^*} = (H^1(\omega))^n.$$

It is equivalent to say that the systems (1)-(2) are weakly ω_G –observable if

$$ker\ H^* = ker\, K\nabla^* \chi_\omega{}^* = \{0\}.$$

• If The systems (1)–(2) are is weakly ω_G –observable, then $x_0(\xi\,,0)$ is given by

$$x_0 = (K^*K)^{-1}K^*y = K^\dagger y,$$

where $K^\dagger$ is the pseudo-inverse of the operator K (see ref. [15, 17]).

• A sensor (D, f) is gradient strategic on ω (ω_G-strategic) if the observed system is weakly ω_G-observable.

As well known the observability [8, 18] and asymptotic observability [14, 16, 18-20] are important concepts to estimate the unknown state of the considered dynamic system from the input and output functions. Thus, These notions are studied and introduced to the regional distributed parameter systems analysis with different characterizations by El-Jai , Zerrik and Al-Saphory *et al.* in many paper for example [2, 4-7, 9-11, 13, 15, 17, 21] in connection with strategic sensors.

Definition 2.1: The system (1) is said to be asymptotically regionally gradient stable (asymptotically ω_G-stable) if the operator A generates a semi-group which is asymptotically gradient stable on the $(H^1(\omega))^n$. It is easy to see that the system (1) is asymptotically ω_G-stable, if and only if for some positive constants $M_{\omega_G}, \alpha_{\omega_G}$, we have

$$\|\chi_\omega \nabla S_A(.)\|_{\mathcal{L}\left(H^1(\Omega),\left(H^1(\Omega)\right)^n\right)} \le M_{\omega_G} e^{-\alpha_{\omega_G} t}, \forall t \ge 0$$

If $(S_A(t))_{t\geq 0}$ is ω_G-stable semi-group in $(H^1(\omega))^n$, then for all $x_0 \in H^1(\Omega)$, the solution of associated system satisfies

$$\lim_{t\to\infty} \|\nabla x(.,t)\|_{(H^1(\omega))^n} = \lim_{t\to\infty} \|\chi_\omega \nabla S_A(.)x_0\|_{(H^1(\omega))^n} = 0 \quad (5)$$

Definition 2.2: The systems (1)-(2) are said to be asymptotically regionally gradient detectable (asymptotically ω_G-detectable) if there exists an operator $H_{\omega_G}: R^q \to (H^1(\omega))^n$ such that $\left(A - H_{\omega_G}C\right)$ generates a strongly continuous semi-group $(S_{H_{\omega_G}}(t))_{t\geq 0}$ which is asymptotically G-stable on $(H^1(\omega))^n$.

Remark 2.3: In this paper, we only need the relation (5) to be true on a sub-region ω of the region Ω

$$\lim_{t\to\infty} \|\nabla x(.,t)\|_{(H^1(\omega))^n} = 0.$$

Thus, from the previous results we can deduce the following results:

Corollary 2.4: If The systems (1)–(2) are is exactly ω_G-observable, then it is asymptotically ω_G –detectable.

Corollary 2.5: For every $x^* \in (H^1(\omega))^n$ there exists $\gamma > 0$, such that

$$\|\nabla x(.,t)\|_{(H^1(\omega))^n} \leq \gamma \|K \nabla^* \chi_\omega{}^* x^*\|_O = 0$$

Remark 2.6: For parabolic systems, the notion of asymptotic ω_G –detectability is far less restrictive than the exact ω_G-observability.

3. Sensors and asymptotic ω_{GFO} –Observer

In this section we present the sufficient conditions which are guarantee the existence of an asymptotic regional gradient full-order observer (asymptotic ω_{GFO} –Observer) which allows to construct an ω_{GFO} –estimator of the state $\chi_\omega \nabla T x(\xi\,,t)$.

3.2 Definitions and characterizations

Definition 3.1: Suppose there exists a dynamical system with state $z(.,t) \in Z$ given by

$$\hat{S}\begin{cases} \frac{\partial \hat{z}}{\partial t}(\xi,t) = A\hat{z}(\xi,t) + Bu(t) + H_{\omega_G}(Cx(\xi,t) - C\hat{z}(\xi,t)) & Q \\ \hat{z}(\xi,0) = \hat{z}_0(\xi) & \Omega \\ \hat{z}(\eta,t) = 0 & \Sigma \end{cases} \quad (6)$$

In this case the operator F_{ω_G} in general case [17] is given by $F_{\omega_G} = A - H_{\omega_G}C$ where $T = I$ the identity operator. Thus the operator $A - H_{\omega_G}C$ generator a strongly continuous semi-group $(S_{A-H_{\omega_G}C}\ (t))_{t\geq 0}$ on separable Hilbert space Z which is asymptotically ω_G –stable.

Thus, $\exists M_{A-H_{\omega_G}C}, \alpha_{A-H_{\omega_G}C} > 0$ such that

$$\left\| S_{A-H_{\omega_G}C}\ (.) \right\| \leq M_{A-H_{\omega_G}C}\ e^{-\alpha_{A-H_{\omega_G}C}\ t}, \forall t \geq 0.$$

and let $G_{\omega_G} \in L(U, Z), H_{\omega_G} \in L(\mathcal{O}\ , Z)$ such that the solution of (6) similar to (3)

$$z(\xi\ , t) = S_{A-H_{\omega_G}C}(t)z(\xi) + \int_0^t S_{A-H_{\omega_G}C}(t-\tau)[Bu(\tau) + H_{\omega_G}y(\tau)]d\tau$$

Definition 3.2: The system (6) defines asymptotic identity (full-order) ω_{GFO} –estimator for $z(\xi, t) = \chi_\omega \nabla T x(\xi\ , t) = Ix(\xi\ , t) \in (H^1(\omega))^n$ where $x(\xi, t)$ is the solution of the systems (1)-(2) if $\lim_{t\to\infty} \|z(.\ , t) - x(\xi\ , t)\|_{(H^1(\omega))^n} = 0$, and $\chi_\omega \nabla I$ maps $D(A)$ into $D(A - H_{\omega_G}C)$ where $z(\xi, t)$ is the solution of system (6).

Remark 3.3: The dynamic system (6) specifies an asymptotic ω_{GFO} –observer of the systems given by (1) and (2) if the following holds:

1- There exists $M_{\omega_G} \in L(R^q, (H^1(\omega))^n)$ and $N_{\omega_G} \in L((H^1(\omega))^n)$ such that
   $M_{\omega_G}C + N_{\omega_G} = I_{\omega_G}$.
2- $A - F_{\omega_G} = H_{\omega_G}\text{C}$ and $G_{\omega_G} = \text{B}$.
3- The system (6) defines an asymptotic ω_{GFO} –estimator for $x(\xi\ , t)$.

The object of an asymptotic ω_{GFO}- observer is to provide an approximation to the original system state gradient. This approximation is given by

$$\hat{x}(t) = M_{\omega_G}y(t) + N_{\omega_G}z(t).$$

Definition 3.4: The systems (1)-(2) are asymptotically ω_{GFO}-observable, if there exists a dynamic system which is asymptotic ω_{GFO}-observer for the original system.

3.2 Asymptotic ω_{GFO}-Observer reconstruction

In this case, we need to consider $\chi_\omega \nabla T = I$ and $Z = X$, then the operator observer equation becomes as $F_{\omega_G} = A - H_{\omega_G}C$ where A and C are known. Thus, the operator H_{ω_G} must be determined such that the operator F_{ω_G} is asymptotically ω_G-stable. This observer is an extension of asymptotic observer as in [2, 10-11, 16]. Now Consider again system (1) together with output function (2) described by the following form

$$\begin{cases} \frac{\partial x}{\partial t}(\xi,t) = Ax(\xi,t) + Bu(t) & Q \\ x(\xi,0) = x_0(\xi) & \Omega \\ x(\eta,t) = 0 & \Sigma \\ y(t) = Cx(.,t) & Q \end{cases} \quad (7)$$

Let ω be a given subdomain of Ω and suppose that $I \in \mathcal{L}((H^1(\Omega))^n)$, and $\chi_\omega \nabla Tx(\xi,t) = \chi_\omega \nabla x(\xi,t)$ there exists a system with state $z(\xi,t)$ such that $z(\xi,t) = \chi_\omega \nabla Tx(\xi,t) = T_\omega x(\xi,t)$ with $T_{\omega_G} = I_{\omega_G}$, where I_{ω_G} is the identity operator with respect to regional gradient state estimator. Then

$$z(\xi,t) = x(\xi,t) \quad (8)$$

From equation (7) and (8) we have

$$\begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} C \\ I_{\omega_G} \end{bmatrix} x$$

If we assume that there exist two bounded linear operators $M_{\omega_G}: \mathcal{O} \to ((H^1(\omega))^n$ and
$N_{\omega_G}: ((H^1(\omega))^n \to ((H^1(\omega))^n$, such that $M_{\omega_G}C + N_{\omega_G}T_\omega = I_{\omega_G}$ then by deriving $z(\xi,t)$ in (8) we have

$$\frac{\partial z}{\partial t}(\xi,t) = I_{\omega_G}\frac{\partial x}{\partial t}(\xi,t) = \chi_\omega \nabla TAx(\xi,t) + \chi_\omega \nabla TBu(t)$$
$$= \chi_\omega \nabla I_{\omega_G} AM_{\omega_G} y(\xi,t) + \chi_\omega \nabla I_{\omega_G} AN_{\omega_G} z(\xi,t) + \chi_\omega \nabla I_{\omega_G} Bu(t)$$

Since the operator $T_{\omega_G} = I_{\omega_G}$, then we have

$$\frac{\partial z}{\partial t}(\xi,t) = AN_{\omega_G} z(\xi,t) + Bu(t) + AM_{\omega_G} y(\xi,t)$$

Therefore

$$\frac{\partial \hat{z}}{\partial t}(\xi,t) = F_{\omega_G}\hat{z}(\xi,t) + G_{\omega_G}u(t) + H_{\omega_G}y(.,t)$$

and since $A - F_{\omega_G} = H_{\omega_G}\mathrm{C}$ and $G_{\omega_G} = \mathrm{B}$ then we have

$$\hat{S}\begin{cases} \frac{\partial \hat{z}}{\partial t}(\xi,t) = A\hat{z}(\xi,t) + Bu(t) + H_{\omega_G}(cx(\xi,t) - c\hat{z}(\xi,t)) & Q \\ \hat{z}(\xi,0) = \hat{z}_0(\xi) & \Omega \\ \hat{z}(\eta,t) = 0 & \Sigma \end{cases} \quad (9)$$

Let us consider a complete sets of eigenfunctions φ_{nj} in $(H^1(\Omega))^n$ orthonormal to $(H^1(\omega))^n$ associated with the eigenvalue λ_n of multiplicity r_n and suppose the system (1) has unstable mode. Then, the sufficient condition of an ω_{GFO}-observer is formulated in the following main result.

Theorem 3.5: Suppose that there are q zone sensors $(D_i, f_i)_{1 \le i \le q}$ and the spectrum of A contains J eigenvalues with non-negative real parts. Then the dynamic system (9) is ω_{GFO}-observer system for the system (7), that is $\lim_{t\to\infty}[z(\xi,t) - \hat{z}(\xi,t)] = 0$, if :

1-There exists $M_{\omega_G} \in L(R^q, (H^1(\omega))^n)$ and $N_{\omega_G} \in L((H^1(\omega))^n)$ such that $M_{\omega_G}C + N_{\omega_G} = I_{\omega_G}$.
2- $A - F_{\omega_G} = H_{\omega_G}\mathrm{C}$, $G_{\omega_G} = \mathrm{B}$.
3- $q \geq m$
4- rank $G_m = m_m$, $\forall m, m = 1, \dots, J$ with

$$G_m = (G_m)_{ij} = \begin{cases} \psi_{mj}(b_i), f_i(.) >_{L^2(D_i)} & \text{for zone sensors} \\ \psi_{mj}(b_i) & \text{for pointwise sensors} \\ < \dfrac{\partial \psi_{mj}}{\partial v}, f_i(.) >_{L^2(\Gamma_i)} & \text{for boundary zone sensors} \end{cases}$$

where sup $m_m = m < \infty$ and $\mathrm{j} = 1, \dots, m_m$.

Proof:

First step: The proof is limited to the case of pointwise sensors. Under the assumptions of section 2, the system (1) can be decomposed by the projections P and $I - P$ on two parts, unstable and stable. The state vector may be given by $x(\xi, t) = [x_1(\xi, t), x_2(\xi, t)]^{tr}$ where $x_1(\xi, t)$ is the state component of the unstable part of the system (1) may be written in the form

$$\begin{cases} \frac{\partial x_1}{\partial t}(\xi, t) = A_1 x_1(\xi, t) + PBu(t) & Q \\ x_1(\xi, 0) = x_{01}(\xi) & \Omega \\ x_1(\eta, t) = 0 & \Theta \end{cases} \tag{10}$$

and $x_2(\xi, t)$ is the component state of the stable part of the system (1) given by

$$\begin{cases} \frac{\partial x_2}{\partial t}(\xi, t) = A_2 x_2(\xi, t) + (I - P)Bu(t) & Q \\ x_2(\xi, 0) = x_{02}(\xi) & \Omega \\ x_2(\eta, t) = 0 & \Theta \end{cases} \tag{11}$$

The operator A_1 is represented by a matrix of order $(\sum_{m=1}^{J} m_m, \sum_{m=1}^{J} m_m)$ given by $A_1 = \mathrm{diag}[\lambda_1, \dots, \lambda_1, \lambda_2, \dots, \lambda_2, \dots, \lambda_J, \dots, \lambda_J]$ and $PB = [G_1^{tr}, G_2^{tr}, \dots, G_J^{tr}]$. The condition (4) of this theorem, allows that the suit $(D_i, f_i)_{1 \leq i \leq q}$ of sensors is ω_G-strategic for the unstable part of the system (1), the subsystem (10) is weakly ω_G-observable [4] and since it is finite dimensional, then it is exactly ω_G-observable [5]. Therefor it is asymptotically ω_G-detectable, and hence there exists an operator H_ω^1 such that $(A_1 - H_\omega^1 C)$ which is satisfied the following:

$\exists, M_{A_1 - H_\omega^1 C}, \alpha_{A_1 - H_\omega^1 C} > 0$ such that $\left\| e^{(A_1 - H_\omega^1 C)t} \right\|_{(H^1(\omega))^n} \leq M_{A_1 - H_\omega^1 C}, e^{-\alpha_{A_1 - H_\omega^1 C} t}$

and then we have

$$\|x_1(., t)\|_{(H^1(\omega))^n} \leq M_{A_1 - H_\omega^1 C}, e^{-\alpha_{A_1 - H_\omega^1 C}} \|Px_0(.)\|_{(H^1(\omega))^n}.$$

Since the semi-group generated by the operator A_2 is stable on $(H^1(\omega))^n$, then there exist $M_{A_2-H_\omega^2 C}, \alpha_{A_2-H_\omega^2 C} > 0$ [20] such that

$$\|x_2(.,t)\|_{(H^1(\omega))^n} \leq M_{A_1-H_\omega^1 C,} e^{-\alpha_{A_2-H_\omega^2 C}(t)} \|(I-P)x_0(.)\|_{(H^1(\omega))^n} + \int_0^t M_{M_{A_1-H_\omega^1 C,}} e^{-\alpha_{A_2-H_\omega^2 C}(t)} \|(I-P)x_0(.)\|_{(H^1(\omega))^n} \|u(\mathcal{T})\| \, d\mathcal{T}$$

and therefor $x(\xi,t) \to 0$ when $t \to \infty$. Finally, the system (9) are asymptotically ω_G-detectable.

Second step: From equation (8), we have $z(\xi,t) = x(\xi,t)$ with the observer error is given by the following form

$$e(\xi,t) = z(\xi,t) - \hat{z}(\xi,t)$$

Where $\hat{z}(\xi,t)$ is a solution of the dynamic system (9). Derive the above equation, and by using equation (8) and condition 2, we can get the following forms

$$\begin{aligned}
\frac{\partial e}{\partial t}(\xi,t) &= \frac{\partial z}{\partial t}(\xi,t) - \frac{\partial \hat{z}}{\partial t}(\xi,t) = \frac{\partial x}{\partial t}(\xi,t) - \frac{\partial \hat{z}}{\partial t}(\xi,t)\\
&= I_{\omega_G} A x(\xi,t) + I_{\omega_G} B u(t) - F_\omega \hat{z}(\xi,t) - G_\omega u(t) - H_\omega C x(\xi,t)\\
&= A x(\xi,t) - (A - H_{\omega_G} C)\hat{z}(\xi,t) - H_{\omega_G} C x(\xi,t)\\
&= \left(A - H_{\omega_G} C\right)(z(\xi,t) - \hat{z}(\xi,t))\\
&= \left(A - H_{\omega_G} C\right) e(\xi,t)
\end{aligned}$$

Thus, from the first part of this proof we obtain $e(\xi,t) = \left(A - H_{\omega_G} C\right) e(0,t)$ is asymptotically ω_G-stable with $e(0,t) = z_0(\xi) - \hat{z}_0(\xi)$.

$$\|e(\xi,t)\|_{(H^1(\omega))^n)} \leq M_{A-H_{\omega_G}C} e^{-\alpha_{A-H_{\omega_G}C} t} \|z_0(\xi) - \hat{z}_0(\xi)\|_{(H^1(\omega))^n)}$$

therefore $\lim_{t\to\infty} e(\xi,t) = 0$. Now, let the approximate solution to the gradient state of the original system is

$\hat{x}(\xi,t) = M_{\omega_G} y(.,t) + N_{\omega_G} \hat{z}(\xi,t)$ with $M_{\omega_G} = o$ and $N_{\omega_G} = I_{\omega_G}$, then we have

$$\hat{x}(\xi,t) = \hat{z}(\xi,t)$$

Now, we can calculate the error of gradient state estimator

$$\begin{aligned}
\hat{e}_{I_{\omega_G}}(\xi,t) &= x(\xi,t) - \hat{x}(\xi,t) = x(\xi,t) - x(\xi,t) + x(\xi,t) - \hat{x}(\xi,t)\\
&= \hat{x}(\xi,t) - \hat{z}(\xi,t) = e(\xi,t) = \left(A - H_{\omega_G} C\right) e(0,t)
\end{aligned}$$

is asymptotically ω_G-stable with $e(0,t) = z_0(\xi) - \hat{z}_0(\xi)$. Consequently we get

$$\lim_{t\to\infty} \|x(.,t) - \hat{x}(\xi\,,t)\|_{(H^1(\omega))^n} = \lim_{t\to\infty} \left\|I_{\omega_G} x(.,t) - \hat{z}(\xi\,,t)\right\|_{(H^1(\omega))^n} = 0.$$

Then, the dynamical system (9) is ω_{GFO}-0bserver to the system (7). ■

Corollary 4.4 From the previous results, we can deduce that**:**

1. Theorem 3.5 gives the sufficient conditions which guarantee the dynamic system (9) is a ω_{GFO}-observer for the system (7).
2. If a system which is an Ω_{GFO}-observer, then it is ω_{GFO}-observer for system (7).
3. If a system is ω_{GFO}-observer, then it is ω^1_{GFO}-observer for every subset ω^1 of ω_G, but the converse is not true [6].

4. Application to asymptotic ω_{GFO}-observer in diffusion system

In this section we consider the distributed diffusion systems defined on Ω_k; where $1 \le k \le 2$. Various results related to different types of sensor have been extended. In the case of two-dimensional, we take $\Omega_2 =]0, a_1[\times]0, a_2[$ and $Q_2 = \Omega_2 \times]0, \infty[$, $\Sigma_2 = \overline{\Omega}_2 \times]0, \infty[$, with boundaries $\Theta_2 = \partial\Omega_2 \times]0, \infty[$.

4.1. Two-dimensional system with rectangular domain

4.1.1. Case of zone sensors

Consider a two dimensional system defined in $\Omega_2 =]0, a_1[\times]0, a_2[$ by parabolic equation

$$\begin{cases} \frac{\partial x}{\partial t}(\xi_1, \xi_2, t) = \frac{\partial^2 x}{\partial \xi_1^2}(\xi_1, \xi_2, t) + \frac{\partial^2 x}{\partial \xi_2^2}(\xi_1, \xi_2, t) & Q_2 \\ x(\xi_1, \xi_2, 0) = x_0(\xi_1, \xi_2) & \Omega_2 \\ x(\xi, \eta, t) = 0 & \Sigma_2 \end{cases} \tag{12}$$

Augmented with output function measured by internal or boundary zone sensors

$$y(.,t) = \int_D x(\xi_1, \xi_2, t) f(\xi_1, \xi_2) d\xi_1 d\xi_2 \text{ or } (y(.,t) = \int_\Gamma x(\eta_1, \eta_2, t) f(\eta_1, \eta_2) d\eta_1 d\eta_2) \tag{13}$$

Where $D \subseteq \Omega$ and $\Gamma, \overline{\Gamma} \subseteq \partial\Omega$, see (Figure 3).

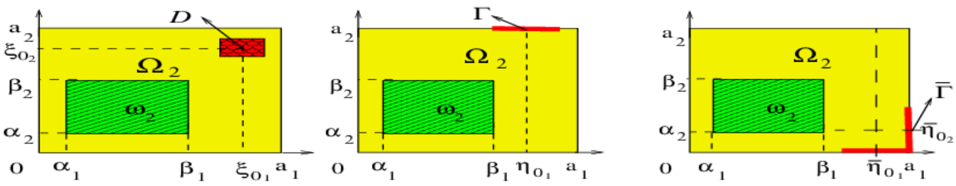

Fig. 3: shows the domain Ω_2 region ω_2 and locations of internal (boundary) zone sensors

Let $\omega =]\alpha_1, \beta_1[\times]\alpha_2, \beta_2[$ be a subregion of Ω. The eigenfunctions of the operator $(\frac{\partial^2}{\partial \xi_1^2} + \frac{\partial^2}{\partial \xi_2^2})$ are defined by

$$\varphi_{ij}(\xi_1, \xi_2) = \frac{2}{\sqrt{(\beta_1-\alpha_1)(\beta_2-\alpha_2)}} sin\, i\pi \frac{\xi_1-\alpha_1}{\beta_1-\alpha_1} sin j\pi \frac{\xi_2-\alpha_2}{\beta_2-\alpha_2}$$

Associated with the eigenvalues

$$\lambda_{ij} = -(\frac{i^2}{(\beta_1-\alpha_1)^2} + \frac{j^2}{(\beta_2-\alpha_2)^2})\pi^2$$

Now, consider the dynamical system

$$\begin{cases} \frac{\partial z}{\partial t}(\xi_1, \xi_2, t) = \frac{\partial^2 z}{\partial \xi_1^2}(\xi_1, \xi_2, t) + \frac{\partial^2 z}{\partial \xi_2^2}(\xi_1, \xi_2, t) - H_{\omega_G}\big(C\, z(\xi_1, \xi_2, t) - y(t)\big) & Q_2 \\ z(\xi_1, \xi_2, 0) = z_0(\xi_1, \xi_2) & \Omega_2 \\ z(\eta_1, \eta_2, t) = 0 & \Sigma_2 \end{cases} \tag{14}$$

And suppose that the sensors is ω_G-strategic for unstable subsystem part of the system (8), then we have the following results:

Proposition 4.1:

1- Internal zone case: Suppose that f_1 is symmetric about ξ=ξ_{01} and f_2 is symmetric about ξ=ξ_{02}, then the dynamic system (14) is ω_{GFO}-observer systems (12)-(13) if $\frac{i(\xi_{01}-\alpha_1)}{\beta_1-\alpha_1}$ and $\frac{j(\xi_2-\alpha_2)}{\beta_2-\alpha_2} \notin$ N for every $i, j = 1, \ldots, J$.

2- One side boundary zone case: Suppose that $\Gamma \subset \partial\Omega$ and f is symmetric with respect to $\eta_1 = \eta_{0_1}$, then the dynamic system (14) is ω_{GFO}-observer for the systems (12)-(13) if $i(\eta_1 - \alpha_1)/(\beta_1 - \alpha_1) \notin N$ for every i, $i = 1, , \ldots, J$.

3- Two side boundary zone case:

Let $\overline{\Gamma} = \overline{\Gamma}_1 \times \overline{\Gamma}_2 = (\overline{\eta}_{0_1} - l_1, a) \times \{0\} \cup \{a_1\} \times [0, \overline{\eta}_{0_2} + l_2] \subset \partial\Omega$ and $f|_{\Gamma_1}$ is symmetric with respect to $\eta_1 = \overline{\eta}_{0_1}$, and the function $f|_{\Gamma_2}$ is symmetric with respect to $\eta_2 = \overline{\eta}_{0_2}$, then the dynamic system (14) is ω_{GFO}--observer for the system (12)-(13), if $i(\overline{\eta}_{0_1} - \alpha_1)/(\beta_1 - \alpha_1)$ and $j(\overline{\eta}_{0_2} - \alpha_2)/(\beta_2 - \alpha_2) \notin N$ for every *i, j =1 ,..., J*.

4.1.2. Case of pointwise sensors

Consider again the systems (12)-(13) augmented with output function measured by internal or boundary pointwise sensors (Figure 4).

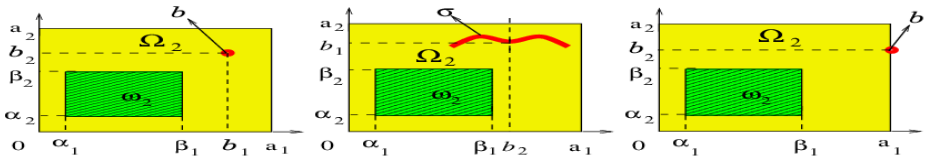

Fig. 4: Rectangular domain and locations *b, σ* of pointwise sensors

$$\boldsymbol{y(.,t) = \int_D x(\xi_1,\xi_2,t)\delta_b(\xi_1-b_1,\xi_2-b_2)d\xi_1 d\xi_2 (y(.,t) = \int_{\partial\Omega}\frac{\partial x}{\partial\vartheta}(\eta_1,\eta_2,t) f(\eta_1,\eta_2)d\eta_1 d\eta_2} \quad (15)$$

Proposition 4.2:

1- The internal pointwise case:

If $i(b_1-\alpha_1)/(\beta_1-\alpha_1)$ and $j(b_2-\alpha_2)/(\beta_2-\alpha_2)\notin N$, for every $i,j = 1,\dots,J$, then the dynamic system (14) is ω_{GFO}-observer for the systems (12)-(15).

2- Filament pointwise case: Suppose that the observation is given by the filament sensor (σ,δ_σ), where $\sigma = Im(\gamma)$ is symmetric with respect to the line $b=(b_1,b_2)$. The dynamic system (14) is ω_{GFO}-observer for the system (12)-(15), if $i(b_1-\alpha_1)/(\beta_1-\alpha_1)$ and $j(b_2-\alpha_2)/(\beta_2-\alpha_2)\notin N$, for every i, j=1,. . . , J.

3- The boundary pointwise case:

If $i(b_1-\alpha_1)/(\beta_1-\alpha_1)$ and $j(b_2-\alpha_2)/(\beta_2-\alpha_2)\notin N$, for $every\ i,j = 1,\dots,J$, then the dynamic system (14) is ω_{GFO}-observer for the system (12)-(15).

4.2. Two-dimensional systems with circular domain

Remark 4.3: We can extend these results to the case of two dimensional systems with circular domain in different sensor structures as in [2, 10].

4.3. One-dimensional systems domain case

Remark 4.4: We can extend the above results of the two dimensional systems (12)-(13) to case of one dimensional systems case if we take $\Omega_1 =]0,a[$. We denote $Q_1 = \Omega_1 \times]0,\infty[$, $\Sigma_1 = \overline{\Omega_1} \times]0,\infty[$ with boundaries $\Theta_1 = \partial\Omega_1 \times]0,\infty[$ as in ref.s [2, 4-7, 9-11, 13, 15, 17, 21].

Remark 4.5: We know that the previous results have been developed with Dirichlet boundary conditions, then we can extend with Neumann or mixed boundary conditions as in [1, 17].

5. Conclusion

The concept studied in this paper is related to the ω_{GFO}-observer in connection with sensors structure for a class of distributed parameter systems. More precisely, we have been given a sufficient condition for existing an ω_{GFO}-observer which allows to estimate the gradient state in a subregion ω. For future work, one can be extension these result to the problem of regional boundary gradient observer in connection with the sensors structures as in [5].

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