

On the k -Jacobsthal Lucas Numbers of Arithmetic Indexes

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Abstract

In this paper, the k -Jacobsthal Lucas numbers of arithmetic indexes of the form $an + r$, where n is a natural number and r is less than a are investigated. A formula is proven for the sum of these numbers and by using this formula we deduced the sums of the first k -Jacobsthal Lucas numbers, even and odd k - Jacobsthal Lucas numbers. The same formula for the alternated k - Jacobsthal Lucas numbers are also found. Later, the generating function of these numbers are obtained. Finally, some relations between the k - Jacobsthal numbers and the k - Jacobsthal Lucas numbers are derived.

Mathematics Subject Classifications: 11B83, 15B39

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1 Introduction

The Fibonacci sequence and the Lucas sequence are used in abundant applications of many science areas. Fibonacci numbers have been generalized by different authors. Some authors have generalized the Fibonacci sequence by preserving the recurrence relation and altering the first two terms of the sequence, while others have generalized the Fibonacci sequence by preserving the first two terms but altering the recurrence relation slightly.

There are many articles in the literature that study on the different number sequences. More recently, Pell, Pell-Lucas, Jacobsthal, Jacobsthal-Lucas sequences have been generalized for any positive real number k . Also the studies of the k -Fibonacci sequence, the k -Lucas sequence, the k -Pell sequence, the k -Pell-Lucas sequence, the modified k -Pell sequence, the k -Jacobsthal sequence and the k -Jacobsthal Lucas sequence are appeared. Falcon and Plaza [2] have found several formulae for k -Fibonacci numbers with indexes in an arithmetic sequence. Falcon [3] defined the k -Lucas number with indexes in an arithmetic sequence. Also, deduced generating function and several sum formulae for these numbers with indexes in an arithmetic sequence. In [4] some properties of k -Jacobsthal numbers with arithmetic indexes is studied.

From these sequences, Jacobsthal and Jacobsthal Lucas numbers are given by the recurrence relations $j_n = j_{n-1} + 2j_{n-2}$, $j_0 = 0$, $j_1 = 1$ and $c_n = c_{n-1} + 2c_{n-2}$, $c_0 = 2$, $c_1 = 1$ for $n \geq 2$, respectively.

2 k -Jacobsthal Lucas Sequences

The second order recurrence sequence has been generalized in two ways mainly, first by preserving the initial conditions and second by preserving the recurrence relation. We used the first way for the following definition

Definition 1 Let be $n \in \mathbb{N}$, $k > 0$ any real number. Then k -Jacobsthal sequence $\{\hat{j}_{k,n}\}_{n \in \mathbb{N}}$ and k -Jacobsthal Lucas sequence $\{\hat{c}_{k,n}\}_{n \in \mathbb{N}}$ are defined by the following equations:

$$\hat{j}_{k,n} = k\hat{j}_{k,n-1} + 2\hat{j}_{k,n-2}, \quad \hat{j}_{k,0} = 0, \quad \hat{j}_{k,1} = 1 \quad (2.1)$$

$$\hat{c}_{k,n} = k\hat{c}_{k,n-1} + 2\hat{c}_{k,n-2}, \quad \hat{c}_0 = 2, \quad \hat{c}_1 = k. \quad (2.2)$$

First few terms of the k -Jacobsthal Lucas number sequences are

$$\begin{aligned} \hat{c}_{k,0} &= 2, \quad \hat{c}_{k,1} = k, \quad \hat{c}_{k,2} = k^2 + 4, \quad \hat{c}_{k,3} = k^3 + 6k, \\ \hat{c}_{k,4} &= k^4 + 8k^2 + 8, \quad \hat{c}_{k,5} = k^5 + 1k^3 + 20k, \end{aligned}$$

Recurrences (2.1) and (2.2) involve the characteristic equation

$$x^2 - kx - 2 = 0$$

with roots

$$\alpha = \frac{k + \sqrt{k^2 + 8}}{2}, \quad \beta = \frac{k - \sqrt{k^2 + 8}}{2}$$

so that

$$\alpha + \beta = k, \quad \alpha\beta = -2, \quad \alpha - \beta = \sqrt{k^2 + 8}.$$

Binet Forms Binet's formulas for k -Jacobsthal, k -Jacobsthal-Lucas sequences are defined by the following equation:

$$\hat{j}_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}$$

and

$$\hat{c}_n = \alpha^n + \beta^n.$$

k -Jacobsthal Lucas sequence satisfies the following properties as below:

$$\hat{c}_{k,n} = \hat{j}_{k,n+1} + 2\hat{j}_{k,n-1}, \tag{2.3}$$

$$\hat{j}_{k,n}\hat{c}_{k,n} = \hat{j}_{k,2n}, \tag{2.4}$$

$$2\hat{j}_{m+n} = \hat{c}_n\hat{j}_m + \hat{j}_m\hat{c}_n, \tag{2.5}$$

Convolution Formula

$$\hat{c}_{m+n+1} = \hat{j}_{m+1}\hat{c}_{n+1} + 2\hat{j}_m\hat{c}_n, \tag{2.6}$$

Catalan Identity

$$\hat{c}_{k,n-r}\hat{c}_{k,n+r} - \hat{c}_{k,n}^2 = (-2)^{n-r} (\hat{c}_{k,r}^2 - 4(-2)^r) \tag{2.7}$$

D' Ocagne Identity

$$\hat{c}_{k,m}\hat{c}_{k,n+1} - \hat{c}_{k,m+1}\hat{c}_{k,n}\hat{c}_{k,n}^2 = (-2)^n \sqrt{k^2 + 8} \left(\hat{c}_{k,m-n} - 2^{n-m+1} \left(k + \sqrt{k^2 + 8} \right)^{m-n} \right) \tag{2.8}$$

$$\hat{c}_{k,n}^2 = (k^2 + 8)\hat{j}_{k,n}^2 + 4(-2)^n, \tag{2.9}$$

$$\sum_{j=0}^n \hat{c}_{k,j} = \frac{\hat{c}_{k,n+1} + 2\hat{c}_{k,n} + k - 2}{k + 1} \tag{2.10}$$

$$\sum_{j=0}^{\infty} \hat{c}_{k,j}x^j = \frac{2 - kx}{1 - kx - 2x^2} \tag{2.11}$$

3 On the k - Jacobsthal Lucas Numbers of Arithmetic Index of $an + r$

In this section we shall derive some formulae for the k -Jacobsthal Lucas numbers with index in an arithmetic sequence, say $an + r$ for fixed integer a and r such that $0 \leq r \leq a - 1$. First we prove the following lemma that will be used later.

Theorem 2 (*The k -Jacobsthal Lucas Numbers of Arithmetic Index*)

Let $a, r \in \mathbb{N}$ and $0 \leq r \leq a - 1$, then

$$\hat{c}_{k,a(n+1)+r} = \hat{c}_{k,a} \hat{c}_{k,an+r} - (-2)^a \hat{c}_{k,a(n-1)+r}. \tag{3.1}$$

In [5] it is proved that the equation is satisfied for k -Jacobsthal numbers

$$\hat{j}_{k,a(n+1)+r} = (2\hat{j}_{k,a-1} + \hat{j}_{k,a+1})\hat{j}_{k,an+r} - (-2)^a \hat{j}_{k,a(n-1)+r}.$$

Then by using this equality we derive

$$\begin{aligned} \hat{c}_{k,a(n+1)+r} &= 2\hat{j}_{k,a(n+1)+r-1} + \hat{j}_{k,a(n+1)+r+1} \\ &= 2((2\hat{j}_{k,a-1} + \hat{j}_{k,a+1})\hat{j}_{k,an+r-1} - (-2)^a \hat{j}_{k,a(n-1)+r-1}) \\ &\quad + (2\hat{j}_{k,a-1} + \hat{j}_{k,a+1})\hat{j}_{k,an+r+1} - (-2)^a \hat{j}_{k,a(n-1)+r+1} \\ &= \hat{c}_{k,a} (2\hat{j}_{k,an+r-1} + \hat{j}_{k,an+r+1}) - (-2)^a (2\hat{j}_{k,a(n-1)+r-1} + \hat{j}_{k,a(n-1)+r+1}) \\ &= \hat{c}_{k,a} \hat{c}_{k,an+r} - (-2)^a \hat{c}_{k,a(n-1)+r} \end{aligned}$$

For $r = 0$, we have

$$\hat{c}_{k,a(n+1)} = \hat{c}_{k,a} \hat{c}_{k,an} - (-2)^a \hat{c}_{k,a(n-1)}.$$

If $a = 2p + 1$, then an odd k -Jacobsthal Lucas number can be expressed in the form

$$\hat{c}_{k,(2p+1)(n+1)} = \hat{c}_{k,a} \hat{c}_{k,(2p+1)n} - (-2)^a \hat{c}_{k,(2p+1)(n-1)}.$$

If $a = 2p$, then an even k -Jacobsthal Lucas number can be expressed in the form

$$\hat{c}_{k,2p(n+1)} = \hat{c}_{k,a} \hat{c}_{k,2pn} - (-2)^a \hat{c}_{k,2p(n-1)}.$$

Theorem 3 (*Generating Functions of the Sequence $\{\hat{c}_{k,an+r}\}$*)

Let $C(k, x, a, r)$ be the generating function of the sequence $\{\hat{c}_{k,an+r}\}$. For $a, r \in \mathbb{N}$ and $0 \leq r \leq a - 1$, we have

$$C(k, x, a, r) = \sum_{j=0}^{\infty} \hat{c}_{k,a_j+r} x^j = \frac{\hat{c}_{k,r} + x(\hat{c}_{k,a+r} - \hat{c}_{k,a} \hat{c}_{k,r})}{1 - \hat{c}_{k,a} x + (-2)^a x^2}. \tag{3.2}$$

Proof.

$$C(k, x, a, r) = \hat{c}_{k,r} + \hat{c}_{k,a+r}x + \hat{c}_{k,2a+r}x^2 + \dots$$

Then multiplying by $\hat{c}_{k,a}x$ and $(-2)^a x^2$ respectively. We have

$$-\hat{c}_{k,a}x C(k, x, a, r) = -\hat{c}_{k,a}\hat{c}_{k,r}x - \hat{c}_{k,a}\hat{c}_{k,a+r}x^2 - \hat{c}_{k,a}\hat{c}_{k,2a+r}x^3 - \dots$$

$$(-2)^a x^2 C(k, x, a, r) = (-2)^a \hat{c}_{k,r}x^2 + (-2)^a \hat{c}_{k,a+r}x^3 + (-2)^a \hat{c}_{k,2a+r}x^4 + \dots$$

By adding these equalities, we obtain

$$\begin{aligned} (1 - \hat{c}_{k,a}x + (-2)^a x^2) C(k, x, a, r) &= \hat{c}_{k,r} + x(\hat{c}_{k,a+r} - \hat{c}_{k,a}\hat{c}_{k,r}) + x^2(\hat{c}_{k,2a+r} - \hat{c}_{k,a}\hat{c}_{k,a+r} - (-2)^a \hat{c}_{k,r}) + \dots \\ &= \hat{c}_{k,r} + x(\hat{c}_{k,a+r} - \hat{c}_{k,a}\hat{c}_{k,r}) + 0x^2 + 0 + \dots \\ C(k, x, a, r) &= \frac{\hat{c}_{k,r} + x(\hat{c}_{k,a+r} - \hat{c}_{k,a}\hat{c}_{k,r})}{1 - \hat{c}_{k,a}x + (-2)^a x^2}. \end{aligned}$$

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As a particular case for $r = 0, a = 1$ we have the generating function of the k - Jacobsthal Lucas sequence as below:

$$C(k, x, 1, 0) = \sum_{j=0}^{\infty} \hat{c}_{k,j}x^j = \frac{2 - kx}{1 - kx - 2x^2}$$

and the generating function for the classical Jacobsthal Lucas sequence is $\frac{2-x}{1-x-2x^2}$.

As a particular case for $r = 0, a = 2$ we have the generating function of the k - Jacobsthal Lucas sequence for even terms as below:

$$C(k, x, 2, 0) = \sum_{j=0}^{\infty} \hat{c}_{k,2j}x^j = \frac{2 - kx}{1 - kx + 4x^2}$$

As a particular case for $r = 1, a = 2$ we have the generating function of the k - Jacobsthal Lucas sequence for even terms as below:

$$C(k, x, 2, 1) = \sum_{j=0}^{\infty} \hat{c}_{k,2j+1}x^j = \frac{k - 2kx}{1 - (k^2 + 4)x + 4x^2}$$

Theorem 4 (Sum of the k -Jacobsthal Lucas Numbers of Arithmetic Index)

Let $a, r \in \mathbb{N}$ and $0 \leq r \leq a - 1$, then

$$\sum_{j=0}^n \hat{c}_{k,a+j+r} = \frac{\hat{c}_{k,r} - (-2)^r \hat{c}_{k,a-r} - \hat{c}_{k,a(n+1)+r} + (-2)^a \hat{c}_{k,an+r}}{1 - \hat{c}_{k,a} + (-2)^a}. \quad (3.3)$$

$$\begin{aligned}
 \sum_{j=0}^n \hat{c}_{k,aj+r} &= \sum_{j=0}^n (\alpha^{aj+r} + \beta^{aj+r}) \\
 &= \alpha^r \left(\frac{1 - (\alpha^a)^{n+1}}{1 - \alpha^a} \right) + \beta^r \left(\frac{1 - (\beta^a)^{n+1}}{1 - \beta^a} \right) \\
 &= \left(\frac{\alpha^r - \alpha^{a(n+1)+r} - \alpha^r \beta^a - \alpha^{a(n+1)+r} \beta^a + \beta^r - \beta^{a(n+1)+r} - \alpha^a \beta^r + \alpha^a \beta^{a(n+1)+r}}{1 + \hat{c}_{k,a} + (-2)^a} \right) \\
 &= \frac{\hat{c}_{k,r} - (-2)^r \hat{c}_{k,a-r} - \hat{c}_{k,a(n+1)+r} + (-2)^a \hat{c}_{k,an+r}}{1 - \hat{c}_{k,a} + (-2)^a}
 \end{aligned}$$

As a particular case for $r = 0, a = 1$ we have the sum of the k - Jacobsthal Lucas sequence as below:

$$\sum_{j=0}^n \hat{c}_{k,j} = \frac{\hat{c}_{k,n+1} + 2\hat{c}_{k,n} + k - 2}{k + 1}.$$

and the sum of the classical Jacobsthal Lucas sequence is $\sum_{j=0}^n \hat{c}_j = \frac{\hat{c}_{n+1} + 2\hat{c}_n - 1}{2}$.

As a particular case for $r = 0, a = 2p + 1$ we have the sum of the first odd numbers for k - Jacobsthal Lucas sequence as below:

$$\sum_{j=0}^n \hat{c}_{k,(2p+1)j} = \frac{2 - \hat{c}_{k,2p+1} - \hat{c}_{k,(2p+1)(n+1)} + (-2)^{2p+1} \hat{c}_{k,(2p+1)n}}{1 - \hat{c}_{k,2p+1} + (-2)^{2p+1}}$$

As particular case for $r = 0, a = 2p$ we have the sum of the first even numbers for k - Jacobsthal Lucas sequence as below:

$$\sum_{j=0}^n \hat{c}_{k,(2p)j} = \frac{2 - \hat{c}_{k,2p} - \hat{c}_{k,2p(n+1)} + (-2)^{2p} \hat{c}_{k,2pn}}{1 - \hat{c}_{k,2p} + (-2)^{2p}}$$

In this case for $p = 1$ we obtain the sum of the first even numbers for k - Jacobsthal Lucas sequence as below:

$$\sum_{j=0}^n \hat{c}_{k,2j} = \frac{-2 - k^2 - \hat{c}_{k,2(n+1)} + 4\hat{c}_{k,2n}}{1 - k^2}$$

Theorem 5 (Recurrence Law for the sequence of sums of k -Jacobsthal Lucas Numbers of Arithmetic Index)

Let $a, r \in \mathbb{N}$ and $0 \leq r \leq a - 1$, and we assume that $S_{k,an+r} = \sum_{j=0}^n \hat{c}_{k,aj+r}$, then the following is verified:

$$S_{k,a(n+1)+r} = (1 + \hat{c}_{k,a})S_{k,an+r} - (\hat{c}_{k,a} + (-2)^a)S_{k,a(n-1)+r} + (-2)^a S_{k,a(n-2)+r}.$$

Proof.

$$\begin{aligned}
 S_{k,an+r} &= \sum_{j=0}^n \hat{c}_{k,aj+r} = \hat{c}_{k,r} + \hat{c}_{k,a+r} + \sum_{j=2}^n (\hat{c}_{k,a} \hat{c}_{k,a(j-1)+r} - (-2)^a \hat{c}_{k,a(j-2)+r}) \\
 &= \hat{c}_{k,r} + \hat{c}_{k,a+r} + \hat{c}_{k,a} \sum_{j=1}^{n-1} \hat{c}_{k,aj+r} - (-2)^a \sum_{j=0}^{n-2} \hat{c}_{k,aj+r} \\
 &= \hat{c}_{k,r} + \hat{c}_{k,a+r} + \hat{c}_{k,a} (S_{k,a(n-1)+r} - \hat{c}_{k,r}) - (-2)^a S_{k,a(n-2)+r} \\
 &= (1 - \hat{c}_{k,a}) \hat{c}_{k,r} + \hat{c}_{k,a+r} + \hat{c}_{k,a} S_{k,a(n-1)+r} - (-2)^a S_{k,a(n-2)+r}
 \end{aligned}$$

Now considering $S_{k,an+r}$ and $S_{k,a(n+1)+r}$:

$$\begin{aligned}
 S_{k,an+r} &= (1 - \hat{c}_{k,a}) \hat{c}_{k,r} + \hat{c}_{k,a+r} + \hat{c}_{k,a} S_{k,a(n-1)+r} - (-2)^a S_{k,a(n-2)+r} \\
 S_{k,a(n+1)+r} &= (1 - \hat{c}_{k,a}) \hat{c}_{k,r} + \hat{c}_{k,a+r} + \hat{c}_{k,a} S_{k,an+r} - (-2)^a S_{k,a(n-1)+r}
 \end{aligned}$$

by eliminating the terms $(1 - \hat{c}_{k,a}) \hat{c}_{k,r} + \hat{c}_{k,a+r}$, it is deduced:

$$S_{k,a(n+1)+r} = (1 + \hat{c}_{k,a}) S_{k,an+r} - (\hat{c}_{k,a} + (-2)^a) S_{k,a(n-1)+r} + (-2)^a S_{k,a(n-2)+r}.$$

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Theorem 6 (Sum of Alternated k -Jacobsthal Lucas Numbers of Arithmetic Index)

Let $a, r \in \mathbb{N}$ and $0 \leq r \leq a - 1$, then we derive

$$\sum_{j=0}^n (-1)^j \hat{c}_{k,aj+r} = \frac{\hat{c}_{k,r} + (-2)^r \hat{c}_{k,a-r} + (-1)^n \hat{c}_{k,a(n+1)+r} + (-1)^n (-2)^a \hat{c}_{k,an+r}}{1 + \hat{c}_{k,a} + (-2)^a} \tag{3.4}$$

Proof.

$$\begin{aligned}
 \sum_{j=0}^n (-1)^j \hat{c}_{k,aj+r} &= \sum_{j=0}^n (-1)^j (\alpha^{aj+r} + \beta^{aj+r}) \\
 &= \alpha^r \left(\frac{1 - (-\alpha^a)^{n+1}}{1 + \alpha^a} \right) + \beta^r \left(\frac{1 - (-\beta^a)^{n+1}}{1 + \beta^a} \right) \\
 &= \left(\frac{\alpha^r + (-1)^n \alpha^{a(n+1)+r}}{1 + \alpha^a} \right) + \left(\frac{\beta^r + (-1)^n \beta^{a(n+1)+r}}{1 + \beta^a} \right) \\
 &= \frac{\hat{c}_{k,r} + (-2)^r \hat{c}_{k,a-r} + (-1)^n \hat{c}_{k,a(n+1)+r} + (-1)^n (-2)^a \hat{c}_{k,an+r}}{1 + \hat{c}_{k,a} + (-2)^a}
 \end{aligned}$$

■

As a particular case, for $a = 1$ and $r = 0$,

$$\sum_{j=0}^n (-1)^j \hat{c}_{k,j} = \frac{2 + k + (-1)^n \hat{c}_{k,(n+1)} - 2(-1)^n \hat{c}_{k,n}}{k - 1}$$

As a particular case, for $a = 2p + 1$ and $r = 0$,

$$\sum_{j=0}^n (-1)^j \hat{c}_{k,(2p+1)j} = \frac{2 + \hat{c}_{k,2p+1} + (-1)^n \hat{c}_{k,(2p+1)(n+1)} + (-1)^n (-2)^{2p+1} \hat{c}_{k,(2p+1)n}}{1 + \hat{c}_{k,2p+1} + (-2)^{2p+1}}.$$

Then, if $a = 2, r = 1$, the sum of the first odd alternated k - Jacobsthal Lucas numbers is obtained as the following:

$$\sum_{j=0}^n (-1)^j \hat{c}_{k,2j+1} = \frac{-k + (-1)^n \hat{c}_{k,2n+3} + 4(-1)^n \hat{c}_{k,2n+1}}{k^2 + 9}$$

the sum of the first odd alternated classic Jacobsthal Lucas numbers is obtained

$$\sum_{j=0}^n (-1)^j \hat{c}_{2j+1} = \frac{-1 + (-1)^n \hat{c}_{2n+3} + 4(-1)^n \hat{c}_{2n+1}}{10}$$

As a particular case, for $a = 2p$ and $r = 0$,

$$\sum_{j=0}^n (-1)^j \hat{c}_{k,2pj} = \frac{2 + \hat{c}_{k,2p} + (-1)^n \hat{c}_{k,2p(n+1)} + (-1)^n (-2)^{2p} \hat{c}_{k,2pn}}{1 + \hat{c}_{k,2p} + (-2)^{2p}}$$

Then, if $a = 2, r = 0$, the sum of the first even alternated k - Jacobsthal Lucas numbers is obtained

$$\sum_{j=0}^n (-1)^j \hat{c}_{k,2j} = \frac{k^2 + 6 + (-1)^n \hat{c}_{k,2n+2} + 4(-1)^n \hat{c}_{k,2n}}{k^2 + 9}$$

the sum of the first even alternated classic Jacobsthal Lucas numbers is obtained

$$\sum_{j=0}^n (-1)^j \hat{c}_{2j} = \frac{7 + (-1)^n \hat{c}_{2n+2} + 4(-1)^n \hat{c}_{2n}}{10}$$

Theorem 7 (A Relation Between Some k -Jacobsthal and k -Jacobsthal Lucas Numbers)

For $r \geq 1$, we obtain

$$\frac{\hat{J}_{k,2^r n}}{\hat{J}_{k,n}} = \prod_{j=0}^{r-1} \hat{C}_{k,2^j n}. \quad (3.5)$$

Proof.

$$\begin{aligned} \frac{\hat{J}_{k,2^r n}}{\hat{J}_{k,n}} &= \frac{\alpha^{2^r n} - \beta^{2^r n}}{\alpha^n - \beta^n} = \left(\alpha^{2^{r-1} n} + \beta^{2^{r-1} n} \right) \frac{\alpha^{2^{r-1} n} - \beta^{2^{r-1} n}}{\alpha^n - \beta^n} \\ &= \hat{C}_{k,2^{r-1} n} \left(\alpha^{2^{r-2} n} + \beta^{2^{r-2} n} \right) \frac{\alpha^{2^{r-2} n} - \beta^{2^{r-2} n}}{\alpha^n - \beta^n} \\ &= \hat{C}_{k,2^{r-1} n} \hat{C}_{k,2^{r-2} n} \dots \frac{\alpha^{2n} - \beta^{2n}}{\alpha^n - \beta^n} \\ &= \hat{C}_{k,2^{r-1} n} \hat{C}_{k,2^{r-2} n} \dots (\alpha^n + \beta^n) \\ &= \prod_{j=0}^{r-1} \hat{C}_{k,2^j n} \end{aligned}$$

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As a particular case, for $r = 1$

$$\frac{\hat{J}_{k,2n}}{\hat{J}_{k,n}} = \hat{C}_{k,n}$$

Theorem 8 (A Relation Between Some k -Jacobsthal and k -Jacobsthal Lucas Numbers)

For $a \geq 0$, we obtain

$$\hat{C}_{k,(a+4)n} - \hat{C}_{k,an} = \hat{J}_{k,(a+2)n+1} [\hat{J}_{k,2n-1} - \hat{J}_{k,2n+1}] + \hat{J}_{k,(a+2)n} (k^2 + 8) \quad (3.6)$$

Proof.

$$\begin{aligned} \hat{C}_{k,(a+4)n} - \hat{C}_{k,an} &= \hat{J}_{k,(a+4)n+1} + 2\hat{J}_{k,(a+4)n-1} - \hat{J}_{k,an+1} - 2\hat{J}_{k,an-1} \\ &= \hat{J}_{k,(a+2)n+2n+1} + 2\hat{J}_{k,(a+2)n+2n-1} - \hat{J}_{k,(a+2)n+(-2n+1)} - 2\hat{J}_{k,(a+2)n+(-2n-1)} \\ &= \hat{J}_{k,(a+2)n+1} \hat{J}_{k,2n+1} + 2\hat{J}_{k,(a+2)n} \hat{J}_{k,2n} + 2 [\hat{J}_{k,(a+2)n+1} \hat{J}_{k,2n-1} + 2\hat{J}_{k,(a+2)n} \hat{J}_{k,2n-2}] \\ &\quad - \hat{J}_{k,(a+2)n+1} \hat{J}_{k,-2n+1} - 2\hat{J}_{k,(a+2)n} \hat{J}_{k,-2n} - 2 [\hat{J}_{k,(a+2)n+1} \hat{J}_{k,-2n-1} + 2\hat{J}_{k,(a+2)n} \hat{J}_{k,-2n-2}] \\ &= \hat{J}_{k,(a+2)n+1} \hat{J}_{k,2n+1} + 2\hat{J}_{k,(a+2)n} \hat{J}_{k,2n} + 2 [\hat{J}_{k,(a+2)n+1} \hat{J}_{k,2n-1} + 2\hat{J}_{k,(a+2)n} \hat{J}_{k,2n-2}] \\ &\quad - \hat{J}_{k,(a+2)n+1} \hat{J}_{k,2n-1} + 2\hat{J}_{k,(a+2)n} \hat{J}_{k,2n} - 2 [\hat{J}_{k,(a+2)n+1} \hat{J}_{k,2n+1} - 2\hat{J}_{k,(a+2)n} \hat{J}_{k,2n+2}] \\ &= \hat{J}_{k,(a+2)n+1} [\hat{J}_{k,2n+1} + 2\hat{J}_{k,2n-1} - \hat{J}_{k,2n-1} - 2\hat{J}_{k,2n+1}] \\ &\quad \hat{J}_{k,(a+2)n} [2\hat{J}_{k,2n} + 4\hat{J}_{k,2n-2} + 2\hat{J}_{k,2n} + 4\hat{J}_{k,2n+2}] \\ &= \hat{J}_{k,(a+2)n+1} [\hat{J}_{k,2n-1} - \hat{J}_{k,2n+1}] + \hat{J}_{k,(a+2)n} (k^2 + 8) \end{aligned}$$

■

Theorem 9 For $|\alpha^{ak}\beta^{a(r-k)}x| < 1$, we get

$$\sum_{i=0}^{\infty} \hat{C}_{k,ai+r}^r x^i = \sum_{k=0}^r \binom{r}{k} \frac{1}{1 - \alpha^k \beta^{r-k} x}.$$

Proof. By using geometric series and Binet formula, we have

$$\begin{aligned} \sum_{i=0}^{\infty} \hat{C}_{k,ai+r}^r x^i &= \sum_{i=0}^{\infty} \sum_{k=0}^p \binom{p}{k} (\alpha^{ai+r})^k (\beta^{ai+r})^{p-k} x^i = \sum_{k=0}^p \binom{p}{k} \alpha^{rk} (\beta^r)^{p-k} \sum_{i=0}^{\infty} [\alpha^{ak} \beta^{a(r-k)} x]^i \\ &= \sum_{k=0}^p \binom{p}{k} (\alpha^k \beta^{p-k})^r \frac{1}{1 - \alpha^{ak} \beta^{a(r-k)} x}. \end{aligned}$$

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References

- [1] S. Falcon, A. Plaza, On the 3-dimensional k-Fibonacci spirals, *Chaos, Soliton Fract.*, **38** (2008), no. 4, 993-1003.
<http://dx.doi.org/10.1016/j.chaos.2007.02.009>
- [2] S. Falcon, A. Plaza, On k-Fibonacci Numbers of Arithmetic Indexes, *Applied Mathematics and Computation*, **208** (2009), no. 1, 180-185.
<http://dx.doi.org/10.1016/j.amc.2008.11.031>
- [3] S. Falcon, On the k-Lucas Numbers of Arithmetic Indexes, *Applied Mathematics*, **3** (2012), no. 10, 1202-1206.
<http://dx.doi.org/10.4236/am.2012.310175>
- [4] A. F. Horadam, Jacobsthal Representation Numbers, *The Fibonacci Quarterly*, **34** (1996), no. 1, 40-54.
- [5] D. Jhala, G.P.S. Rathore, K. Sisodiya, Some Properties of k-Jacobsthal Numbers with Arithmetic Indexes, *Turkish Journal of Analysis and Number Theory*, **2** (2014), no. 4, 119-124.
<http://dx.doi.org/10.12691/tjant-2-4-3>
- [6] D. Jhala, G.P.S. Rathore, Some Properties of the K-Jacobsthal Lucas Sequence, *International Journal of Modern Sciences and Engineering Technology*, **1** (2014), no. 3, 87-92.
- [7] T. Koshy, Fibonacci and Lucas Numbers with Applications, John Wiley and Sons Inc., New York, 2001. <http://dx.doi.org/10.1002/9781118033067>

- [8] F. Koken, D. Bozkurt, On the Jacobsthal Numbers by Matrix Methods, *Int. J. Contemp. Math. Sciences*, **3** (2008), no. 13, 605-614.
- [9] N. J. A. Sloane, *The On-Line Encyclopedia of Integer Sequences*, 2006.
- [10] K. Uslu, Ş. Uygun, The (s,t) Jacobsthal and (s,t) Jacobsthal-Lucas Matrix sequences, *ARS Combinatoria*, **108** (2013), 13-22.
- [11] Ş. Uygun, The (s,t) -Jacobsthal and (s,t) -Jacobsthal Lucas sequences, *Applied Mathematical Sciences*, **9** (2015), no. 70, 3467-3476. <http://dx.doi.org/10.12988/ams.2015.52166>
- [12] Ş. Uygun and H. Eldogan, k -Jacobsthal and k -Jacobsthal Lucas Matrix Sequences, *International Mathematical Forum*, **11** (2016), no. 3, 145-154. <http://dx.doi.org/10.12988/imf.2016.51192>

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