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Some Results for the Hodge Decomposition Theorem in Euclidean Three-Space

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Abstract

The Hodge Decomposition Theorem plays a significant role in the study of partial differential equations. Several interrelated propositions which are required for the proof of the Hodge theorem in Euclidean three-space are introduced and proved in a novel way.

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1 Introduction

Let Ω be a bounded subset of three-space $\mathbb{R}^3$ and let $\mathbf{V}(x, y, z)$ be a vector field on Ω. In applications it is often useful to be able to determine whether $\mathbf{V}(x, y, z)$ is the gradient of a function, or the curl of another vector field, or perhaps a divergence-free field. The answer to such questions is based on an understanding of the relationship between vector calculus and the topology of their domains of definition [**1**]. The Hodge Decomposition theorem addresses these questions by studying the space of vector fields as a decomposition into five mutually orthogonal subspaces that are topologically and analytically significant. This kind of decomposition is useful not only in mathematical terms, but also from the physical point of view in such diverse areas as electrodynamics and fluid dynamics. This theorem also has significant applications in the study of partial differential equations as well [**2-3**]. Here several theorems which can be used to develop the Hodge theorem will be stated and proved and some applications of homology and cohomology theory will be illustrated [**4-6**]. Hodge decomposition also generalizes to a similar but more restrictive proposition on Riemannian manifolds [**2, 7-9**].

2 Preliminaries and Main Results

To state the Hodge theorem, suppose Ω is a compact domain in three-space with smooth boundary $\partial\Omega$. Denote by $V(\Omega)$ the infinite-dimensional vector space of all smooth vector fields on Ω. This means all partial derivatives of all orders exist and are continuous over Ω. Let Ω be endowed with the $L^2(\Omega)$ inner product, which is defined for vector fields $\mathbf{U}, \mathbf{V} \in V(\Omega)$ as

$$\langle \mathbf{U}, \mathbf{V} \rangle = \int_{\Omega} \mathbf{U} \cdot \mathbf{V} \, dv.$$

The purpose here is to give some novel proofs to some of the theorems which are required in the proof of the Hodge Decomposition theorem in three-space.

Hodge Decomposition Theorem. The vector space $V(\Omega)$ is the direct sum of five mutually orthogonal subspaces

$$V(\Omega) = FK \oplus HK \oplus CG \oplus HG \oplus GG. \tag{1}$$

Furthermore, there exist the following set of isomorphisms between homologies: $HK \cong H_1(\Omega) \cong H_2(\Omega, \partial\Omega)$ and $HG \cong H_2(\Omega) \cong H_1(\Omega, \partial\Omega)$. $\Box$

The five different subspaces of $V(\Omega)$ which appear on the right-hand side of (1) must be defined. With the understanding $\mathbf{V} \in V(\Omega)$ in these sets, the fluxless knots are given by $FK = \{\vec{\nabla} \cdot \mathbf{V} = 0, \mathbf{V} \cdot \mathbf{n} = 0, \text{all interior fluxes zero}\}$, harmonic knots are $HK = \{\vec{\nabla} \cdot \mathbf{V} = 0, \vec{\nabla} \times \mathbf{V} = \mathbf{0}, \mathbf{V} \cdot \mathbf{n} = 0\}$ and the curly gradients are defined to be $CG = \{\mathbf{V} = \vec{\nabla}\varphi, \vec{\nabla} \cdot \mathbf{V} = 0, \text{ all boundary fluxes zero}\}$, then harmonic gradients are $HG = \{\mathbf{V} = \vec{\nabla}\varphi, \vec{\nabla} \cdot \mathbf{V} = 0, \varphi \text{ is locally constant on } \partial\Omega\}$ and finally, the grounded gradients are defined as $GG = \{\mathbf{V} = \vec{\nabla}\varphi, \; \varphi|_{\partial\Omega} = 0\}$.

The aim is to prove several theorems which play a basic role in one verion of the proof of (1) which proceeds in well-defined steps. Consider first the subspace of $V(\Omega)$ which consists of all divergence-free vector fields on Ω that are tangent to $\partial\Omega$. This class of vector field arises in the study of incompressible fluid flows with fixed boundaries as well as magnetic fields in plasmas. Let the vector fields referred to as knots be defined as

$$K = \{\mathbf{V} \in V(\Omega) : \vec{\nabla} \cdot \mathbf{V} = 0, \mathbf{V} \cdot \mathbf{n}|_{\partial\Omega} = 0\}. \tag{2}$$

In (2), $\mathbf{n}$ is the unit normal outward everywhere to $\partial\Omega$, and the second condition will be written simply as $\mathbf{V} \cdot \mathbf{n} = 0$ on $\partial\Omega$.

The set of vector fields which are determined as gradients of a scalar function are defined as

$$G = \{\mathbf{V} \in V(\Omega) : \mathbf{V} = \vec{\nabla}\varphi\}, \tag{3}$$

for some smooth, real-valued function φ, which is defined on Ω. A preliminary version of the Hodge theorem can be given in the following form. It asserts the $V(\Omega)$ breaks up orthogonally as a direct sum of K and G, which means all smooth vector fields on Ω can be split up into those that are divergence-free and tangent to the boundary, and those that are gradients of smooth functions.

Proposition 1: The space $V(\Omega)$ is the direct sum of the two orthogonal subspaces (2) and (3) in the form

$$V(\Omega) = K \oplus G. \tag{4}$$

Proof: Let $\mathbf{V}$ be an arbitrary, smooth vector field on Ω, and suppose f is the smooth function $f = \vec{\nabla} \cdot \mathbf{V}$ defined on Ω with g the smooth function defined by $g = \vec{\nabla} \cdot \mathbf{n}$ on $\partial\Omega$. If Ω has components Ω_i, the divergence theorem applied to f on Ω_i gives

$$\int_{\Omega_i} f dv = \int_{\Omega_i} \vec{\nabla} \cdot \mathbf{V}\, dv = \int_{\partial\Omega_i} \mathbf{V} \cdot \mathbf{n}\, da = \int_{\partial\Omega_i} g\, da.$$

By the existence-uniquness theorem for the Neumann problem, this implies that there is a solution φ of the Poisson equation $\Delta\varphi = f$ on Ω with the Neumann boundary condition $\partial\varphi/\partial n = g$ on $\partial\Omega$. Define the vector fields $\mathbf{V}_2 = \vec{\nabla}\varphi$ and $\mathbf{V}_1 = \mathbf{V} - \mathbf{V}_2$, so that on $\partial\Omega$, it follows that

$$\mathbf{V}_2 \cdot \mathbf{n} = \vec{\nabla}\varphi \cdot \mathbf{n} = \frac{\partial\varphi}{\partial n} = g = \mathbf{V} \cdot \mathbf{n}.$$

Since $\mathbf{V} = \mathbf{V}_1 + \mathbf{V}_2$, this result implies that $\vec{\nabla} \cdot \mathbf{V}_1 = 0$. This result is stating that $\mathbf{V}_1$ is divergence-free on Ω, and is tangent to $\partial\Omega$. However, $\mathbf{V}_2$ is a gradient vector field, so this implies decomposition (4).

It remains to show that (4) is an orthogonal direct sum. To carry this out, let $\mathbf{V}_1$ denote a smooth, divergence-free vector field on Ω tangent to $\partial\Omega$, and let $\mathbf{V}_2 = \vec{\nabla}\varphi$ denote any smooth gradient vector field on Ω. Using the product rule $\vec{\nabla} \cdot (\varphi\mathbf{V}_1) = (\vec{\nabla}\varphi) \cdot \mathbf{V}_1 + \varphi(\vec{\nabla} \cdot \mathbf{V}_1)$, we get

$$\langle \mathbf{V}_1, \mathbf{V}_2 \rangle = \int_\Omega \mathbf{V}_1 \cdot \mathbf{V}_2\, dv = \int_\Omega \mathbf{V}_1 \cdot \vec{\nabla}\varphi\, dv = \int_\Omega \vec{\nabla} \cdot (\varphi\mathbf{V}_1)\, dv - \int_\Omega \varphi(\vec{\nabla} \cdot \mathbf{V}_1)\, dv.$$

Since $\mathbf{V}_1$ satisfies $\vec{\nabla} \cdot \mathbf{V}_1 = 0$, this reduces to

$$\int_\Omega \mathbf{V}_1 \cdot \mathbf{V}_2\, dv = \int_\Omega \vec{\nabla} \cdot (\varphi\mathbf{V}_1)\, dv = \int_{\partial\Omega} \varphi\mathbf{V}_1 \cdot \mathbf{n}\, da = 0.$$

The last integral vanishes since $\mathbf{V}_1$ is tangent to $\partial\Omega$. Hence the summands K and G in (4) are orthogonal and in particular their sum is direct which proves (4). □

It is necessary to create more tools at this point which will be needed to refine the mathematical description of $V(\Omega)$ to the form (1). Let Σ denote any smooth, orientable surface in Ω whose boundary $\partial\Sigma$ lies in the boundary $\partial\Omega$ of domain Ω. Then Σ is called a cross-sectional surface and one writes $(\Sigma, \partial\Sigma) \subset (\Omega, \partial\Omega)$. Now Σ can be oriented by picking one of its two unit normal vector fields $\mathbf{n}$ on Σ. For any vector field $\mathbf{V}$ on Ω, define the flux of $\mathbf{V}$ through Σ to be

$$\Phi = \int_{\Sigma} \mathbf{V} \cdot \mathbf{n} \, da. \tag{5}$$

Suppose $\mathbf{V}$ is divergence-free and tangent to $\partial\Omega$. Then the value of Φ depends only on the homology class of Σ in the relative homology group $H_2(\Omega, \partial\Omega)$. For example, if Ω is an n-holed solid torus, then $H_2(\Omega, \partial\Omega)$ is generated by disjoint oriented cross-sectional disks $\Sigma_1, \cdots, \Sigma_n$ which are positioned so that cutting Ω along these disks produces a simply-connected region. The fluxes $\Phi_1, \cdots, \Phi_n$ of $\mathbf{V}$ through these disks produces a simply-connected region. The fluxes $\Phi_1, \cdots, \Phi_n$ of $\mathbf{V}$ through these disks determine the flux of $\mathbf{V}$ through any other cross-sectional surface.

The following integral formula and the related expression for its curl play a very prominent role in the decompositions. Let Ω be a compact domain in 3-space which has a smooth boundary $\partial\Omega$ and $\mathbf{V}(\mathbf{x}) \in V(\Omega)$. Define a new vector field for $\mathbf{y} \in \Omega$ as follows,

$$\mathbf{B}(\mathbf{V})(\mathbf{y}) = \frac{1}{4\pi} \int_{\Omega} \mathbf{V}(\mathbf{x}) \times \frac{\mathbf{y} - \mathbf{x}}{|\mathbf{y} - \mathbf{x}|^3} \, dv_x. \tag{6}$$

If $\mathbf{V}$ is thought of as a current throughout Ω, (6) constitutes the Biot-Savart law which gives the resulting magnetic field $\mathbf{B}$ in $\mathbb{R}^3$.

Vector field $\mathbf{B}$ is well-defined on all of $\mathbb{R}^3$ as (6) converges for every $\mathbf{y} \in \mathbb{R}^3$. Also $\mathbf{B}$ is continuous on all of $\mathbb{R}^3$, though its derivatives experience a jump discontinuity as one crosses $\partial\Omega$. Thus $\mathbf{B}$ is C^∞ on Ω and on the closure Ω' of $\mathbb{R}^3 - \Omega$.

Theorem 1. Vector field $\mathbf{B}$ is divergence-free so that

$$\vec{\nabla}_{\mathbf{y}} \cdot \mathbf{B}(\mathbf{V}) = 0, \tag{7}$$

on Ω and Ω', for all $\mathbf{V} \in V(\Omega)$.

Proof: Set $R = |\mathbf{y} - \mathbf{x}|$ and let $V_j(\mathbf{x})$ be the components of $\mathbf{V}$. The integrand of $\mathbf{B}$ in (6) is

$$\mathbf{V}(\mathbf{x}) \times \frac{\mathbf{R}}{R^3} = \frac{1}{R^3}[(V_2R_3 - V_3R_2)\hat{\imath} + (V_3R_1 - V_1R_3)\hat{\jmath} + (V_1R_2 - V_2R_1)\hat{k}].$$

Since only the components of $\mathbf{R}$ contain the $\mathbf{y}$ variable, it follows that $\partial(R^{-3})/\partial y_i = -3R_i R^{-5}$, $i = 1, 2, 3$. Taking the operator $\vec{\nabla}_{\mathbf{y}}$ inside the integrand, it is found to annihilate the integrand, which implies that

$$\vec{\nabla}_{\mathbf{y}} \cdot (\mathbf{V}(\mathbf{x}) \times \frac{\mathbf{R}}{R^3}) = 0.$$

□

Theorem 2. Let $\mathbf{B}$ be the vector field (6), then its curl is given by the following expression

$$\vec{\nabla}_{\mathbf{y}} \times \mathbf{B}(\mathbf{V})(\mathbf{y}) = \mathbf{V}(\mathbf{y})\delta_{\mathbf{y},\Omega} + \frac{1}{4\pi}\vec{\nabla}_{\mathbf{y}} \int_{\Omega} \frac{\vec{\nabla}_{\mathbf{x}} \cdot \mathbf{V}(\mathbf{x})}{|\mathbf{y}-\mathbf{x}|}\, dv_x - \frac{1}{4\pi}\vec{\nabla}_{\mathbf{y}} \int_{\partial\Omega} \frac{\mathbf{V}(\mathbf{x}) \cdot \mathbf{n}}{|\mathbf{y}-\mathbf{x}|}\, da_x, \tag{8}$$

where the operator $\vec{\nabla}_{\mathbf{x}}$ differentiates with respect to $\mathbf{x}$ while $\vec{\nabla}_{\mathbf{y}}$ differentiates with respect to the $\mathbf{y}$ variable coordinates. Moreover, let us define $\delta_{\mathbf{y},\Omega} = 1$ for $\mathbf{y} \in \Omega$ and $\delta_{\mathbf{y},\Omega} = 0$ when $\mathbf{y} \notin \Omega$.

Proof: It is the case that (6) can be written in the form

$$\mathbf{B}(\mathbf{V})(\mathbf{y}) = -\frac{1}{4\pi} \int_{\Omega} \mathbf{V}(\mathbf{x}) \times \vec{\nabla}_{\mathbf{y}} \frac{1}{|\mathbf{y}-\mathbf{x}|}\, dv_x. \tag{9}$$

Define $R = |\mathbf{y} - \mathbf{x}|$ and so, using a basic vector identity, equation (9) becomes

$$\vec{\nabla}_{\mathbf{y}} \times \mathbf{B}(\mathbf{V})(\mathbf{y}) = -\frac{1}{4\pi} \int_{\Omega} \vec{\nabla}_{\mathbf{y}} \times \mathbf{V}(\mathbf{x}) \times \vec{\nabla}_{\mathbf{y}}(\frac{1}{R})\, dv_x$$

$$= -\frac{1}{4\pi} \int_{\Omega} \vec{\nabla}^2_{\mathbf{y}} \mathbf{V}(\mathbf{x})(\frac{1}{R})\, dv_x + \frac{1}{4\pi} \int_{\Omega} (\mathbf{V}(\mathbf{x}) \cdot \vec{\nabla}_{\mathbf{y}})\vec{\nabla}_{\mathbf{y}}(\frac{1}{R})\, dv_x. \tag{10}$$

In each integral of (10), $\mathbf{V}(\mathbf{x})$ commutes with the differential operators in the $\mathbf{y}$ variable, so upon using $\vec{\nabla}_{\mathbf{y}}(1/R) = -\vec{\nabla}_{\mathbf{x}}(1/R)$, we obtain

$$\vec{\nabla}_{\mathbf{y}} \times \mathbf{B}(\mathbf{V})(\mathbf{y}) = -\frac{1}{4\pi} \int_{\Omega} \mathbf{V}(\mathbf{x})\Delta_{\mathbf{y}}(\frac{1}{R})\, dv_x + \frac{1}{4\pi}\vec{\nabla}_{\mathbf{y}} \int_{\Omega} (\mathbf{V}(\mathbf{x}) \cdot \vec{\nabla}_{\mathbf{y}})(\frac{1}{R})\, dv_x$$

$$= -\frac{1}{4\pi} \int_{\Omega} \mathbf{V}(\mathbf{x})\Delta_{\mathbf{y}}(\frac{1}{R})\, dv_x - \frac{1}{4\pi}\vec{\nabla}_{\mathbf{y}} \int_{\Omega} (\mathbf{V}(\mathbf{x}) \cdot \vec{\nabla}_{\mathbf{x}} \frac{1}{R})\, dv_x$$

$$= -\frac{1}{4\pi} \int_{\Omega} \mathbf{V}(\mathbf{x})\Delta_{\mathbf{y}}(\frac{1}{R})\, dv_x - \frac{1}{4\pi}\vec{\nabla}_{\mathbf{y}} \int_{\Omega} (\vec{\nabla}_{\mathbf{x}} \cdot (\frac{\mathbf{V}(\mathbf{x})}{R}) - (\vec{\nabla}_{\mathbf{x}} \cdot \mathbf{V}(\mathbf{x}))(\frac{1}{R}))\, dv_x. \tag{11}$$

The product rule

$$\vec{\nabla}_{\mathbf{x}} \cdot (\frac{\mathbf{V}(\mathbf{x})}{R}) = \vec{\nabla}_{\mathbf{x}} \cdot \mathbf{V}(\mathbf{x})(\frac{1}{R}) + \mathbf{V}(\mathbf{x}) \cdot \vec{\nabla}_{\mathbf{x}} \frac{1}{R}$$

has been substituted to arrive at (11). The first term in (11) can be simplified by substituting

$$\Delta_{\mathbf{y}}(\frac{1}{R}) = -4\pi\delta^{(3)}(\mathbf{x}-\mathbf{y})$$

and integrating over Ω to obtain $\mathbf{V}(\mathbf{y})\delta_{\mathbf{y},\Omega}$ as in (8). To finish the proof, Gauss's Theorem is used to write the second integral in (11) as an integral over $\partial\Omega$ to obtain the final result

$$\vec{\nabla}_{\mathbf{y}}\times\mathbf{B}(\mathbf{V})(\mathbf{y}) = \mathbf{V}(\mathbf{y})\delta_{\mathbf{y},\Omega}+\frac{1}{4\pi}\vec{\nabla}_{\mathbf{y}}\int_{\Omega}\vec{\nabla}_{\mathbf{x}}\cdot\mathbf{V}(\mathbf{x})(\frac{1}{R})\,dv_x-\frac{1}{4\pi}\vec{\nabla}_{\mathbf{y}}\int_{\partial\Omega}\frac{\mathbf{V}(\mathbf{x})\cdot\mathbf{n}}{R}\,da_x$$

$$= \mathbf{V}(\mathbf{y})\delta_{\mathbf{y},\Omega}+\frac{1}{4\pi}\vec{\nabla}_{\mathbf{y}}\int_{\Omega}\frac{\vec{\nabla}_{\mathbf{x}}\cdot\mathbf{V}(\mathbf{x})}{|\mathbf{y}-\mathbf{x}|}\,dv_x-\frac{1}{4\pi}\vec{\nabla}_{\mathbf{y}}\int_{\partial\Omega}\frac{\mathbf{V}(\mathbf{x})\cdot\mathbf{n}}{|\mathbf{y}-\mathbf{x}|}\,da_x. \qquad (12)$$

□

This theorem contains an enormous amount of information. If the vector field $\mathbf{V}$ is divergence-free, then the first integral on the right of (12) vanishes. If $\mathbf{V}$ is tangent to $\partial\Omega$, the second integral vanishes. If both conditions apply, then $\mathbf{V}\in K$, the set of knots, and it is found that on Ω, the image of curl contains $FK\oplus HK$.

The absolute homology of Ω consists of vector spaces $H_i(\Omega)$ for $i=0,1,2,3$ while the relative homology of Ω modulo its boundary consists of the vector spaces $H_i(\Omega,\partial\Omega)$, and only homology with real coefficients is used.

The absolute homology vector space $H_0(\Omega)$ is generated by equivalence classes of points in Ω, with two points equivalent if they can be connected by a path in Ω; $H_1(\Omega)$ is generated by equivalence classes of oriented loops in Ω, with two loops being equivalent if their difference is the boundary of an oriented surface in Ω. The space $H_2(\Omega)$ is generated by equivalence classes of closed oriented surfaces in Ω, with two such surfaces equivalent if their difference is the boundary of some oriented subregion of Ω. The space $H_3(\Omega)$ is always zero. Similar descriptions can be given for the spaces $H_i(\Omega,\partial\Omega)$, but Poincaré duality provides the following isomorphisms:

$$H_0(\Omega)\cong H_3(\Omega,\partial\Omega),\quad H_1(\Omega)\cong H_2(\Omega,\partial\Omega),$$

$$H_2(\Omega)\cong H_1(\Omega,\partial\Omega),\quad H_3(\Omega)\cong H_0(\Omega,\partial\Omega).$$

Alexander duality provides the isomorphisms

$$H_0(\Omega)\cong H_2(\Omega'),\qquad H_1(\Omega)\cong H_1(\Omega'),\qquad H_2(\Omega)\cong \tilde{H}_0(\Omega'),$$

where $\tilde{H}_0$ has dimension reduced by one. The isomorphisms here state that the homology of the closed complementary domain Ω' depends only on the homology of Ω, and not on how Ω is embedded in three-space. This is quite different from how fundamental groups behave.

Thus homology of the domain of definition plays a significant role here. For example, if $\mathbf{V}$ has zero curl, then the circulation of $\mathbf{V}$ around C depends only on the homology class of C in $H_1(\Omega)$. A consequence of Stokes theorem, for if the oriented loops C and C' together bound a surface S, $\partial S = C - C'$, then we have

$$\int_C \mathbf{V}\cdot d\mathbf{s} - \int_{C'} \mathbf{V}\cdot d\mathbf{s} = \int_{C-C'} \mathbf{V}\cdot d\mathbf{s} = \int_{\partial S} \mathbf{V}\cdot d\mathbf{s} = \int_S \vec{\nabla}\times\mathbf{V}\cdot\mathbf{n}\,da = 0. \quad (13)$$

Also $\mathbf{V}$ can be integrated along an oriented path P. If the end points of P lie on $\partial\Omega$ and $\mathbf{V}$ has zero curl and is orthogonal to $\partial\Omega$, then the value of the integral $\int_P \mathbf{V}\cdot d\mathbf{s}$ depends only on the relative homology class of P in $H_1(\Omega, \partial\Omega)$, again a consequence of Stokes theorem.

The next proposition requires the fact that the harmonic knots HK on Ω are rich enough to reflect a significant portion of its topology. It must be shown that the subspace of divergence-free vector fields that are tangent to the boundary of Ω is the orthogonal direct sum of the subspace of fluxless knots and the subspace of harmonic knots.

Proposition 2. The subspace K is the direct sum of the orthogonal subspaces FK and HK

$$K = FK \oplus HK. \quad (14)$$

Proof: Let $\mathbf{V}$ be a divergence-free vector field defined in Ω and tangent to its boundary. Let $\Sigma_1, \cdots, \Sigma_k$ be a family of cross-sectional surfaces in Ω that form a basis for the relative homology $H_2(\Omega, \partial\Omega)$. Let $\Phi_1, \cdots, \Phi_k$ be the fluxes through these surfaces.

Now since $HK \cong H_1(\Omega) \cong H_2(\Omega, \partial\Omega)$, there exists a harmonic knot $\mathbf{V}_H$ in Ω with precisely these flux values. Define $\mathbf{V}_F = \mathbf{V} - \mathbf{V}_H$ then $\mathbf{V}_F$ is fluxless. Hence, every divergence-free vector field $\mathbf{V}$ defined in Ω and tangent to its boundary can be written as the sum of a fluxless knot and a harmonic knot. In fact, the fluxless knots are orthogonal to harmonic knots.

To see this, let $\mathbf{V} = FK$ be a fluxless knot, so $\vec{\nabla}\cdot\mathbf{V} = 0$, and $\mathbf{V}\cdot\mathbf{n} = 0$ and all interior fluxes are zero. By the theorem, it is possible to write $\mathbf{V} = \vec{\nabla}\times\mathbf{U}$ where $\vec{\nabla}\cdot\mathbf{U} = 0$ and $\mathbf{U}\times\mathbf{n} = \mathbf{0}$. Let $\mathbf{W} \in HK$ be a harmonic knot so that $\vec{\nabla}\cdot\mathbf{W} = 0$, $\vec{\nabla}\times\mathbf{W} = \mathbf{0}$ and $\mathbf{W}\cdot\mathbf{n} = 0$. By means of the identity $\vec{\nabla}\cdot(\mathbf{U}\times\mathbf{W} = (\vec{\nabla}\times\mathbf{U})\cdot\mathbf{W} - \mathbf{U}\cdot(\vec{\nabla}\times\mathbf{W})$, then since $\vec{\nabla}\times\mathbf{W} = \mathbf{0}$, it is found that

$$\langle\mathbf{V},\mathbf{W}\rangle = \langle\vec{\nabla}\times\mathbf{U},\mathbf{W}\rangle = \int_\Omega (\vec{\nabla}\times\mathbf{U})\cdot\mathbf{W}\,dv = \int_\Omega [\vec{\nabla}\cdot(\mathbf{U}\times\mathbf{W}) + \mathbf{U}\cdot(\vec{\nabla}\times\mathbf{W})]\,dv$$

$$= \int_\Omega \vec{\nabla}\cdot(\mathbf{U}\times\mathbf{W})\,dv = \int_{\partial\Omega} (\mathbf{U}\times\mathbf{W})\cdot\mathbf{n}\,da = 0.$$

The last integral vanishes since vector field $\mathbf{U}$ is orthogonal to $\partial\Omega$. □

Moreover, the subspace G of gradient vector fields can be decomposed as follows.

Proposition 3. The subspace G is the direct sum of two orthogonal subspaces

$$G = DFG \oplus GG. \tag{15}$$

Proof: Consider a gradient vector field $\mathbf{V} = \vec{\nabla}\varphi$, where φ is any smooth function on Ω. Let φ_1 be a solution of Laplace's equation on Ω which satisifes the Dirichlet boundary condition $\varphi_1|_{\partial\Omega} = \varphi|_{\partial\Omega}$. Set $\varphi_2 = \varphi - \varphi_1$ so that $\mathbf{V}_1 = \vec{\nabla}\varphi_1$ and $\mathbf{V}_2 = \vec{\nabla}\varphi_2$ satisfy $\mathbf{V} = \mathbf{V}_1 + \mathbf{V}_2$.

Since φ_1 is harmonic $\vec{\nabla} \cdot \mathbf{V}_1 = \Delta\varphi_1 = 0$, it follows that $\mathbf{V} \in DFG$. Similarly, $\varphi_2|_{\partial\Omega} = \varphi|_{\partial\Omega} - \varphi_1|_{\partial\Omega} = 0$, which implies that $\mathbf{V}_2 \in GG$. Consequently, the subspaces DFG and GG span G.

It has to be shown that the divergence-free gradients are orthogonal to the grounded gradients. Let $\mathbf{V} \in DFG$ so that $\mathbf{V} = \vec{\nabla}\varphi$ and $\vec{\nabla} \cdot \mathbf{V} = \Delta\varphi = 0$. Moreover, let $\mathbf{W} \in GG$ and take the function ψ satisfy $\mathbf{W} = \vec{\nabla}\psi$ with $\psi|_{\partial\Omega} = 0$. Starting with the identity $\vec{\nabla} \cdot (\psi\vec{\nabla}\varphi) = \vec{\nabla}\psi \cdot \vec{\nabla}\varphi + \psi\Delta\varphi$, then using the fact φ is harmonic, we have

$$\langle \mathbf{V}, \mathbf{W} \rangle = \int_\Omega \mathbf{V}\cdot\mathbf{W}\, dv = \int_\Omega \vec{\nabla}\varphi\cdot\vec{\nabla}\psi\, dv = \int_\Omega \vec{\nabla}\cdot(\psi\vec{\nabla}\varphi)\, dv = \int_{\partial\Omega} (\psi\vec{\nabla}\varphi)\cdot\mathbf{n}\, da = 0.$$

The last integral is zero since $\psi|_{\partial\Omega} = 0$ and so the integrand vanishes on $\partial\Omega$.
□

Proposition 4.

$$\text{im curl} = FK \oplus HK \oplus CG. \tag{16}$$

Proof: It is clear first of all that

$$\text{im curl} \subset FK \oplus HK \oplus CG. \tag{17}$$

To obtain (17), suppose that $\mathbf{V} = \vec{\nabla} \times \mathbf{U}$, hence it follows that $\vec{\nabla} \cdot \mathbf{V} = 0$ and so the flux of $\mathbf{V}$ through every closed surface in Ω is zero. Suppose φ is a solution of $\Delta\varphi = 0$ with Neumann boundary condition $\partial\varphi/\partial n = \mathbf{V} \cdot \mathbf{n}$ on $\partial\Omega$, which implies that $\mathbf{V}_2 = \vec{\nabla}\varphi$ lies in CG. The vector field $\mathbf{V}_1 = \mathbf{V} - \mathbf{V}_2$ satisfies $\vec{\nabla} \cdot \mathbf{V} = \vec{\nabla} \cdot \mathbf{V} - \vec{\nabla} \cdot \mathbf{V}_2 = -\Delta\varphi = 0$. Thus, $\mathbf{V}$ is divergence free and tangent to $\partial\Omega$ since $\mathbf{V}_1 \cdot \mathbf{n} = \mathbf{V} \cdot \mathbf{n} - \vec{\nabla}\varphi \cdot \mathbf{n} = \vec{\nabla}\varphi \cdot \mathbf{n} = 0$. Hence, $\mathbf{V}$ resides in $FK \oplus HK$, and since we can write $\mathbf{V} = \mathbf{V}_1 + \mathbf{V}_2$, this establishes (17).

Suppose $\mathbf{V} \in FK \oplus HK \oplus CG$ which implies that $\vec{\nabla} \cdot \mathbf{V} = 0$ and the flux of $\mathbf{V}$ through each component of $\partial\Omega$ vanishes. It is to be shown that $\mathbf{V}$ can be expressed as a curl, $\mathbf{V} = \vec{\nabla} \times \mathbf{U}$ for some vector field $\mathbf{U}$, and the way to do this is to make use of Theorem 2 for divergence-free fields

$$\vec{\nabla}_{\mathbf{y}} \times \mathbf{B}(\mathbf{V})(\mathbf{y}) = \mathbf{V}(\mathbf{y})\delta_{\mathbf{y},\Omega} - \frac{1}{4\pi}\vec{\nabla}_{\mathbf{y}} \int_{\partial\Omega} \frac{\mathbf{V}(\mathbf{x})}{|\mathbf{y} - \mathbf{x}|}\, da_x. \tag{18}$$

To show that vector field $\mathbf{V}$ can be expressed as a curl, it suffices to show that the last term in (18) is a curl.

To this end, construct a new domian set Ω^* by looking at a ball β large enough to contain Ω in its interior, and then removing the interior of Ω. The boundary components of Ω^* then will consist of the boundary components of Ω union the boundary of the ball, $\partial\beta$.

Now a Neumann problem is solved for the Laplacian on Ω^*, that is, a harmonic function φ^* is to be found on Ω^* such that $\partial\varphi^*/\partial n^* = -\mathbf{V}\cdot\mathbf{n}$ on each of the boundary components that Ω^* shares with Ω, and moreover, we have $\partial\varphi^*/\partial n^* = 0$ on the boundary of the ball. Suppose we let $\mathbf{V} = \vec{\nabla}\varphi^*$, the curl equation for $\mathbf{V}^*$ is

$$\vec{\nabla}_{\mathbf{y}} \times \mathbf{B}(\mathbf{V}^*)(\mathbf{y}) = \mathbf{V}^*(\mathbf{y})\delta_{\mathbf{y},\Omega^*} - \frac{1}{4\pi}\vec{\nabla}_{\mathbf{y}} \int_{\partial\Omega^*} \frac{\mathbf{V}^*(\mathbf{x})\cdot\mathbf{n}^*}{|\mathbf{y}-\mathbf{x}|}\, da_x.$$

In the domain complementary to Ω^*, namely $\Omega^{*'}$, this equation takes the form

$$\begin{aligned}\vec{\nabla}_{\mathbf{y}} \times \mathbf{B}(\mathbf{V}^*)(\mathbf{y}) &= -\frac{1}{4\pi}\vec{\nabla}_{\mathbf{y}} \int_{\partial\Omega^*} \frac{\mathbf{V}^*(\mathbf{x})\cdot\mathbf{n}^*}{|\mathbf{y}-\mathbf{x}|}\, da_x = -\frac{1}{4\pi}\vec{\nabla}_{\mathbf{y}} \int_{\partial\Omega^*} \frac{\vec{\nabla}\varphi^*\cdot\mathbf{n}^*}{|\mathbf{y}-\mathbf{x}|}\, da_x \\ &= \frac{1}{4\pi}\vec{\nabla}_{\mathbf{y}} \int_{\partial\Omega} \frac{\mathbf{V}(\mathbf{x})\cdot\mathbf{n}}{|\mathbf{y}-\mathbf{x}|}\, da_x - \frac{1}{4\pi}\vec{\nabla}_{\mathbf{y}} \int_{\partial\beta} \frac{\mathbf{V}^*(\mathbf{x})\cdot\mathbf{n}^*}{|\mathbf{y}-\mathbf{x}|}\, da_x \\ &= \frac{1}{4\pi} \int_{\partial\Omega} \frac{\mathbf{V}(\mathbf{x})\cdot\mathbf{n}}{|\mathbf{y}-\mathbf{x}|}\, da_x. \end{aligned} \tag{19}$$

The second integral over $\partial\beta$ here vanishes since $\mathbf{V}^*\cdot\mathbf{n}^* = 0$ on the boundary of the ball. Since $\Omega \subset \Omega^{*'}$, this result holds in Ω as well. Thus, in Ω the result (18) and (19) can be added together to obtain

$$\vec{\nabla}_{\mathbf{y}}(\mathbf{B}(\mathbf{V}) - \mathbf{B}(\mathbf{V}^*))(\mathbf{y}) = \mathbf{V}(\mathbf{y}) - \frac{1}{4\pi}\vec{\nabla}_{\mathbf{y}} \int_{\partial\Omega} \frac{\mathbf{V}(\mathbf{x})\cdot\mathbf{n}}{|\mathbf{y}-\mathbf{x}|}\, da_x + \frac{1}{4\pi}\vec{\nabla}_{\mathbf{y}} \int_{\partial\Omega} \frac{\mathbf{V}(\mathbf{x})\cdot\mathbf{n}}{|\mathbf{y}-\mathbf{x}|}\, da_x = \mathbf{V}(\mathbf{y}). \tag{20}$$

Equation (20) illustrates that $\mathbf{V}(\mathbf{y})$ is in the image of the operator curl, and hence the statement of the theorem follows. □

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