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Decomposition of Groups and Top Couples

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Abstract

We have endowed various categories of groups with topologies [12]. The purpose of this paper is to introduce on these categories others topologies which are more suitable to study well-known problems in groups theory. We use this framework to define a notion of prime ideal and to provide a decomposition of a large class of groups into a product of prime ideals. Remark that a similar question has been studied in [5] by Kurata. We remark that these topologies can be extended to other categories like the categories of commutative algebras, associative algebras and left symmetric algebras.

Definition 1.

A top couple (C, D) is defined by:

A subcategory C of the category of groups, a subclass D of the class of objects of C which satisfies the following properties:

T1. Let G, G' be objects of C such that G' is in D, if there exists an injective morphism $i : G \to G'$, then G is in D.

T2. Let G be an object of D, I and J two normal subgroups of G such that $I \cap J = 1$, then $I = 1$ or $J = 1$.

T3. Let G be an object of C, the normal subgroup I of G is an ideal of G if and only if the quotient G/I is an object of C; we suppose that the inverse image of an ideal by a morphism of C is an ideal.

Remark.

Let $[I, J]$ be the subgroup generated by the commutators $[x, y] = xyx^{-1}y^{-1}$.

$x \in I$ and $y \in J$.In [12], we have defined a notion of Top couple where we have replaced the axiom T2 by the axiom T'2 as follows: Let G be an object of D, I, J two normal subgroups of G, $[I, J] = 1$ implies $I = 1$ or $J = 1$; remark that $[I, J] \subset I \cap J$. This new definition enables to obtain more examples of Top couples which are eventually commutative and non trivial. We start by our first example:

Let G be a group, we denote by C_G the comma category over G, the objects of C_G are morphisms $f_H : G \to H$. We denote such an object by (f_H, H). A morphism between (f_H, H) and (f_L, L) is a morphism of groups $f : H \to L$ such that $f \circ f_H = f_L$. Let (H, ϕ_H) be an object of C_G and x an element of H, we denote by $G(x)$ the subgroup of H generated by $\{gxg^{-1}, g \in G\}$. A non trivial element x of H is a divisor of zero if and only if there exists a non trivial element y of H such that $G(x) \cap G(y) = \{1\}$ and $[G(x), G(y)] = 1$. We denote by D_G the subclass of the class of objects of C_G whose objects do not have divisors of zero.

Proposition 1. *The couple (C_G, D_G) is a Top couple.*

Proof. Let us verify the property T1: Let H and H' be elements of C_G, suppose that H' is an object of D_G and there exists an injective morphism $i : H \to H'$. If x, y are elements of H such that $G(x) \cap G(y) = 1$ and $[G(x), G(y)] = 1$, we also have $G(i(x)) \cap G(i(y)) = 1$ and $[G(i(x)), G(i(y))] = 1$ since i is a G-morphism. Since H' does not have divisors of zero, we deduce that $i(x) = 1$ or $i(y) = 1$. This implies that $x = 1$ or $y = 1$ since i is injective.

The verification of T2:

Let H be an object D_G, I and J two normal subgroups of H such that $I \cap J = 1$. Suppose that I and J are not trivial. Let x be a non trivial element of I and y be a non trivial element of J, we have $G(x) \subset I$ and $G(y) \subset J$, this implies that $G(x) \cap G(y) \subset I \cap J = 1$ and $[G(x), G(y)] \subset [I, J] \subset I \cap J = 1$. Since H does not have divisors of zero, we deduce that $x = 1$ or $y = 1$. This is a contradiction.

Verification of T3:

Let $f : H \to H'$ be a morphism of C_G, and I an ideal of H'; $f^{-1}(I)$ is an ideal of H since we can endow $H/f^{-1}(I)$ with the structure induced by the morphism $p \circ f_H$, where $p : H \to H/f^{-1}(I)$ is the canonical projection.

Definitions 2. Let (C, D) be a Top couple, and H an object of C, an ideal P of H is prime if and only if H/P is an object of D.

For every normal subgroup I of H, we denote by $V_H(I)$ the set of prime ideals which contain I.

Proposition 2. *Let (C, D) be a Top couple and H an object of C. For every normal subgroups I, J of H, we have $V_H(I \cap J) = V_H(I) \bigcup V_H(J)$.*

Let $(I_a)_{a \in A}$ be a family of normal subgroups of H, and I_A the normal subgroup generated by $(I_a)_{a \in A}$, we have $V_H(I_A) = \cap_{a \in A} V(I_a)$.

Proof. Firstly, we show that $V_H(I \cap J) = V_H(I) \bigcup V_H(J)$. Let P be an element of $V_H(I \cap J)$ suppose that P does not contain neither I nor J. Let $x \in I$, $y \in J$ which are not elements of P. We denote by $u(x)$ the normal subgroup of H generated by x. We have $u(x) \cap u(y) \subset I \cap J \subset P$. This implies that $x \in P$ or $y \in P$ and $V_H(I \cap J) \subset V_H(I) \bigcup V_H(J)$. Since $I \cap J \subset I$ and $I \cap J \subset J$, we deduce that $V_H(I) \subset V_H(I \cap J)$ and $V_H(J) \subset V_H(I \cap J)$. This implies that $V_H(I \cap J) = V_H(I) \bigcup V_H(J)$.

Now we show that $V_H(I_A) = \cap_{a \in A} V(I_a)$ Let $P \in V_H(I_A)$. For every $a \in A$, $I_a \subset I_A \subset P$. This implies that $P \subset \cap_{a \in A} V_H(I_a)$. Let $P \in \cap_{a \in A} V_H(I_a)$, for every $a \in A$, $I_a \subset P$; this implies that $I_A \subset P$.

Remark.

The proposition 2 shows that the space $Spec_G(H)$ of prime ideals is endowed with a topology whose closed subsets are the subsets $V_H(I)$ and the empty subset of H.

Let x and y be divisors of zero in the G-group H; the subgroup of H generates by $G(x)$ and $G(y)$ is isomorphic to the direct product $G(x) \times G(y)$.

This leads to the following definitions:

Definitions 3. Let H be an element of C_G, the adjoint representation $Ad : G \rightarrow Aut(H)$ is the morphism which associates to $g \in G$ the automorphism of H defined by $Ad(g)(h) = ghg^{-1}, h \in H$.

Let H be an object of C_G; a non trivial subgroup H' of H stable by the adjoint representation is G-decomposable if and only if there exists two non trivial subgroups H_1 and H_2 of H stable by the adjoint representations and an isomorphism of groups $f : H' \rightarrow H_1 \times H_2$ which commutes with the adjoint representation.

An object H of C_G is locally G-indecomposable if every non trivial subgroup of H is not G-decomposable.

If G is the trivial group, we will omit the suffix G in the previous definitions, for example, we will speak of decomposable groups and locally indecomposable groups.

Proposition 3. *A G-group H does not have divisors of zero if and only if H is locally G-indecomposable.*

Proof. Suppose that the G-group H does not have divisors of zero, let L be a subgroup stable by the adjoint action; suppose that L is isomorphic to the product of the non trivial subgroups L_1 and L_2 stable by the adjoint representation. Let $x_1 \in L_1$ and $x_2 \in L_2$ be non trivial elements; $(x_1, 1)$ and $(1, x_2)$ are divisors of zero. This is a contradiction.

Conversely, suppose that the G-group H is locally indecomposable; let x and y be divisors of zero; the subgroup of H generates by $G(x)$ and $G(y)$ is a subgroup of G which is the direct product of the subgroups $G(x)$ and $G(y)$ which are stable by the adjoint action. This is a contradiction.

Remark.

Let G be a group, to study the geometry of objects of C_G, it is very important to know objects without divisors of zero. Firstly, we are going to study these objects for $G = 1$. We are also going to classify finitely generated nilpotent groups who do not have divisors of zero. Remark that finite groups without divisors of zero have been classified by Marin when $G = 1$; to present his result, let us recall that the quaternionic group Q_n (n is an integer superior or equal to 3) is a finite group of order 2^n with the presentation:

$$< x, y : x^{2^{n-1}} = 1, y^2 = x^{2^{n-2}}, y^{-1}xy = x^{-1} >$$

Theorem Marin [6]. *Suppose that $G = 1$; a finite group H is indecomposable if and only if:*

1. H is isomorphic to Z/p^n for some prime p.

2. H is isomorphic to $Q_n, n \geq 3$.

3. H is isomorphic to an extension of Z/q^b by Z/p^a where p and q are different prime integers such that p is odd, q^b divides $p - 1$ and the image of Z/q^b in $(Z/p^a)^*$ has order q^b.

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Proposition 4. *Suppose that $G = 1$, let H be a group without divisors of zero. The rank of every commutative subgroup of H is inferior to 1. In particular the rank of the center $C(H)$ is inferior to 1. If the center is not trivial, for every $y \in H$, there exists $n \in N$ such that y^n is an element of $C(H)$ and distinct of the identity. If the order of the center $C(H)$ is finite, then the order of every element of H is finite and in this case the order of such an element is p^n where p is a prime integer.*

Proof. If the rank of a commutative subgroup L is strictly greater than 1, there exists non trivial elements x, y in L such that $[x, y] = 1$ and $(x) \cap (y) = 1$. Where (x) is the subgroup of H generated by x. This is in contradiction with the fact that H does not have zero divisors. Let $z \in C(H)$ be a non trivial element, for every element $x \in H$, we have $[x, z] = 1$, since H does not have divisors of zero, we deduce that $(x) \cap (z)$ is not the trivial group.

Suppose that the center of H has a finite order, for any element $x \in H$, there exists an integer n such that $x^n \in C(H)$, x^n and henceforth x has a finite order. If the order of z is the product nm of two integers n and m which are relatively prime, then z^n and z^m are divisors of zero.

Theorem 1. *Suppose that $G = 1$, let H be a finitely generated nilpotent group without divisors of zero. Then H is finite or H is isomorphic to Z.*

Proof. Let H be a non trivial finitely generated nilpotent group. recall that the derivative sequence of H is defined by $H^0 = H$, and $H^{(n)} = [H, H^{(n-1)}]$. There exists n such that $H^{(n)} = 1$, and $H^{(n-1)}$ is not trivial and contained in the center of H. The proposition 4 shows that the rank of $H^{(n-1)}$ is 1. Suppose that there exists an element x of $H^{(n-1)}$ which has a finite order, then every element of H has a finite order. The subgroup $H^{(n-2)}$ is finite since it is the extension of a commutative finite group by a commutative finite group; recursively, we obtain that H is finite.

Suppose now that $C(H)$ has infinite order and the rank of H is different of 1. We have $[H, H^{(n-2)}] = H^{(n-1)}$. This implies the existence of an element $x \in H$ and $y \in H^{(n-2)}$ such that $[x, y] \in H^{(n-1)}$ and is distinct of the neutral element and has an infinite order. Remark that $[x, y] = h$ is in the center of H. There exists integers n, m such that $x^n \in C(H)$ and $y^m \in C(H)$. We have $1 = x^n y^m x^{-n} y^{-m} = h^{mn} y^m x^n x^{-n} y^{-m} = h^{nm}$. This implies that the order of h is finite. This is a contradiction with the hypothesis.

Corollary 1. *A finitely generated locally indecomposable whose commutator subgroup is nilpotent is a finite group or is a finite extension of Z.*

Proof. Let H be a finitely generated locally indecomposable whose commutator subgroup is nilpotent. Then $[H, H]$ is a locally indecomposable nilpotent group. Suppose that $[H, H]$ is infinite, thus $[H, H] = Z$. Let x be an element of H; $Ad_x : [H, H] \to [H, H]$ defined by $Ad_x(y) = xyx^{-1}$ has order inferior to 2 since the group of automorphisms of Z is isomorphic to $Z/2$. Let y be a non trivial element of $[H, H]$, we deduce that for every $x \in H$, $[x^2, y] = 1$. Since H does not have divisors of zero, it results that there exists n, m such that $x^{2n} = y^m$. Thus the quotient $H/[H, H]$ is finite since it is a finitely generated commutative group and each of its element has a finite order.

Suppose that $[H, H]$ is finite and for every $x \in H$, Ad_x is an automorphism of a finite group, thus there exists n such that Ad_{x^n} is the identity. Let z be a non trivial element of $[H, H]$, $[x^n, z] = 1$, since H does not have divisors of zero, we deduce that there exists m such that $x^{nm} \in [H, H]$; thus every element of H has a finite order. Since H is solvable, we deduce that H is finite.

Corollary 2. *A subgroup I of a finitely generated commutative group H is a prime ideal if and only if G/H is isomorphic either to Z or to Z/p^n where p is a prime.*

Proof. Let I be a prime ideal of the finitely generated commutative group H, if H/I is finite, Marin [6] implies that H/I is isomorphic to Z/p^n where n is a prime. If H/I is infinite, since H/I is nilpotent, the theorem 1 implies that H/I is isomorphic to Z.

Remark.

Suppose that $H = Z$ the group of relative integers. Let I be a ideal of H, we know that I is a subgroup generated by a positive integer n, write $n = \prod_{i \in I} p_i^{n_i}$. Let p be a prime number and a and integer, the prime ideal (p^a) generated by p^a is an element of $V((n))$ if and only if p^a divides n.

We are going to present other examples of locally indecomposable groups Recall that the Tarski group is an infinite group H such that there exists a prime integer p such that every proper of H different of the trivial subgroup is isomorphic to the cyclic group Z/p. The Tarski group is known to be simple. Olshans'kii [8] and have shown the existence of Tarski groups for $p > 10^{75}$.

Adyan and Lysenok [1] and have generalized the construction of Ovshan'skii and shown that for $n > 1003$ there exists non commutative groups H such that every proper subgroup of H is isomorphic to a subgroup isomorphic to Z/n, we will call these groups Adyan-Lysenok groups.

Remark that the Adyan-Lysenok groups H defined for $n = p^m$ is a domain for $G = 1$: Let x, y divisors of zero in H, since the subgroup $< x, y >$ generated by x and y is a commutative subgroup we deduce that $< x, y >$ is isomorphic to a subgroup of Z/p^m. This is in contradiction with the fact that $< x > \cap < y >$ is trivial.

More domains can be constructed by using the following proposition:

Proposition 5. *The free product two locally indecomposable groups is a locally indecomposable group.*

Proof. Let G and H be two locally indecomposable groups. Let x and y be divisors of zero, then since $xy = yx$, the corrollary [7] 4.1.6 p.187 shows either:

- x and y are conjugated in the same factor of G or H. This is impossible since G and H are locally indecomposable

- x and y are the power of the same element. This is in contradiction with the fact that x and y are divisors of zero.

Definition 4. Let H be an element of $C(G)$, we denote by $Rad_G(H)$ the intersection of all the prime ideals of H.

Recall that a topological space X is irreducible if and only if it is not the union of two proper subsets.

We say that an ideal I is a radical ideal if it is the intersection of all the prime which contains I.

Proposition 6. *Let H be an element of $C(G)$, and I a radical ideal of H, then $V_H(I)$ is irreducible if and only if I is a prime.*

Proof. Suppose that I is a prime, and $V_H(I) = V_H(J) \bigcup V_H(K)$ where $V_H(J)$ and $V_H(K)$ are proper subsets, since I is a prime, I is an element of $V_H(I)$. This implies that $I \in V_H(J)$ or $V_H(K)$. If I is an element of $V_H(J)$, then $V_H(I) \subset V_H(J)$; if $I \in V_H(K)$, then $V_H(I) \subset V_H(K)$.This is a contradiction with the fact that $V_H(J)$ and $V_H(K)$ are proper subsets of $V_H(I)$.

Suppose that $V_H(I)$ is irreducible; let x, y be elements of H such that $[G(x), G(y)] \subset I$ and $G(x) \cap G(y) \subset I$. Let $u(x)$ be the normal subgroup generated by x, $u(x) \cap u(y)$ and $[u(x), u(y)]$ are contained in I. Since $V_H(u(x) \cap$

$u(y)) = V_H(u(x)) \bigcup V_H(u(y))$, this implies that $V_H(I) = V_H(u(x)) \cap V_H(I) \bigcup V_H(u(y)) \cap V_H(I)$. Since $V_H(I)$ is irreducible, we deduce that $V_H(I)$ is contained in $V_H(u(x))$ or is contained in $V_H(u(y))$. If $V_H(I)$ is contained in $V_H(u(x))$, the $\cap_{P \in V_H(I)} P = I$ contains $u(x)$. It results that $x \in I$ since I is a radical ideal. Similarly, if $V_H(I) \subset V_H(u(y))$ we deduce that $y \in I$.

Definition 5. Recall that a space is Noetherian if and only if every ascending chain of closed subsets $Z_0 \subset Z_1 ... \subset Z_n \subset ...$ stabilizes, this is equivalent to saying that there exists i such that for every $n > i, Z_n = Z_i$. We deduce that the topological space $Spec_G(H)$ is Noetherian if and only if a descending chain of normal subgroups of H $(I_n)_{n \in N}$ such that $I_{n+1} \subset I_n$ stabilizes.

Remark.

Let x be an element of G, $G(x)$ is a normal subgroup of G.

A maximal normal subgroup I of G is a prime ideal since for every element x of G/I, $G(x)$ is a normal subgroup of G/I. Thus $G(x) = G/I$ since G/I is simple.

Now we show the decomposition theorem:

Theorem 2. *Suppose that $Spec_G(G)$ is Noetherien and $Rad_G(G) = 1$, then G is the product of groups $G_1 \times ... \times G_n$ such that for every i, the subgroup H_i of H generated by $G_j, j \in \{1, ..., n\} - \{i\}$ is a prime. Moreover, this decomposition is unique up to the permutation of the G_i.*

Proof. Suppose that $Spec_G(G)$ is Noetherian, then $Spec_G(G)$ is union of maximal irreducible components $(V_G(H_i))_{i=1,...,n}$.

The intersection $\cap_{i=1,...,n} H_i = 1$. This is due to the fact that $V_G(\cap_{i=1,...,n} H_i) = V_G(H_1) \bigcup ... \bigcup V_G(H_n) = Spec_G(G)$ and $Rad_G(G) = 1$.

We write $G_i = \cap_{j \in \{1,...,n\} - \{i\}} H_j$. We are going to show that G is isomorphic to the direct product $G_1 \times ... \times G_n$.

Firstly, remark that $G_i \cap G_j = \cap_{k=1,..,n} H_k = 1$ if $i \neq j$. Since the subgroup G_i are normal, for $i \neq j$, we have $[G_i, G_j] \subset G_i \cap G_j = 1$. This implies that the subgroup L of H generated by $(G_i)_{\{i=1,...,n\}}$ is isomorphic to the direct product $G_1 \times G_2 \times ... \times G_n$. It remains to shows that G is equal to its subgroup L.

We have $V_G(G_i) = \bigcup_{j \in \{1,..,n\}, j \neq i} V_G(H_i)$. This implies that $V_G(L) = \cap_{i=1,...,n} \bigcup_{j \in \{1,..,n\}, j \neq i} V_G(H_i)$ is empty. We deduce that $L = H$, otherwise L would have been contained in a maximal ideal which would have been an element of $V_G(L)$.

We show now that the subgroup L_i of generated by $(G_j)_{j \neq i}$ is H_i. For every $j \neq i$, $G_j \subset H_i$. Suppose that there exists an element $x \in H_i$ which is not in L_i. Since $H = G_1 \times ... \times G_n$, we can write $x = (x_1, ..., x_n), x_j \in G_j$ and $x_i \neq 1$, we have $x_j \in H_i, j \neq i$. This implies that $x_i \in H_i$. This is a contradiction since $H_i \cap G_i = \{1\}$.

We show now that the decomposition is unique. Suppose that there are two decompositions $H = G_1 \times ... \times G_n$ and $H = U_1 \times ... \times U_m$ such that the group H_i generated by $1 \times ..G_j \times 1$, $j \neq i$ is a prime ideal, the group L_i generated by $1 \times ..U_j \times .. \times 1$, $j \neq i$ is also a prime ideal. Then $\bigcup_{i=1,..,n} V_G(H_i)$ and $\bigcup_{i=1,..,m} V_G^*(L_i)$ are decomposition of $Spec_G(G)$ as union of irreducible components. Since this decomposition is unique, we deduce that $n = m$, and up to permutation that $V_G(H_i) = V_G(L_i)$, since U_i and H_i are prime, we deduce that $H_i = L_i$. This implies that $G_i \simeq G/H_i$ and $U_i \simeq G/L_i$ are isomorphic.

Corollary 3. *Suppose that G is a finite group and $Rad_G(G) = 1$, then G is a product of indecomposable subgroups.*

Some generalizations.

Let A be a commutative ring, in classical algebraic geometry a prime ideal P of A is an ideal P such for every elements $a, b \in A$, $ab \in P$ implies that $a \in P$ or $b \in P$. Inspired by the topologies defined above, we define the following notion:

Definitions 5. Let A be a ring non necessarily commutative, a, b elements of A. We denote by $I(a)$ the two-sided ideal generated by a. A two-sided ideal P of the ring A is a prime ideal if for every elements $a, b \in A$, $I(a) \cap I(b) \in P$ implies that $a \in P$ or $b \in P$.

Let I be a two-sided ideal of A, we denote by $V(I)$ the set of prime ideals of A which contain I.

Proposition 7. *Let I, J be two-sided ideals of A, we have: $V(I \cap J) = V(I) \bigcup V(J)$. Let $(I_a)_{a \in A}$ be family of ideals of A which generates the ideal I_A, we have $V(I_A) = \cap_{a \in A} V(I_a)$.*

Proof. Firstly, we show that $V(I \cap J) = V(I) \bigcup V(J)$. Since $I \cap J \subset I$ and $I \cap J \subset J$, we have $V(I) \subset V(I \cap J)$ and $V(J) \subset V(I \cap J)$. Let P be an element of $V(I \cap J)$, suppose that P does not contain I and J. Let $a \in I, b \in J$ be elements which are not in P, $I(a) \cap I(b) \subset I \cap J$. This is a contradiction since P is a prime ideal.

Let P be an element of $V(I_A)$, since $I_A \subset P$, $I_a \subset P$ for every $a \in A$, this implies that $P \in \cap_{a \in A} V(I_a)$. Conversely, let $P \in \cap_{a \in A} V(I_a)$, for every $a \in A$, $I_a \subset P$. This implies that $I_A \subset P$.

Remark.

Suppose that A is a commutative algebra, the notion of prime ideal obtained here is different from the classical notion of prime. As we have seen, if $A = Z$, Z/p^n is a prime ideal.

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