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On the Basis Property of Eigenfunction of the Frankl Problem with Nonlocal Parity Conditions of the Third Kind

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Abstract

In the present paper, we obtain the eigenvalues and eigenfunctions of the Frankl problem with a nonlocal parity condition of the third kind.we prove the minimalist, the completeness and Riesz basis of the eigenfunctions corresponding to the eigenvalues of the problem in the space $L_2(D_+)$.

1. INTRODUCTION

The primary Frankl problem was inquired in [1]. The problem with a nonlocal boundary condition of the second kind was stadid in [2]. In the present paper, we assume boundary canditions of the third kind that when y is limited to zero and also in $x = o$ x=o the function values are linearly dependent in the elliptic and Hyperbolic regions. In the proof of principal theorem we investigate the minimalist, the completeness and Riesz basis of a specified system of cosines.

Definition 1. System $\{x_n\}_{n\in N} \subset X$ is called complete in X if $\overline{L[\{x_n\}_{n\in N}]} = X$.

Definition 2. System $\{x_n\}_{n\in N} \subset X$ is called minimal in X if $x_k \notin \overline{L[\{x_n\}_{n\neq k}]}, \forall k \in N$.

Remark. If the system $\{x_n\}_{n\in N}$ is minimal in $L_p(I)$, then it is also minimal in $L_p(J)$ for $J \supset I$; and if it is complete in $L_p(I)$, then it is also complete in $L_p(J)$; for $J \subset I$.

2. THE FRANKL PROBLEM WITH NONLOCAL CONDITION OF THE THIRD KIND

The Frankl problem is to seek a solution for equation

$$u_{xx} + sgn(y)u_{yy} + \mu^2 sgn(x+y)u = 0 \qquad (1)$$

in $D_+ \cup D_{-1} \cup D_{-2}$ with the boundary conditions

$$u(1,\theta) = 0, \qquad \theta \in [0, \frac{\pi}{2}] \qquad (2)$$

$$\frac{\partial u}{\partial x}(0, y) = 0, \qquad y \in (-1,1) \qquad (3)$$

$$\frac{\partial u}{\partial y}(x,+0) = \frac{\partial u}{\partial y}(x,-0) \qquad (4)$$

$$\kappa u(0, y) = u(0,-y), \qquad y \in [0,1] \qquad (5)$$

$$\kappa u(x,+0) = u(x,-0) \qquad (6)$$

The function $u(x, y) \in C^0(D_+ \cup D_{-1} \cup D_{-2}) \cap C^2(D_+) \cap C^2(D_-)$. which areas D_+, D_{-1} and D_{-2} are defined as follows:

$$D_+ = \left\{(r,\theta): \quad 0 < r < 1 \quad , \quad 0 < \theta < \frac{\pi}{2}\right\}$$

$$D_{-1} = \left\{(x, y): \quad -y < x < y+1, \quad \frac{-1}{2} < y < 0\right\}$$

$$D_{-2} = \left\{(x, y): \quad x-1 < y < -x, \quad 0 < x < \frac{1}{2}\right\}$$

Theorem 1.The eigenvalues and eigenfunctions of problem (1)-(6) show by two series.In the first series, the eigenvalues $\lambda_{nk} = \mu_{nk}^2$ are found from the equation

$$J_{4n}(\mu_{nk}) = 0$$

such that $n = 0,1,2,..., k = 1,2,...$ and the $J_\alpha(z)$ are the Bessel functions [3, p. 12], and the eigenfunctions are provided by the regulations

$$u_{nk}(r,\theta) = A_{nk} J_{4n}(\mu_{nk} r)\cos 4n(\frac{\pi}{2} - \theta) \qquad \text{in} \quad D_+$$

$$u_{nk}(\rho,\psi)=\kappa A_{nk}J_{4n}(\mu_{nk}\rho)\cosh 4n\psi \qquad \text{in} \qquad D_{-1}$$
$$u_{nk}(R,\varphi)=\kappa A_{nk}J_{4n}(\mu_{nk}R)\cosh 4n\varphi \qquad \text{in} \qquad D_{-2}$$

that we use of polar coordinate system

$$r^2=x^2+y^2 \qquad x=r\cos\theta \qquad y=r\sin\theta$$

for $0\le\theta\le\frac{\pi}{2}$ and $0\le r\le 1$ in D_+,

of cartesian coordinate system

$$\rho^2=x^2-y^2 \qquad x=\rho\cosh\psi \qquad y=\rho\sinh\psi$$

for $-\infty<\psi<0$ and $0<\rho<1$ in D_{-1}, and

$$R^2=y^2-x^2 \qquad x=R\sinh\varphi \qquad y=-R\cosh\varphi$$

for $0<\varphi<\infty$ and $0<R<1$ in D_{-2}. In the second series, the eigenvalues $\tilde{\lambda}_{nk}=\tilde{\mu}_{nk}^2$ are resulted from the equation

$$J_{4(n+\Delta)}(\tilde{\mu}_{nk})=0 \qquad n=0,1,... \qquad k=1,2,...$$

and the eigenfunctions are determined by the relations

$$\tilde{u}_{nk}(r,\theta)=\tilde{A}_{nk}J_{\tilde{\alpha}_n}(\tilde{\mu}_{nk}r)\cos\tilde{\alpha}_n(\frac{\pi}{2}-\theta) in D_+$$
$$\tilde{u}_{nk}(\rho,\psi)=\tilde{A}_{nk}J_{\tilde{\alpha}_n}(\tilde{\mu}_{nk}\rho)(\kappa\frac{\kappa^2-1}{\kappa^2+1}\cosh\tilde{\alpha}_n\psi-\frac{2\kappa}{\kappa^2+1}\sinh\tilde{\alpha}_n\psi) \qquad in D_{-1}$$
$$\tilde{u}_{nk}(R,\varphi)=\kappa\tilde{A}_{nk}J_{\tilde{\alpha}_n}(\tilde{\mu}_{nk}R)\cosh\tilde{\alpha}_n\varphi \qquad in D_{-2}$$

where;

$$\tilde{\alpha}_n=4(n+\Delta) \quad , \quad \Delta=\frac{1}{\pi}\arcsin\frac{\kappa}{\sqrt{1+\kappa^2}} \quad , \quad \Delta\in(\frac{-1}{2},\frac{1}{2})$$

Theorem 2. The system of functions

$$\left\{\cos 4n(\frac{\pi}{2}-\theta)\right\}_{n=0}^{\infty} \quad , \quad \left\{\cos 4(n+\Delta)(\frac{\pi}{2}-\theta)\right\}_{n=1}^{\infty}$$

is complete and a Riesz basis in the space $L_2(0,\frac{\pi}{2})$ for $\Delta\in(\frac{-1}{4},\frac{1}{2})$.

for $\Delta<\frac{-1}{4}$ the system is not complete but is minimal, for $\Delta>\frac{3}{4}$ is complete but is not minimal, and if $\Delta=\frac{-1}{4}$ is complete and minimal.

Proof. The proof of this theorem we use the convergence function

$$f(\theta)=\sum_{n=0}^{\infty}A_n\cos 4n(\frac{\pi}{2}-\theta)+\sum_{n=1}^{\infty}B_n\cos 4(n+\Delta)(\frac{\pi}{2}-\theta)$$

in $L_2(0,\frac{\pi}{2})$, Riesz basis the system $\{\sin(n+\Delta)(\pi-4\theta)\}_{n=0}^{\infty}$ for $\Delta\in(\frac{-1}{4},\frac{3}{4})$ and [3].

Thorem 3. The system eigenfunction

$$u_{nk}(r,\theta)=A_{nk}J_{4n}(\mu_{nk}r)\cos 4n(\frac{\pi}{2}-\theta)$$
$$\tilde{u}_{nk}(r,\theta)=\tilde{A}_{nk}J_{\tilde{\alpha}_n}(\tilde{\mu}_{nk}r)\cos\tilde{\alpha}_n(\frac{\pi}{2}-\theta)$$

is complete and basis in the space $L^2(D_+)$, therefore

$$\int_{D_+} f(r,\theta)u_{nk}(r,\theta)rd\theta dr=0,$$
$$\int_{D_+} f(r,\theta)\tilde{u}_{nk}(r,\theta)rd\theta dr=0$$

and $f\in L^2(D_+)$ then $f=0$ in D_+.

Proof. Using fobini theorem and Lebesgue's integral for any $n,k=1,2,...$ we have

$$0=\int_{D_+} f(r,\theta)u_{nk}(r,\theta)rd\theta dr$$
$$=\int_0^1\left(rJ_{4n}(\mu_{nk}r)\int_0^{\frac{\pi}{2}} f(r,\theta)\cos(4n)(\frac{\pi}{2}-\theta)d\theta\right)dr$$

again since $f\in L^2(D_+)$ so;

$$\int_0^1\int_0^{\frac{\pi}{2}} r\,|\,f(r,\theta)\,|^2\,d\theta dr<\infty$$

Insomuch system $\{\sqrt{r}J_{4n}(\mu_{nk}r)\}_{k=1}^{\infty}$ in $L^2(0,1)$ is orthogonal and complete, it is enough to prove;

$$\sqrt{r}\int_0^{\frac{\pi}{2}} f(r,\theta)\cos(4n)(\frac{\pi}{2}-\theta)d\theta\in L^2(0,1)$$

Using the Holder inequality

$$\left|\sqrt{r}\int_0^{\frac{\pi}{2}} f(r,\theta)\cos(4n)(\frac{\pi}{2}-\theta)d\theta\right|^2$$

$$< \left| \sqrt{r} \left\{ \int_0^{\frac{\pi}{2}} f^2(r,\theta) d\theta \right\}^{\frac{1}{2}} \left\{ \int_0^{\frac{\pi}{2}} \cos^2 4n(\frac{\pi}{2} - \theta) d\theta \right\}^{\frac{1}{2}} \right|^2$$

$$= |r| \left| \left\{ \int_0^{\frac{\pi}{2}} f^2(r,\theta) \right\}^{\frac{1}{2}} \right|^2 \left| \left\{ \int_0^{\frac{\pi}{2}} \cos^2 4nd\theta \right\}^{\frac{1}{2}} \right|^2$$

$$< r \int_0^{\frac{\pi}{2}} | f^2(r,\theta) | d\theta \int_0^{\frac{\pi}{2}} \frac{1+\cos 8n\theta}{2} d\theta$$

$$< \frac{1}{2} r \int_0^{\frac{\pi}{2}} | f^2(r,\theta) | d\theta \int_0^{\frac{\pi}{2}} d\theta$$

$$= \frac{\pi}{4} r \int_0^{\frac{\pi}{2}} | f^2(r,\theta) | d\theta = \frac{\pi}{4} r \int_0^{\frac{\pi}{2}} | f(r,\theta) |^2 \, d\theta$$

with the integration interval $(0,1)$,

$$\int_0^1 \left| \sqrt{r} \int_0^{\frac{\pi}{2}} f(r,\theta) \cos(4n)(\frac{\pi}{2} - \theta) d\theta \right|^2 dr < \frac{\pi}{4} \int_0^1 r \int_0^{\frac{\pi}{2}} | f(r,\theta) |^2 \, d\theta dr$$

Thus

$$\int_0^1 \left| \sqrt{r} \int_0^{\frac{\pi}{2}} f(r,\theta) \cos(4n)(\frac{\pi}{2} - \theta) d\theta \right|^2 dr \le \frac{\pi}{4} \int_0^1 \int_0^{\frac{\pi}{2}} r \, | f(r,\theta) |^2 \, dr d\theta < \infty$$

This inequality is equivalent to

$$\left\{ \int_0^1 \sqrt{r} \left| \int_0^{\frac{\pi}{2}} f(r,\theta) \cos(4n)(\frac{\pi}{2} - \theta) d\theta \right|^2 \right\}^{\frac{1}{2}} < \infty$$

Also system $\{\sqrt{r} J_{4n}(\mu_{nk} r)\}$ is orthognal and complete for $k = 1,2,...$ in $L^2(0,1)$ of relation

$$\int_0^1 \left(\sqrt{r} J_{4n}(\mu_{nk} r) \sqrt{r} \int_0^{\frac{\pi}{2}} f(r,\theta) \cos 4n(\frac{\pi}{2} - \theta) d\theta \right) dr = 0$$

imply that

$$\sqrt{r}\int_0^{\frac{\pi}{2}} f(r,\theta)\cos 4n(\frac{\pi}{2}-\theta)d\theta = 0$$

According to theorem 2, we conclude that $f(r,\theta)=0$ in $L^2(0,1)$. Similarly, if we consider the above calculations for sequence $\{\cos[4(n+\Delta)](\frac{\pi}{2}-\theta)\}$ for $n=1,2,...$ we have;

$$\sqrt{r}\int_0^{\frac{\pi}{2}} f(r,\theta)\cos[4(n+\Delta)](\frac{\pi}{2}-\theta)d\theta = 0$$

Because completeness [4(n+)](2-)_n=0^, $f(r,\theta)=0$ in $L^2(0,1)$ .The proof of the theorem is complete.

Theorem 3. The system of eigenfunctions u_{nk} and $\tilde{u}_{nk}$ of the problem (1)-(6) is a Riesz basis in the space $L_2(D_+)$ where,

$$A_{nk}^2 = (\int_0^1 J_{4n}^2(\mu_{nk}r)rdr)^{-1} \qquad \tilde{A}_{nk} = (\int_0^1 J_{4(n+\Delta)}^2(\tilde{\mu}_{nk}r)rdr)^{-1}$$

Proof. Theorem 3 results from Theorem 2 and the completeness and orthognality of the system $\{A_{nk}\sqrt{r}J_{4n}(\mu_{nk}r)\}_{k=1}^{\infty}$ for $n0$ and $\{\tilde{A}_{nk}\sqrt{r}J_{4(n+\Delta)}(\tilde{\mu}_{nk}r)\}_{k=1}^{\infty}$ for $n1$ in $L_2(0,1)$.

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