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Finite Semigroups whose Semigroup Algebra over a Field Has a Trivial Right Annihilator[1]

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Abstract

An element a of a semigroup algebra $\mathbb{F}[S]$ over a field $\mathbb{F}$ is called a right annihilating element of $\mathbb{F}[S]$ if $xa = 0$ for every $x \in \mathbb{F}[S]$, where 0 denotes the zero of $\mathbb{F}[S]$. The set of all right annihilating elements of $\mathbb{F}[S]$ is called the right annihilator of $\mathbb{F}[S]$. In this paper we show that, for an arbitrary field $\mathbb{F}$, if a finite semigroup S is a direct product or semilattice or right zero semigroup of semigroups S_i such that every semigroup algebra $\mathbb{F}[S_i]$ has a trivial right annihilator, then the right annihilator of the semigroup algebra $\mathbb{F}[S]$ is also trivial.

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1 Introduction

Let S be a finite semigroup and $\mathbb{F}$ a field. By the *semigroup ring* $\mathbb{F}[S]$ of S over $\mathbb{F}$ we mean the set of all formal sums of the form $\Sigma_{s\in S}\alpha_s s,\ (\alpha_s \in \mathbb{F})$ with the following equality, addition and multiplication:

$$\Sigma_{s\in S}\alpha_s s = \Sigma_{s\in S}\beta_s s \ \leftrightarrow\ (\forall s \in S)\ \alpha_s = \beta_s;$$

$$(\Sigma_{s\in S}\alpha_s s) + (\Sigma_{s\in S}\beta_s s) = \Sigma_{s\in S}(\alpha_s + \beta_s)s;$$

$$(\Sigma_{s\in S}\alpha_s s)(\Sigma_{s\in S}\beta_s s) = \Sigma_{s\in S}\gamma_s s,$$

where $\gamma_s = 0$ if $s \notin S^2$ and $\gamma_s = \Sigma_{\{x,y\in S, xy=s\}}\alpha_x\beta_y$ if $s \in S^2$.

$\mathbb{F}[S]$ is also a vector space over $\mathbb{F}$ with the following multiplication by a scalar: for $a \in \mathbb{F}[S]$, $\alpha \in \mathbb{F}$, $\alpha(\Sigma_{s\in S}\alpha_s s) = \Sigma_{s\in S}(\alpha\alpha_s)s$. This vector space has a basis S' which is a subsemigroup of the multiplicative semigroup of $\mathbb{F}[S]$, and S' is isomorphic to S. Thus $\mathbb{F}[S]$ is a *semigroup algebra* of S over $\mathbb{F}$.

An element a of a semigroup algebra $\mathbb{F}[S]$ is called a *right annihilating element* of $\mathbb{F}[S]$ if $xa = 0$ for every $x \in \mathbb{F}[S]$, where 0 is the zero of $\mathbb{F}[S]$. The set of all right annihilating elements of $\mathbb{F}[S]$ (which is an ideal of $\mathbb{F}[S]$) is called the *right annihilator* of $\mathbb{F}[S]$. The right annihilator of a semigroup algebra $\mathbb{F}[S]$ is said to be trivial if it contains only the zero element of $\mathbb{F}[S]$.

In this paper we consider finite semigroups whose semigroup algebra $\mathbb{F}[S]$ over a field $\mathbb{F}$ has a trivial right annihilator. The main results of the paper are Theorem 3.1, 4.1, and 5.1, which show that the property of having trivial right annihilator is preserved by some important general constructions.

For the notions not defined here, we refer to [1], [4], [5], [7] and [8].

2 The general case

First of all we provide an example which shows that the right annihilator of $\mathbb{F}[S]$ may depends on the field $\mathbb{F}$.

Example 1 Let S be the multiplicative semigroup of all singular 2×2 matrices over the two-element field $\mathbb{F}_2$. It is clear that the elements of S are the $\mathbf{0}$ matrix and the dyads $\mathbf{v}^T\mathbf{w}$, where $\mathbf{v}, \mathbf{w}$ are nonzero row vectors of length 2 over $\mathbb{F}_2$. It is easy to show that the equations of type $xa = b$ or $ax = b$

($a, b \in S$ are fixed and x is an unknown element of S) have an even number of solutions. This implies that $\Sigma_{s\in S} s$ is in the right annihilator of $\mathbb{F}_2[S]$. Thus the right annihilator of $\mathbb{F}_2[S]$ is not trivial.

Let $\mathbb{F}$ be a field of characteristic $\neq 2$. We show that the right annihilator of $\mathbb{F}[S]$ is trivial. Suppose for contradiction that $\Sigma_{s\in S}\alpha_s s$ is a right annihilating element of the semigroup ring $\mathbb{F}[S]$ for some constants $\alpha_s \in \mathbb{F}$, not all 0. As the elements of S are dyads, $\Sigma_{s\in S}\alpha_s s$ can be written in the following form. $\mathbf{A} = \alpha_0 \mathbf{0} + \Sigma_{\mathbf{v},\mathbf{w}} \alpha_{\mathbf{v},\mathbf{w}} (\mathbf{v}^T \mathbf{w})$. We claim that, for every fixed $\mathbf{w} \neq qbf0$, we have $\alpha_{\mathbf{v}^1,\mathbf{w}} + \alpha_{\mathbf{v}^2,\mathbf{w}} = 0,\ \alpha_{\mathbf{v}^1,\mathbf{w}} + \alpha_{\mathbf{v}^3,\mathbf{w}} = 0,\ \alpha_{\mathbf{v}^2,\mathbf{w}} + \alpha_{\mathbf{v}^3,\mathbf{w}} = 0$, where $\mathbf{v}^1, \mathbf{v}^2, \mathbf{v}^3$ the 3 nonzero vectors from $\mathbb{F}_2^2$. (The validity of these equations follows if we compute $\mathbf{w}^T \mathbf{v}^i \mathbf{A}$, and consider the coefficient of $\mathbf{w}^T \mathbf{w}$ in the result.) The determinant of this linear system is ± 2, hence we have $\alpha_{\mathbf{v}^1,\mathbf{w}} = \alpha_{\mathbf{v}^2,\mathbf{w}} = \alpha_{\mathbf{v}^3,\mathbf{w}} = 0$ over every field of characteristic $\neq 2$.

The above construction can be generalized. Let $p > 2$ be a prime and S be the semigroup of all 2×2 matrices over $\mathbb{F}_p$ whose rank is at most 1. We have $|S| = p^3 + p^2 - p$. Moreover $\mathbb{F}_p[S]$ has a nontrivial right annihilator (that is S is not $\mathcal{R}_{\mathbb{F}_p}$-independent), and $\mathbb{F}[S]$ has a trivial right annihilator (that is S is $\mathcal{R}_{\mathbb{F}}$-independent) if the characteristic of $\mathbb{F}$ is not p. To show these two last assertions, it is sufficient to show that the matrix $\mathbf{M}$ defined below is singular over $\mathbb{F}_p$ and non-singular over every field of characteristic $\neq p$. Set $t = p^2 - 1$. Let $\mathbf{M} = \{m_{\mathbf{v},\mathbf{w}}\}$ be a $t \times t$ matrix whose rows and columns are indexed by nonzero vectors $\mathbf{v}, \mathbf{w} \in \mathbb{F}_p^2$ (here bold letters $\mathbf{v}, \mathbf{w}$, etc. stand for non-zero row vectors of length 2 over the finite field $\mathbb{F}_p$). We set $m_{\mathbf{v},\mathbf{w}} = 1$ if the dot product $\mathbf{v} \cdot \mathbf{w} = 1$, and $m_{\mathbf{v},\mathbf{w}} = 0$ otherwise. $\mathbf{M}$ is a square $0, 1$-matrix.

1. We claim that $\mathbf{M}$ is singular over $\mathbb{F}_p$. Indeed, the sum of the rows of $\mathbf{M}$ is the vector $(p, p, \ldots, p)$. This follows because for every nonzero $\mathbf{w} \in \mathbb{F}_p^2$ we have p vectors $\mathbf{v}$ with $\mathbf{v} \cdot \mathbf{w} = 1$.

2. On the other hand, $\mathbf{M}$ is nonsingular over a field $\mathbb{F}$ which has characteristic different from p. This follows if we show that the t-vector $V_{\mathbf{w}}$ which has p at the coordinate $\mathbf{w}$ and 0 elsewhere is in the row space of $\mathbf{M}$ over $\mathbb{Z}$. Let $U_{\mathbf{w}}$ be the sum of all rows of which have index $\mathbf{u}$ such that $\mathbf{u} \cdot \mathbf{w} = 0$. It is immediate that the $\mathbf{w}'$ component of $U_{\mathbf{w}}$ is 0 if $\mathbf{w}'$ is a multiple of $\mathbf{w}$, and is 1 otherwise. Set now $W_{\mathbf{w}}$ to be the sum of the rows of $\mathbf{M}$ with index $\mathbf{u}$ such that $\mathbf{u} \cdot \mathbf{w} = 1$. One readily sees that $W_{\mathbf{w}} = V_{\mathbf{w}} + U_{\mathbf{w}}$, hence $V_{\mathbf{w}} = W_{\mathbf{w}} - U_{\mathbf{w}}$.□

Let S be a finite semigroup and $\mathbb{F}$ an arbitrary field. By an S*-matrix* over $\mathbb{F}$ we mean a mapping of the Descartes product $S \times S$ into $\mathbb{F}$. The set $\mathbb{F}_{S\times S}$ of all S-matrices over $\mathbb{F}$ is an algebra over $\mathbb{F}$ under the usual addition and multiplication of matrices and the product of matrices by scalars.

For an element $s \in S$, let $\mathbf{R}^{(s)}$ denote the S-matrix defined by

$$\mathbf{R}^{(s)}((x, y)) = \begin{cases} 1, & \text{if } xs = y \\ 0 & \text{otherwise,} \end{cases}$$

where 1 and 0 denote the identity element and the zero element of $\mathbb{F}$, respectively. This matrix will be called the right matrix of the element s of S. The mapping $\mathcal{R}_{\mathbb{F}}: \ s \mapsto \mathbf{R}^{(s)}$ is a matrix representation of S over $\mathbb{F}$ (see, for example, Exercise 4 in §3.5 of [1]). $\mathcal{R}_{\mathbb{F}}$ describes in terms of matrices $\mathbf{R}^{(s)}$ the maps $x \mapsto xs$. Thus, it is essentially the right regular representation. This representation of a finite semigroup S is faithful if and only if S is left reductive (that is, for every $a, b \in S$, the assumption that $sa = sb$ holds for all $s \in S$ implies $a = b$).

Theorem 2.1 *The semigroup algebra $\mathbb{F}[S]$ of a finite semigroup S over a field $\mathbb{F}$ has a trivial right annihilator if and only if the system $\{\mathbf{R}^{(s)}, s \in S\}$ of right matrices of elements s of S is linearly independent over $\mathbb{F}$.*

Proof. Let S be a finite semigroup and $\mathbb{F}$ a field. It is immediate that $\Sigma_{s\in S}\beta_s s \quad (\beta_s \in \mathbb{F})$ is a right annihilating element of the semigroup algebra $\mathbb{F}[S]$ if and only if $\Sigma_{s\in S}\beta_s\mathbf{R}^{(s)} = \mathbf{0}$. Thus the right annihilator of $\mathbb{F}[S]$ contains only the zero if and only if the system $\{\mathbf{R}^{(s)}, s \in S\}$ of right matrices of the elements s of S is linearly independent over $\mathbb{F}$. □

Definition 2.2 *Let $\mathbb{F}$ be a field. We shall say that elements $s_1, \ldots, s_k$ of a finite semigroup S are $\mathcal{R}_{\mathbb{F}}$-independent (in S) if the system $\{\mathbf{R}^{(s_1)}, \ldots, \mathbf{R}^{(s_k)}\}$ of right matrices of elements $s_1, \ldots, s_k$ of S over $\mathbb{F}$ is linearly independent over $\mathbb{F}$. If all the elements of a finite semigroup S are $\mathcal{R}_{\mathbb{F}}$-independent (in S) then we shall say that S is an $\mathcal{R}_{\mathbb{F}}$-independent semigroup.*

With this terminology, the right annihilator of $\mathbb{F}[S]$ is trivial if and only if S is an $\mathcal{R}_{\mathbb{F}}$-independent semigroup (see Theorem 2.1).

Lemma 2.3 *Every element of a finite semigroup is $\mathcal{R}_{\mathbb{F}}$-independent for an arbitrary field $\mathbb{F}$.*

Proof. It is obvious, because a right matrix is non-zero. □

Lemma 2.4 *Two elements of a finite semigroup are $\mathcal{R}_{\mathbb{F}}$-independent for a field $\mathbb{F}$ if and only if their right matrices over $\mathbb{F}$ are different.*

Proof. Let a and b be two elements of a finite semigroup S. Assume $\mathbf{R}^{(b)} = \alpha\mathbf{R}^{(a)}$ for some element α of an arbitrary field $\mathbb{F}$. It is easy to see that $\alpha = 1$, the identity element of $\mathbb{F}$, and so $\mathbf{R}^{(b)} = \mathbf{R}^{(a)}$. Thus the assertion of the lemma is obvious. □

Lemma 2.5 *Three elements of a finite semigroup are $\mathcal{R}_{\mathbb{F}}$-independent for a field $\mathbb{F}$ if and only if their right matrices over $\mathbb{F}$ are pairwise different.*

Proof. Let S be a finite semigroup and a, b, c be arbitrary elements of S. If they are $\mathcal{R}_{\mathbb{F}}$-independent, then their right matrices over $\mathbb{F}$ are different. Conversely, assume that the right matrices $\mathbf{R}^{(a)}, \mathbf{R}^{(b)}, \mathbf{R}^{(c)}$ are pairwise different. Assume for contradiction, that the elements a, b, c are not $\mathcal{R}_{\mathbb{F}}$-independent. Then one of their right matrices, for example $\mathbf{R}^{(a)}$, can be expressed as a linear combination of the others, that is, $\mathbf{R}^{(a)} = \beta\mathbf{R}^{(b)} + \gamma\mathbf{R}^{(c)}$ for some $\beta, \gamma \in \mathbb{F}$. Then $a - \beta b - \gamma c$ is a right annihilating element of $\mathbb{F}[S]$. By Lemma 2.3 and Lemma 2.4, $\beta, \gamma \neq 0$. As the matrices $\mathbf{R}^{(a)}$ and $\mathbf{R}^{(b)}$ are different, there is an element $x \in S$ such that $xa \neq xb$. As $x(a-\beta b-\gamma c) = 0$, we get $xa = \beta xb+\gamma xc$. This is impossible. Indeed, if xc is different from xa and xb, then xa, xb and xc are linearly independent in $\mathbb{F}[S]$ by definition. If $xc = xa$ then we obtain that xb is a multiple of xa, a contradiction again. The case $xc = xb$ is similar. □

The next example shows that four elements of a finite semigroup are not necessarily $\mathcal{R}_{\mathbb{F}}$-independent for a field $\mathbb{F}$ even if the right matrices are pairwise different.

Example 2. Let S be the multiplicative semigroup of all singular 2×2 matrices over the two-element field $\mathbb{F}_2$ (see Example 1). Let $\mathbf{v}^1 = [1, 0]$, $\mathbf{v}^2 = [0, 1]$, $\mathbf{v}^3 = [1, 1]$ be the nonzero vectors from $\mathbb{F}_2^2$, and let $\mathbf{R} = \mathbf{0} + (\mathbf{v}^{1T}\mathbf{v}^1) + (\mathbf{v}^{2T}\mathbf{v}^1) + (\mathbf{v}^{3T}\mathbf{v}^1)$. Then, in $\mathbb{F}_2[S]$, $\mathbf{0R} = \mathbf{0} + \mathbf{0} + \mathbf{0} + \mathbf{0} = \mathbf{0}$. and $(\mathbf{v}^{iT}\mathbf{v}^j)\mathbf{R} = 2(\mathbf{v}^{iT}\mathbf{v}^1) + 2\mathbf{0} = \mathbf{0}$. Thus $\mathbf{R}$ is a right annihilating element of $\mathbb{F}_2[S]$, and so $\mathbf{R}^{(\mathbf{0})} + \mathbf{R}^{(\mathbf{v}^{1T}\mathbf{v}^1)} + \mathbf{R}^{(\mathbf{v}^{2T}\mathbf{v}^1)} + \mathbf{R}^{(\mathbf{v}^{3T}\mathbf{v}^1)} = \mathbf{0}$. Thus the right matrices $\mathbf{R}^{(\mathbf{0})}, \mathbf{R}^{(\mathbf{v}^{1T}\mathbf{v}^1)}, \mathbf{R}^{(\mathbf{v}^{2T}\mathbf{v}^1)}, \mathbf{R}^{(\mathbf{v}^{3T}\mathbf{v}^1)}$ are linearly dependent over $\mathbb{F}_2$. They are linearly independent over every field of characteristic $\neq 2$; hence also they are pairwise different. We note that $A = \{\mathbf{0}, \mathbf{v}^{1T}\mathbf{v}^1, \mathbf{v}^{2T}\mathbf{v}^1, \mathbf{v}^{3T}\mathbf{v}^1\}$ is a subsemigroup of S which is not a left reductive semigroup. Thus, for every field $\mathbb{F}$, A is not $\mathcal{R}_{\mathbb{F}}$-independent as a semigroup, but the elements of A as the elements of S are $\mathcal{R}_{\mathbb{F}}$-independent (in S) for every field $\mathbb{F}$ of characteristic $\neq 2$. □

Lemma 2.6 *Let a, b, c, d be arbitrary elements of a finite semigroup whose right matrices are pairwise different over a field $\mathbb{F}$ and these matrices are linearly dependent over $\mathbb{F}$. Then there are elements $\alpha, \beta, \gamma, \delta$ in $\mathbb{F}$ such that $\alpha\mathbf{R}^{(a)} + \beta\mathbf{R}^{(b)} + \gamma\mathbf{R}^{(c)} + \delta\mathbf{R}^{(d)} = \mathbf{0}$ and two of the coefficients are equal to 1 and the others are equal to -1, where 1 is the identity element of $\mathbb{F}$.*

Proof. Let a, b, c, d be elements of a finite semigroup S whose right matrices are pairwise different over a field $\mathbb{F}$ and they are not independent over $\mathbb{F}$. Then one of them can be expressed (over $\mathbb{F}$) as a linear combination of the others. Assume, for example, $\mathbf{R}^{(a)} = \beta\mathbf{R}^{(b)}+\gamma\mathbf{R}^{(c)}+\delta\mathbf{R}^{(d)}$ for some $\beta, \gamma, \delta \in F$. By Lemma 2.3, Lemma 2.4 and Lemma 2.5, β, γ, δ are nonzero elements of $\mathbb{F}$. As $\mathbf{R}^{(a)} \neq \mathbf{R}^{(b)}$, there is an element $h \in S$ such that $ha \neq hb$. Then we have, in $\mathbb{F}[S]$, $ha-\beta hb = \gamma hc+\delta hd$. This is possible only if $\{ha, \beta hb\} = \{\gamma hc, \delta hd\}$.

Without loss of generality we may assume $ha = \gamma hc$. Hence $\gamma = 1$, moreover $hb = hd$ and $-\beta = \delta$. Also, as $\mathbf{R}^{(b)} \neq \mathbf{R}^{(d)}$, we have an element $h' \in S$ such that $h'b \neq h'd$. From $\mathbf{R}^{(a)} = \beta\mathbf{R}^{(b)} + \gamma\mathbf{R}^{(c)} + \delta\mathbf{R}^{(d)}$, we obtain $h'a - h'c = \beta h'b + \delta h'd$, implying that $\{h'b, h'd\} = \{h'a, h'c\}$ and $\{\beta, \delta\} = \{1, -1\}$. The statement now follows with $\alpha = 1$. $\square$

Corollary 2.7 *Let s_1, s_2, s_3, s_4 be elements of a finite semigroup S such that their right matrices over a field $\mathbb{F}$ are pairwise different but not linearly independent over $\mathbb{F}$. For an element $t \in S$, let C_t denote the sequence of the elements ts_i $(i = 1, 2, 3, 4)$ of S. If an element of S is in C_t, then it occurs in C_t either two times or four times.*

Proof. By Lemma 2.6, we may assume that $s_1 + s_2 - s_3 - s_4$ is a right annihilating element of the semigroup ring $\mathbb{F}[S]$. Thus, for every $t \in S$, $ts_1 + ts_2 = ts_3 + ts_4$. Hence $\{ts_1, ts_2\} = \{ts_3, ts_4\}$. This gives the claim. $\square$

Theorem 2.8 *If S is a left reductive semigroup containing at most four elements then the right annihilator of $\mathbb{F}[S]$ is trivial for any field $\mathbb{F}$.*

Proof. Let S be a left reductive semigroup. First consider the case when S contains at most three elements. As S is left reductive, the right matrices of the elements of S over an arbitrary field $\mathbb{F}$ are pairwise different. Then, by Lemma 2.3, Lemma 2.4 and Lemma 2.5, the elements of S are $\mathcal{R}_{\mathbb{F}}$-independent, that is, the right annihilator of $\mathbb{F}[S]$ is trivial by Theorem 2.1.

Consider the case when S contains four elements. In [2], we can find the Cayley-tables of all non-isomorphic and non-antiisomorphic 4-element semigroups (126 Cayley-tables). Thus the Cayley table of a 4-element semigroup is one of the mentioned Cayley-tables or the dual of one of them. Among the 126 Cayley-tables there are 38 which correspond to left reductive semigroups. Among the reflected Cayley-tables there are 69 which correspond to a left reductive semigroup. It is a matter of checking to see that every Cayley-table corresponding to a left reductive semigroup S contains at least one row in which some element of S occurs once. From this it follows, by Corollary 2.7, that every left reductive 4-element semigroup is $\mathcal{R}_{\mathbb{F}}$-independent. Consequently, our assertion follows from Theorem 2.1. $\square$

Example 3. Let $S = \{a, b, c, d, e\}$ be a semigroup defined by the following Cayley table (see [2]; page 167, the last Cayley table in row 7):

As the columns of the Cayley-table are pairwise distinct, S is left reductive. It is a matter of checking to see that, for every field $\mathbb{F}$, $d + a - b - c$ is in the right annihilator of $\mathbb{F}[S]$. Thus the right annihilator of $\mathbb{F}[S]$ is not trivial. $\square$

	a	b	c	d	e
a	b	b	a	a	b
b	b	b	b	b	b
c	b	b	c	c	b
d	b	b	d	d	b
e	a	b	a	b	e

3 Finite semilattice of finite semigroups

A semigroup S is called a *band* if every element of S is idempotent. A commutative band is called a *semilattice*. We say that a semigroup S is a semilattice I of subsemigroups S_i $(i \in I)$ of S if there is a congruence α of S such that the α classes are the subsemigroups S_i, and the factor semigroup S/α is isomorphic to the semilattice I.

Theorem 3.1 *Let $\mathbb{F}$ be an arbitrary field. If a finite semigroup S is a semilattice I of semigroups S_i $(i \in I)$ such that every semigroup algebra $\mathbb{F}[S_i]$ $(i \in I)$ has a trivial right annihilator then the right annihilator of the semigroup algebra $\mathbb{F}[S]$ is trivial.*

Proof. Let $\mathbb{F}$ be an arbitrary field and S a finite semigroup which is a semilattice I of semigroups S_i $(i \in I)$. Assume that every semigroup algebra $\mathbb{F}[S_i]$ $(i \in I)$ has a trivial annihilator. On the semilattice I, the relation $j \leq k$ if and only if $jk = j$ is a partial order. As I is finite, every non-empty subset A of I has an element which is maximal in A.

Let r be an arbitrary right annihilating element of $\mathbb{F}[S]$. As the semigroup algebra $\mathbb{F}[S]$ is the direct sum of the subalgebras $\mathbb{F}[S_i]$ $(i \in I)$, there are elements $r_i \in \mathbb{F}[S_i]$ $(i \in I)$ such that $r = \sum_{i \in I} r_i$. Let $I' = \{i \in I : \ r_i \neq 0\}$. Assume $I' \neq \emptyset$. Let i_0 be an element of I' which is maximal in I' under the above mentioned partial order $\leq$. Let $x \in S_{i_0}$ be an arbitrary element. Then $0 = xr = x(\sum_{i \in I'} r_i) = \sum_{i \in I'} xr_i = xr_{i_0} + \sum_{i \in (I' - \{i_0\})} xr_i$. For every $i \in (I' - \{i_0\})$, $xr_i \notin \mathbb{F}[S_{i_0}]$. Indeed, if xr_i was in $\mathbb{F}[S_{i_0}]$ for some $i \in (I' - \{i_0\})$, then we would have $i_0 i = i_0$ in I, that is, $i_0 \leq i$ which would imply $i = i_0$, because i_0 is a maximal element in I'. It would be a contradiction. Thus the above equation implies $xr_{i_0} = 0$. Hence r_{i_0} is a right annihilating element of $\mathbb{F}[S_{i_0}]$. Thus $r_{i_0} = 0$. We get $i_0 \notin I'$ which is a contradiction. Hence $I' = \emptyset$ and so $r = 0$. Consequently, the right annihilator of $\mathbb{F}[S]$ contains only the zero element. $\square$

Corollary 3.2 *For an arbitrary semilattice S and an arbitrary field $\mathbb{F}$, the semigroup algebra $\mathbb{F}[S]$ has a trivial right annihilator.*

Proof. As the semigroup algebra of a one-element semigroup over every field has a trivial right annihilator, our assertion follows from Theorem 3.1. $\square$

Remark The assertion of Corollary 3.2 also follows from Theorem 5.27 of [3] and Proposition 2.1 of this paper. By Theorem 5.27 of [3], if S is a finite band then $\mathbb{F}[S]$ is semisimple if and only if S is a semilattice. Thus, for every semilattice S, $\mathbb{F}[S]$ is semisimple (that is, the radical of $\mathbb{F}[S]$ is trivial) and so the right annihilator of $\mathbb{F}[S]$ is trivial. $\square$

Corollary 3.3 *Let S be a finite semigroup and $\mathbb{F}$ a field. If the right annihilator of $\mathbb{F}[S]$ is trivial then the right annihilator of $\mathbb{F}[S^0]$ is trivial, where S^0 denotes the semigroup obtained from S by adjoining a zero if necessary ([4]).*

Proof. If $S = S^0$ then the assertion is obvious. If $S \neq S^0$ then S^0 is a semilattice of S and a one-element semigroup. Thus the assertion follows from Theorem 3.1. $\square$

An element x of a semigroup S is said to be a *left cancellable element* of S if $xa = xb$ implies $a = b$ for every $a, b \in S$.

Lemma 3.4 *If a finite semigroup S has a left cancellable element then $\mathbb{F}[S]$ has a trivial right annihilator for an arbitrary field $\mathbb{F}$.*

Proof. Let $\mathbb{F}$ be an arbitrary field and S a finite semigroup. Assume that $u = \Sigma_{s\in S}\alpha_s s$ is a right annihilating element of $\mathbb{F}[S]$ for some $\alpha_s \in \mathbb{F}$. Let x be a left cancellable element of S. Then $0 = xu = \Sigma_{s\in S}\alpha_s(xs)$. As the multiplication (from the left) by x is a permutation of S, $\Sigma_{s\in S}\alpha_s(xs) = \Sigma_{s\in S}\beta_s s$, where the sequence $\{\beta_s\}$ is a permutation of the sequence $\{\alpha_s\}$. Thus $\alpha_s = 0$ for every $s \in S$ and so $u = 0$. $\square$

Corollary 3.5 *If a finite semigroup S has a left identity element then $\mathbb{F}[S]$ has a trivial right annihilator for any field $\mathbb{F}$.*

Proof. As a left identity element of a semigroup S is a left cancellable element of S, the assertion follows from Lemma 3.4. $\square$

Corollary 3.6 *Let $\mathbb{F}$ be a field and S a finite semigroup which is a semilattice of monoids S_i ($i = 1, 2, \ldots, n$). Then the the right annihilator of $\mathbb{F}(S)$ is trivial.*

Proof. By Corollary 3.5, the right annihilator of a finite monoid is trivial. Then our assertion follows from Theorem 3.1. $\square$

A regular semigroup S whose idempotent elements are in the centre of S is called a *Clifford semigroup*.

Corollary 3.7 *For every field $\mathbb{F}$ and every finite Clifford semigroup S, the right annihilator of $\mathbb{F}[S]$ is trivial.*

Proof. Let S be a finite Clifford semigroup. By Theorem 2.1 of [4], S is a finite semilattice of finite groups. By Corollary 3.6, the right annihilator of $\mathbb{F}[S]$ is trivial for an arbitrary field $\mathbb{F}$. □

A semigroup S is said to be *left [right] separative* if, for every $a, b \in S$, $ab = a^2$ and $ba = b^2$ [$ab = b^2$ and $ba = a^2$] imply $a = b$. A semigroup is said to be *separative* if it is both left and right separative.

A semigroup S is said to be *medial* if it satisfies the identity $axyb = ayxb$.

Corollary 3.8 *If S is a finite left separative medial semigroup then the right annihilator of $\mathbb{F}[S]$ is trivial for an arbitray field $\mathbb{F}$.*

Proof. Let S be a finite left separative medial semigroup. Then S is a semilattice of left cancellative (medial) semigroups by Theorem 9.12 of [5]. Then the right annihilator of $\mathbb{F}[S]$ is trivial for an arbitray field $\mathbb{F}$ by Lemma 3.4 and Theorem 3.1. □

Corollary 3.9 *If S is a finite commutative separative semigroup then the right annihilator of $\mathbb{F}[S]$ is trivial for an arbitrary field $\mathbb{F}$.*

Proof. It is obvious by Corollary 3.8. □

4 Finite right zero semigroup of finite semigroups

A semigroup S is called a right (left) zero semigroup if $ab = b$ ($ab = a$) for every $a, b \in S$. We say that a semigroup S is a right zero semigroup Y of subsemigroups S_i ($i \in Y$) of S if there is a congruence α of S such that the α classes are the subsemigroups S_i, and the factor semigroup S/α is isomorphic to the right zero semigroup Y.

Theorem 4.1 *Let $\mathbb{F}$ be a field and S a finite semigroup which is a right zero semigroup I of semigroups S_i ($i \in I$). If the right annihilator of $\mathbb{F}[S_i]$ is trivial for every $i \in I$, then the right annihilator of $\mathbb{F}[S]$ is trivial.*

Proof. Let $\mathbb{F}$ be an arbitrary field and S a finite semigroup which is a right zero semigroup I of semigroups S_i ($i \in I$). Assume that the right annihilator of $\mathbb{F}[S_i]$ is trivial for every $i \in I$. Let r be an arbitrary right annihilating element of $\mathbb{F}[S]$. As S is a disjoint union of the subsemigroups S_i ($i \in I$), there are elements $r_i \in \mathbb{F}[S_i]$ ($i \in I$) such that $r = \sum_{i\in I} r_i$. Let $s \in S$ be an arbitrary element. Then $0 = sr = s(\sum_{i\in I} r_i) = \sum_{i\in I} sr_i$. As S is a right zero semigroup of subsemigroups S_i $(i \in I)$, we have $SS_i \subseteq S_i$ for every $i \in I$. We have $sr_i \in \mathbb{F}[S_i]$ for every $i \in I$. Thus the above equation implies $sr_i = 0$ for every $i \in I$. Hence, for every $i \in I$, r_i is a right annihilating element of $\mathbb{F}[S_i]$. By the assumption, $r_i = 0$ for every $i \in I$. Thus $r = 0$ and so the right annihilator of $\mathbb{F}[S]$ is trivial. □

Corollary 4.2 *For every finite right zero semigroup S and every field $\mathbb{F}$, the right annihilator of $\mathbb{F}[S]$ is trivial.*

Proof. As the right annihilator of $\mathbb{F}[S]$ is trivial for a one-element semigroup S and an arbitrary field $\mathbb{F}$, our assertion follows from Theorem 4.1. □

A band is called a *rectangular band* if it is a direct product of a left zero semigroup and a right zero semigroup.

Corollary 4.3 *Let S be a finite rectangular band. The right annihilator of $\mathbb{F}[S]$ is trivial for a field $\mathbb{F}$ if and only if S is a right zero semigroup.*

Proof. Let $\mathbb{F}$ be an arbitrary field. Assume that the right annihilator of $\mathbb{F}[S]$ is trivial for a rectangular band $S = L \times R$ (L is a left zero semigroup, R is a right zero semigroup). Then, by Theorem 2.1, the representation $\mathcal{R}_{\mathbb{F}}$ of S is faithful and so S is left reductive. Thus $|L| = 1$ and so S is a right zero semigroup.

The converse follows from Corollary 4.2. □

A band is called a *right regular band* if it satisfies the identity $xa = axa$. By Theorem II.1.2 of [8], every band is a semilattice of rectangular bands. It is easy to see that a right regular rectangular band is a right zero semigroup. Thus the right regular bands are semilattices of right zero semigroups.

Corollary 4.4 *If S is a finite right regular band then the right annihilator of $\mathbb{F}[S]$ is trivial for any field $\mathbb{F}$.*

Proof. Let $\mathbb{F}$ be an arbitrary field and S a finite right regular band. Then S is a semilattice of finite right zero semigroups, and our assertion follows from Corollary 4.2 and Theorem 3.1. □

5 Finite direct product of finite semigroups

Theorem 5.1 *Let $\mathbb{F}$ be a field and S a finite semigroup which is a direct product of semigroups S_i $(i \in I)$. If the right annihilator of $\mathbb{F}[S_i]$ is trivial for every $i \in I$ then the right annihilator of $\mathbb{F}[S]$ is trivial.*

Proof. Let $\mathbb{F}$ be a field and $A = \{1, \ldots, m\}$, $B = \{1, \ldots, n\}$ be arbitrary finite semigroups such that both of $\mathbb{F}[A]$ and $\mathbb{F}[B]$ have a trivial right annihilator. Then, by Theorem 2.1, A and B are $\mathcal{R}_{\mathbb{F}}$-independent semigroups, that is, both of the systems $\{\mathbf{A}^{(i)}\ i \in A\}$ and $\{\mathbf{B}^{(j)}\ j \in B\}$ are linearly independent over $\mathbb{F}$, where $\mathbf{A}^{(i)}$ and $\mathbf{B}^{(j)}$ denote the right matrices of the element $i \in A$ and $j \in B$ over $\mathbb{F}$, respectively. It is a matter of checking to see that the right matrix $\mathbf{C}^{((i,j))}$ of the element $(i, j) \in A \times B = \{(1, 1); (1, 2) \ldots ; (1, n); (2, 1); \ldots ; (2, n); \cdots ; (m, 1); \ldots ; (m, n)\}$ is the Kronecker-product $\mathbf{B}^{(j)} \otimes \mathbf{A}^{(i)}$, that is, $\mathbf{C}^{((i,j))}$ is a matrix of blocks $\mathbf{C}^{(i,j)}_{k,t}$ $(k, t \in \{1, \ldots, m\})$ such that $\mathbf{C}^{(i,j)}_{k,t} = a^{(i)}_{k,t}\mathbf{B}^{(j)}$.

Assume $\Sigma_{j=1}^{n}\Sigma_{i=1}^{m}\gamma_{j,i}\mathbf{C}^{((i,j))} = \mathbf{0}_{mn\times mn}$ for some $\gamma_{i,j} \in \mathbb{F}$. Then, for every $k, t \in \{1, \ldots m\}$, $\Sigma_{j=1}^{n}\Sigma_{i=1}^{m}\gamma_{j,i}\mathbf{C}^{(i,j)}_{k,t} = \mathbf{0}_{n\times n}$, that is, $\Sigma_{j=1}^{n}\Sigma_{i=1}^{m}\gamma_{j,i}a^{(i)}_{k,t}\mathbf{B}^{(j)} = \mathbf{0}_{n\times n}$. Then $\Sigma_{j=1}^{n}(\Sigma_{i=1}^{m}\gamma_{j,i}a^{(i)}_{k,t})\mathbf{B}^{(j)} = \mathbf{0}_{n\times n}$ from which we obtain that, for every $j = 1, \ldots, n$ (and every k, t), $\Sigma_{i=1}^{m}\gamma_{j,i}a^{(i)}_{k,t} = 0$, because the matrices $\mathbf{B}^{(1)}, \ldots, \mathbf{B}^{(n)}$ are linearly independent over $\mathbb{F}$. Then, for every $j = 1, \ldots, n$, $\Sigma_{i=1}^{m}\gamma_{j,i}\mathbf{A}^{(i)} = \mathbf{0}_{m\times m}$. As the matrices $\mathbf{A}^{(1)}, \ldots, \mathbf{A}^{(m)}$ are linearly independent over $\mathbb{F}$, we get $\gamma_{j,i} = 0$ for every $i = 1, \ldots, m$ and $j = 1, \ldots, n$. Hence $A \times B$ is $\mathcal{R}_{\mathbb{F}}$-independent. By Theorem 2.1, $\mathbb{F}[A \times B]$ has a trivial right annihilator. The assertion of the theorem follows from this result by induction. □

A direct product of a rectangular band and a group is called a *rectangular group*. A direct product of a right zero semigroup and a group is called a *right group*.

Corollary 5.2 *For a finite rectangular group S, the right annihilator of $\mathbb{F}[S]$ is trivial for a field $\mathbb{F}$ if and only if S is right group.*

Proof. Let S be a finite rectangular group. Then S is a direct product of a group G and a rectangular band $L \times R$ (L is a left zero semigroup, R is a righ zero semigroup).

If the right annihilator of $\mathbb{F}[S]$ is trivial then S is left reductive. It is easy to see that the left reductivity of S implies $|L| = 1$. Then S is a right group.

Conversely, let S be a right group, that is, a direct product of a right zero semigroup R and a group G. By Corollary 4.2, the right annihilator of $\mathbb{F}[L]$ is trivial. By Corollary 3.6, the right annihilator of $\mathbb{F}[G]$ is trivial. Then, by Theorem 5.1, the right annihilator of $\mathbb{F}[S]$ is trivial. □

Corollary 5.3 *If a finite semigroup S is a semilattice of right groups then the right annihilator of $\mathbb{F}[S]$ is trivial for any field $\mathbb{F}$.*

Proof By Corollary 5.2 and Theorem 3.1, it is obvious. □

A semigroup is said to be *left commutative* if it satisfies the identity $abc = bac$. It is clear that every left commutative semigroup is medial.

Corollary 5.4 *If S is a finite left commutative simple semigroup then the right annihilator of $\mathbb{F}[S]$ is trivial for any field $\mathbb{F}$.*

Proof. Let S be a finite left commutative simple semigroup. By Theorem 10.5 of [5], S is a right abelian group, that is, a direct product of a right zero semigroup and a commutative group. Then the right annihilator of $\mathbb{F}[S]$ is trivial by Corollary 5.2 of this paper. $\square$

Corollary 5.5 *If S is a finite left commutative regular semigroup then the right annihilator of $\mathbb{F}[S]$ is trivial for every field $\mathbb{F}$.*

Proof. Let S be a left commutative regular semigroup. Then it is a semilattice of right abelian groups by Corollary 10.1 of [5]. Then our assertion follows from Corollary 5.3 of this paper. $\square$

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