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State Linearization of Multi-Input Nonlinear Differential Algebraic Control Systems

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Abstract

The problem of state linearization of multi-input nonlinear differential algebraic control systems via coordinate transformations is addressed by defining an algorithm allowing to compute explicitly the linearizing state coordinate for index one nonlinear differential algebraic control systems. The algorithm is performed using a maximum of $n-1$ steps (n the dimension of the system).

1. INTRODUCTION

The problem of transforming a nonlinear differential algebraic control system (NDACS)

$$\Sigma : \begin{cases} \dot{x} = f(x,z) + g_1(x,z)u_1 + \cdots + g_m(x,z)u_m \\ 0 = \sigma(x,z) \end{cases} \tag{1.1}$$

into the linear system

$$\Lambda : \begin{cases} \dot{\varpi} = A\varpi + B_1 u_1 + \cdots + B_m u_m \\ 0 = \sigma(\varpi,z) \end{cases} \tag{1.2}$$

by a change of coordinates transformation of the form

$$\varpi = \varphi(x,z), \quad (x,z) \in M \tag{1.3}$$

Where $M = \left\{ (x,z) \middle| (x,z) \subset \mathbb{R}^n \times \mathbb{R}^m, \sigma(x,z) = 0, \text{rank}\left(\frac{\partial \sigma(x,z)}{\partial z} \right) = m \right\}$, is called state linearization problem to the system (1.1). The linearization problem of nonlinear differential algebraic control system is an important one and has been studied sparsely. some investigations have been carried out by McClamroch et al. with constrained mechanical systems [1,2] and also by Kaprielian et al. with an AC/DC power system model [3,4]. Their approaches consist of using transformations to obtain a state realization (state space representation) of the nonlinear descriptor system and then apply differential geometry for linearization. For single-input nonlinear differential algebraic control systems Z. Jaindong and C. Zhaolin et al.[5] have defined $F(x,z) = \begin{pmatrix} I_n \\ -\left(\frac{\partial \sigma}{\partial z}\right)^{-1} \frac{\partial \sigma}{\partial x} \end{pmatrix}$ where (I_n is an $n \times n$ identity matrix) and deal with the index one NDAE locally as the following nonlinear control, $\begin{pmatrix} \dot{x} \\ \dot{z} \end{pmatrix} = F(x,z)f(x,z) + F(x,z)g(x,z)u$ to study the exact feedback linearization for this class of NDAS. On the other hand, C. Chen et al. [6] used the ideas of differential geometric control theory to define M derivative and M bracket in order to investigate the necessary and sufficient geometric conditions for exact feedback linearization of index one single-input nonlinear differential algebraic control systems. The problem of state linearization is solvable if and only if

$(S'1)$ dim span $\left\{ g(x,z), Mad_f\, g(x,z), \ldots, Mad_f^{n-1} g(x,z) \right\} = n$;

$(S'2)$ $\left[Mad_f^q\, g(x,z), Mad_f^r\, g(x,z) \right]_M = 0; \quad 0 \le q < r \le n.$

Although, the conditions $(S'1)$ and $(S'2)$ provide a way of testing whether a given system is state linearizable but they offer little on how to find the linearizing change of coordinates $\varphi(x,z)$ except by solving a systems of partial differential equations (PDEs) which is, in general, not straightforward. For the problem of feedback linearization of single-input nonlinear differential algebraic control systems, Ayad and Nada [7,8] provide a complete solution by defining an algorithm that allows to compute explicitly the linearizing state coordinates and feedback for index one nonlinear differential algebraic control systems. Each algorithm is performed using a maximum of $n - 1$ steps (n being the dimension of the system). The objective of this paper is to provide an algorithm giving linearizing coordinates for index one multi-input nonlinear differential algebraic control systems without solving the partial differential equations. The algorithm based on Frobenius Theorem.

2. Notations and Preliminaries

Consider the index one multi-input nonlinear differential algebraic control systems NDACS (1.1)

$$\Sigma : \begin{cases} \dot{x} = f(x,z) + g_1(x,z)u_1 + \cdots + g_m(x,z)u_m \\ 0 = \sigma(x,z) \end{cases}$$

where $x = (x_1,\ldots,x_n)^T \in \mathbb{R}^n, z = (z_1,\ldots,z_p)^T \in \mathbb{R}^P$ and $u = (u_1,\ldots,u_m)^T \in \mathbb{R}^m$. Also $f(x,z) : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^n$, $g(x,z) : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^n$ and $\sigma(x,z) : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}$ are smooth vector fields. and assume that its linear system

$$\Lambda : \begin{cases} \dot{\varpi} = A\varpi + Bu = A\varpi + B_1u_1 + \cdots + B_m u_m \\ 0 = \sigma(\varpi,z) \end{cases}$$

is controllable, that is, there exist positive integers $r_1 \geq 1,\ldots,r_m \geq 1$ with $r_1 + \cdots + r_m = n$ such that

$$\dim \operatorname{span}\left\{A^k B_i, 0 \leq k \leq r_i - 1, 1 \leq i \leq m\right\} = n.$$

Define the coordinates $\mathrm{x}_k = \left((\mathrm{x}_k^1)^T,\ldots,(\mathrm{x}_k^m)^T\right)^T$ on $\mathbb{R}^n = \mathbb{R}^{r_1} \times \cdots \times \mathbb{R}^{r_m}$, where for any $1 \leq i \leq r$ we set $\mathrm{x}_\mathrm{k}^i = (\mathrm{x}_{\mathrm{k}1}^i,\ldots,\mathrm{x}_{\mathrm{k}r}^i)^T$ and we put

$$\widehat{\mathrm{x}}_{\mathrm{k}i} = \left(\mathrm{x}_{\mathrm{k}1}^1,\ldots,\mathrm{x}_{\mathrm{k}i}^1,\mathrm{x}_{\mathrm{k}1}^2,\ldots,\mathrm{x}_{\mathrm{k}i}^2,\ldots,\mathrm{x}_{\mathrm{k}1}^m,\ldots,\mathrm{x}_{\mathrm{k}i}^m\right)^T.$$

Let the system Σ be denoted in the coordinates x_k by Σ_k

$$\Sigma_\mathrm{k} : \begin{cases} \dot{\mathrm{x}}_\mathrm{k} = \mathrm{f}_\mathrm{k}(\mathrm{x}_\mathrm{k},z) + \mathrm{g}_{\mathrm{k}1}(\mathrm{x}_\mathrm{k},z)u_1 + \cdots + \mathrm{g}_{\mathrm{k}m}(\mathrm{x}_\mathrm{k},z)u_m \\ 0 = \sigma(\mathrm{x}_\mathrm{k},z) \end{cases}$$

and for any $1 \leq i \leq m$ the ith subsystem Σ_k^i by

$$\Sigma_\mathrm{k}^i : \begin{cases} \dot{\mathrm{x}}_\mathrm{k}^i = \mathrm{f}_\mathrm{k}^i(\mathrm{x}_\mathrm{k},z) + g_{\mathrm{k}1}^i(\mathrm{x}_\mathrm{k},z)u_1 + \cdots + g_{\mathrm{k}m}^i(\mathrm{x}_\mathrm{k},z)u_m \\ 0 = \sigma(\mathrm{x}_\mathrm{k},z) \end{cases}$$

For any $1 \leq i \leq m$ and any $1 \leq \mathrm{k} \leq r$ we define A_i^k in the following way: for any $x = (x_1,\ldots,x_n)^T$ we have

$$A_i^\mathrm{k} x = (0,\ldots,0,x_{k+2},\ldots,x_r,0)^T$$

that is, A_i^k is the matrix A_i with the entries in the first k rows being zeros.

Definition 2.1: [9]
The minimum number of times that all or part of the constraint equation must be differentiated with respect to time in order to solve for $\dot{z}$ as a continuous function of x and z is the index of the nonlinear differential algebraic system (1.1).

Definition 2.2: [6]
Let $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ be a smooth vector field and $w : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ a smooth function. The M derivative of w along f is a function $\mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$, written $M_f w$ and defined as $M_f w = E(w) f$, where $E(w) = \dfrac{\partial w}{\partial x} - \dfrac{\partial w}{\partial z}\left(\dfrac{\partial \sigma}{\partial z}\right)^{-1}\dfrac{\partial \sigma}{\partial x}$. If w is

differential k times along f, the function $M_f^k w$ can be defined as $M_f^k w = M_f\left(M_f^{k-1} w\right)$ with $M_f^0 w = w$.

Definition 2.3: [6]

Given two smooth vector fields $f(x,z)$ and $g(x,z)$, both are defined on $\mathbb{R}^n$ then the M bracket is defined as follows:

$Mad_{f(x,z)} g(x,z) = [f(x,z), g(x,z)]_M = E(g)f - E(f)g$.

Repeated M brackets are denoted as

$Mad_{f^k(x,z)} g(x,z) = Mad_f(Mad_{f^{k-1}} g)$, $Mad_{f^1(x,z)} g(x,z) = Mad_f g$ and $Mad_{f^0(x,z)} g(x,z) = g$. Also, $[f(x,z), g(x,z)]_M = -[g(x,z), f(x,z)]_M$ and $[f(x,z), g(x,z)]_M w(x,z) = M_f M_g w - M_g M_f w$.

Theorem 2.4: (Frobenius) [6]

Consider the partial differential equation of function $w(x,z)$ with constraint condition $0 = \sigma(x,z)$

$$E(w)\left[v_1(x,z)\, v_2(x,z)\ \cdots\ v_3(x,z)\right] = 0$$

in which

$$E(w) = \frac{\partial w}{\partial x} - \frac{\partial w}{\partial z}\left(\frac{\partial \sigma}{\partial z}\right)^{-1}\frac{\partial \sigma}{\partial x}$$

where $(x,z) \in \mathbb{R}^n \times \mathbb{R}^m, v_i(x,z)\,(i = 1,2,\ldots,k < n)$ are linearly independent vector fields. If vector field set

$$D = \{v_1(x,z)\ \ v_2(x,z)\ \ \ldots\ \ v_3(x,z)\}$$

is involutive at $(x,z) = (x^0, z^0)$, then there exist certainly $(n-k)$ functions $w^1(x,z), w^2(x,z), \ldots, w^{n-k}(x,z)$ which satisfy given partial differential equation groups and the vectors

$\left[E_1(w^j)\ E_2(w^j)\ \cdots\ E_n(w^j)\right]$ $(j = 1,2,\ldots,(n-k), E_i = \partial/\partial x^i - \Sigma_{k=1}^m r_{k_i}\, \partial/\partial z^i, i = 1,2,\ldots,n)$

are linearly independent at (x^0, z^0).

Theorem 2.5: [7]

Let υ be a smooth vector field on $\mathbb{R}^n$, for any integer $1 \le k \le n$ such that $\upsilon_k(0,0) \ne 0$ and $\omega_k(x,z) = 1/\upsilon_k(x,z)$. The diffeomorphism $\xi = \varphi(x,z)$, where $\varphi : M \to \mathbb{R}^n$ defined by

$$\begin{aligned}\varphi_j(x,z) &= x_j + \sum_{s=1}^{\infty}\frac{(-1)^s x_k^s}{s!} M_{\omega_k \upsilon}^{s-1}(\omega_k \upsilon_j)(x,z), \quad j \ne k \\ \varphi_k(x,z) &= \sum_{s=1}^{\infty}\frac{(-1)^{s+1} x_k^s}{s!} M_{\omega_k \upsilon}^{s-1}(\omega_k)(x,z)\end{aligned} \tag{2.5.1}$$

Satisfies $\varphi^*(\upsilon) = \partial_{\xi_k}$. Moreover, the diffeomorphism $\psi(\xi)$,
where $\psi : \mathbb{R}^n \to M$ defined by

$$\begin{aligned}\psi_j(\xi) &= \xi_j + \sum_{s=1}^{\infty} \frac{\xi_k^s}{s!}\left(\sum_{i=0}^{s-1}(-1)^i C_i^s \partial_{\xi_k}^i M_\upsilon^{s-i-1}(\upsilon_j)(\xi,z)\right)\\ \psi_k(\xi) &= \sum_{s=1}^{\infty} \frac{\xi_k^s}{s!}\left(\sum_{i=0}^{s-1}(-1)^i C_i^s \partial_{\xi_k}^i M_\upsilon^{s-i-1}(\upsilon_k)(\xi,z)\right)\end{aligned} \tag{2.5.2}$$

is the inverse of $\xi = \varphi(x,z)$.

where $\partial_{\xi_k} = \dfrac{\partial}{\partial \xi_k}$, $\partial_{\xi_k}.h = \dfrac{\partial h}{\partial \xi_k}, \ldots, \partial_{\xi_k}^i.h = \dfrac{\partial^i h}{\partial \xi_k^i}$, $\quad i \ge 2$ and $C_i^s = \dfrac{s!}{i!(s-i)!}$.

3. MAIN RESULTS

Definition 3.1: The index one multi-input nonlinear differential algebraic control systems Σ_k is called $(\mathrm{ST})_k$ – linear from if

$$g_{ki} = B_i, Mad_{f_k}^j(g_{ki}) = A^j B_i, 1 \le j \le r-k, 1 \le i \le m.$$

It follows easily that Σ_k is $(\mathrm{ST})_k$ – linear if and only if each subsystem Σ_k^i decomposes as follows

$$\Sigma_k^i : \begin{cases} \dot{x}_{kj}^i = F_{kj}^i(\hat{x}_{kk+1}, z) & \text{if } 1 \le j \le k \\ \dot{x}_{kj}^i = x_{kj+1}^i + F_{kj}^i(\hat{x}_{kk+1}, z) & \text{if } k+1 \le j \le r-1 \\ \dot{x}_{kr}^i = F_{kr}^i(\hat{x}_{kk+1}, z) + u_i & \\ 0 = \sigma(\hat{x}_{kk+1}, z) & \end{cases}$$

A more compact representation of Σ_k^i is obtained as

$$\Sigma_k^i : \begin{cases} \dot{x}_k^i = A_i^k x_k^i + F_k^i(\hat{x}_{kk+1}, z) + b_i u_i, \ x_k^i \in \mathbb{R}^r \\ 0 = \sigma(\hat{x}_{kk+1}, z) \end{cases}$$

where $A_i^k x_k^i = (0, \ldots, 0, x_{kk+2}^i, x_{kk+3}^i, \ldots, x_{kr}^i, 0)^T$ is a vector whose last and first k components are zero. By extension, a compact notation for Σ_k would be

$$\Sigma_k : \begin{cases} \dot{x}_k = A_k x_k + F_k(\hat{x}_{kk+1}, z) + B_1 u_1 + \cdots + B_m u_m \\ 0 = \sigma(\hat{x}_{kk+1}, z) \end{cases}$$

where $x_k \in \mathbb{R}^n$ and $A^k x_k = \left((A_1^k x_k^1)^T, \ldots, (A_m^k x_k^m)^T\right)^T$

and $F_k(\hat{x}_{kk+1}, z) = \left((F_k^1(\hat{x}_{kk+1}, z))^T, \ldots, (F_k^m(\hat{x}_{kk+1}, z))^T\right)^T$.

Theorem 3.2: Consider the index one multi-input NDACS

$$\Sigma_r : \begin{cases} \dot{x}_r = f_r(x_r, z) + g_{r1}(x_r, z)u_1 + \cdots + g_{rm}(x_r, z)u_m \\ 0 = \sigma(x_r, z) \end{cases}$$

that is state linearizable, i.e., such that $(S'1)$ and $(S'2)$ hold. There exists a sequence of explicit coordinates changes $\varphi_{\mathrm{r}}(\mathrm{x}_{\mathrm{r}},z),\varphi_{\mathrm{r-1}}(\mathrm{x}_{\mathrm{r-1}},z),\ldots,\varphi_1(\mathrm{x}_1,z)$ giving rise to a sequence of state $(\mathrm{ST})_k$ - linear systems $\Sigma_{\mathrm{r-1}},\ldots,\Sigma_1$ such that

$$\Sigma_{\mathrm{k-1}} = \varphi_{\mathrm{k}}^{*}(\Sigma_{\mathrm{k}}), \quad 2 \le \mathrm{k} \le r .$$

The $(\mathrm{ST})_k$ -linear system Σ_{k} can be transformed into a $(\mathrm{ST})_{k-1}$ - linear system $\Sigma_{\mathrm{k-1}}$ if and only if

$$\begin{cases} \text{(a)}\ \dfrac{\partial^2 f_{\mathrm{k}}(\widehat{\mathrm{x}}_{\mathrm{k}k+1},z)}{\partial \mathrm{x}^i_{\mathrm{k}k+1}\partial \mathrm{x}^j_{\mathrm{k}k+1}} = 0 & 1 \le i,j \le m \\ \text{(b)}\ \left[\dfrac{\partial f_{\mathrm{k}}}{\partial \mathrm{x}^i_{\mathrm{k}k+1}},\dfrac{\partial f_{\mathrm{k}}}{\partial \mathrm{x}^j_{\mathrm{k}k+1}}\right] = 0 & 1 \le i,j \le m \end{cases} \tag{3.2}$$

Moreover, in the coordinates $\varpi = \varphi_1(\mathrm{x}_1,z)$ the system Σ_{r} takes the linear form

$$\Lambda : \begin{cases} \dot{\varpi} = A\varpi + B_1 u_1 + \cdots + B_m u_m \\ 0 = \sigma(\varpi,z) \end{cases}$$

where $\varpi = \left(\varpi^1_1,\ldots,\varpi^1_r,\varpi^2_1,\ldots,\varpi^2_r,\ldots,\varpi^m_1,\ldots,\varpi^m_r\right)^T$.

The proof of the above theorem follows from the algorithm below.

Algorithm 3.3:

Step r. Consider a state linearizable system Σ denoted in the coordinates $x = \mathrm{x}_{\mathrm{r}}$ by

$$\Sigma_{\mathrm{r}} : \begin{cases} \dot{x}_{\mathrm{r}} = \mathrm{f}_{\mathrm{r}}(\mathrm{x}_{\mathrm{r}},z) + g_{\mathrm{r}1}(\mathrm{x}_{\mathrm{r}},z)u_1 + \cdots + g_{\mathrm{r}m}(\mathrm{x}_{\mathrm{r}},z)u_m \\ 0 = \sigma(\mathrm{x}_{\mathrm{r}},z) \end{cases} .$$

Because Σ_{r} is state linearizable, and hence the distribution $\Delta_{\mathrm{r}} = \left\{g_{\mathrm{r}1},\ldots,g_{\mathrm{r}m}\right\}$ is involutive, we apply Theorem 2.5 to construct a change of coordinates $\mathrm{x}_{\mathrm{r-1}} = \varphi_r(\mathrm{x}_r,z)$ such that $(\varphi_r)^* g_{ri} = B_i$ for all $1 \le i \le m$. The change of coordinates takes the system into $\Sigma_{\mathrm{r-1}} = (\varphi_{\mathrm{r}})^*\Sigma_{\mathrm{r}}$

$$\Sigma_{\mathrm{r-1}} : \begin{cases} \dot{\mathrm{x}}_{\mathrm{r-1}} = \mathrm{f}_{\mathrm{r-1}}(\mathrm{x}_{\mathrm{r-1}},z) + g_{\mathrm{r-1}1}(\mathrm{x}_{\mathrm{r-1}},z)u_1 + \cdots + g_{\mathrm{r-1}m}(\mathrm{x}_{\mathrm{r-1}},z)u_m \\ 0 = \sigma(\mathrm{x}_{\mathrm{r-1}},z) \end{cases}$$

where $\mathrm{f}_{\mathrm{r-1}}(\mathrm{x}_{\mathrm{r-1}},z) = (\varphi_r)^*\mathrm{f}_{\mathrm{r}}(\mathrm{x}_{\mathrm{r}},z)$ and $g_{\mathrm{r-1i}}(\mathrm{x}_{\mathrm{r-1}},z) = (\varphi_r)^* g_{\mathrm{ri}}$.

Step k. Assume that Σ (or Σ_{r}) has been transformed, via explicit coordinates changes, into $(ST)_k$ − linear system

$$\Sigma_{\mathrm{k}} : \begin{cases} \dot{\mathrm{x}}_{\mathrm{k}} = \mathrm{f}_{\mathrm{k}}(\mathrm{x}_{\mathrm{k}},z) + g_{\mathrm{k}1}(\mathrm{x}_{\mathrm{k}},z)u_1 + \cdots + g_{\mathrm{k}m}(\mathrm{x}_{\mathrm{k}},z)u_m \\ 0 = \sigma(\mathrm{x}_{\mathrm{k}},z) \end{cases}$$

with $\mathrm{f}_{\mathrm{k}}(\mathrm{x}_{\mathrm{k}},z) = A^k\mathrm{x}_{\mathrm{k}} + \mathrm{F}_{\mathrm{k}}(\widehat{\mathrm{x}}_{\mathrm{k}k+1},z)$ and $g_{\mathrm{k}i}(\mathrm{x}_{\mathrm{k}},z) = B_i$ for all $1 \le i \le m$. Since Σ (hence Σ_{k}) is state linearizable , then condition $(S'2)$ is satisfied, implying in particular,

$$\left[Mad_{f_k}^{s}(g_{ki}), Mad_{f_k}^{t}(g_{kj})\right] = 0,\ \ s,t \geq 0,\ 1 \leq i, j \leq m. \tag{3.3}$$

Setting $s = r-k$ and $t = r-k-1$ implies

$$\left[[f_k, A^{r-k-1}B_i], A^{r-k-1}B_j\right] = 0,\ \ 1 \leq i, j \leq m\ .$$

This latter condition is equivalent to

$$(SL_{k+1}) \Rightarrow \frac{\partial^2 f_k(\widehat{x}_{kk+1}, z)}{\partial x_{kk+1}^{i} \partial x_{kk+1}^{j}} = 0, \qquad 1 \leq i, j \leq m.$$

The vector field $F_k(\widehat{x}_{kk+1}, z)$ decomposes uniquely as

$$F_k(.) = F_k(\widehat{x}_{kk}, z) + x_{kk+1}^{1}\, \breve{g}_{k1}(\widehat{x}_{kk}, z) + \cdots + x_{kk+1}^{m}\, \breve{g}_{km}(\widehat{x}_{kk}, z),$$

where

$$\breve{F}_k(\widehat{x}_{kk}, z) = \breve{F}_k(x_{k1}^{1}, \ldots, x_{kk}^{1}, x_{k1}^{2}, \ldots, x_{kk}^{2}, \ldots, x_{k1}^{m}, \ldots, x_{kk}^{m}, z)$$

$$\breve{g}_{ki}(\widehat{x}_{kk}, z) = \breve{g}_{ki}(x_{k1}^{1}, \ldots, x_{kk}^{1}, x_{k1}^{2}, \ldots, x_{kk}^{2}, \ldots, x_{k1}^{m}, \ldots, x_{kk}^{m}, z)\ .$$

We deduce from (S1) that the distribution

$$\breve{\Delta}_k = \left\{\breve{g}_{k1}(\widehat{x}_{kk}, z), \breve{g}_{k2}(\widehat{x}_{kk}, z), \ldots, \breve{g}_{km}(\widehat{x}_{kk}, z)\right\}$$

is commutative and of maximal rank m . By Theorem 2.5 we can construct a change of coordinates $x_{k-1} = \varphi_k(x_k, z)$ such that $(\varphi_k)^{*}\breve{g}_{ki} = \partial_{\breve{x}_{k-1i}} = A^{r-k}B_i,\ 1 \leq i \leq m$.

This change of coordinates transforms Σ_k into

$$\Sigma_{k-1} : \begin{cases} \dot{x}_{k-1} = & f_{k-1}(x_{k-1}, z) + g_{k-11}(x_{k-1}, z)u_1 + \cdots + g_{k-1m}(x_{k-1}, z)u_m \\ 0 \ \ = & \sigma(x_{k-1}, z) \end{cases}$$

where $g_{k-1i}(x_{k-1}, z) = (\varphi_k)^{*} g_{ki} = B_i$ and

$$\begin{aligned} f_{k-1}(x_{k-1}, z) &= (\varphi_k)^{*} f_k \\ &= (\varphi_k)^{*}(A^k x_k) + (\varphi_k)^{*}(\breve{F}_k(x_k, z)) + (\varphi_k)^{*}(x_{kk+1}^{m}\breve{g}_{k1}) + \cdots + (\varphi_k)^{*}(x_{kk+1}^{m}\breve{g}_{k1}) \\ &= A^k x_{k-1} + F_{k-1}(\widehat{x}_{k-1k}) + x_{k-1k+1}^{1}A^{r-k}B_1 + \cdots + x_{k-1k+1}^{m}A^{r-k}B_m \end{aligned}$$

Because the first k components of $A^k x_k$ are zero. We also have

$(\varphi_k)^{*}(x_{kk+1}^{i}\breve{g}_{ki}) = (x_{kk+1}^{i} \circ \phi_k^{-1})(\varphi_k)^{*}\breve{g}_{k1} = x_{k-1k+1}^{i}A^{r-k}B_i$

It is straightforward to verify that

$$A^k x_{k-1} + \sum_{i=1}^{m} x_{k-1k+1}^{i} A^{r-k} B_i = A^{k-1} x_{k-1}$$

and hence the system Σ_{k-1} is in $(ST)_{k-1}-$linear form. This ends the general step and shows that a sequence of explicit coordinates changes $\varphi_r(x_r, z), \ldots, \varphi_1(x_1, z)$ can be constructed whose composition $\varphi_1 \circ \cdots \circ \varphi_r(x_r, z)$ takes the original system Σ_r into a linear form.

Example: Consider the index one multi-input nonlinear differential algebraic control system (NDACS)

$$\Sigma_3 : \begin{cases} \dot{x}_3 = f_3(x_3,z) + g_{31}(x_3,z)u_1 + g_{32}(x_3,z)u_2 \\ 0 \;\; = \sigma(x_3,z) \end{cases}$$

Defined in the coordinates $x_3 = \left(x_{31},\ldots,x_{35}\right)^T \in \mathbb{R}^5$ by

$$\begin{cases} \dot{x}_{31} = z + u_2 \\ \dot{x}_{32} = x_{33} - x_{31}x_{34} - x_{34}^3 + x_{34}x_{35} - 2x_{35}^2 - 2x_{34}u_2 \\ \dot{x}_{33} = z\, x_{34} + x_{31}x_{35} + 3x_{34}^2x_{35} - x_{35}^2 + u_1 \\ \dot{x}_{34} = x_{35} \\ \dot{x}_{35} = u_2 \\ 0 \;\; = x_{32} - z \end{cases}$$

where

$$\left(\frac{\partial\sigma}{\partial z}\right)^{-1} = -1$$

$$\frac{\partial\sigma}{\partial x} = (0\;\,1\;\,0\;\,0\;\,0)$$

$$\left(\frac{\partial\sigma}{\partial z}\right)^{-1}\frac{\partial\sigma}{\partial x} = (0\;\,-1\;\,0\;\,0\;\,0)$$

$$f_3(x_3,z) = \left(z, x_{33} - x_{31}x_{34} - x_{34}^3 + x_{34}x_{35} - 2x_{35}^2, z\, x_{34} + x_{31}x_{35} + 3x_{34}^2x_{35} - x_{35}^2, x_{35}, 0\right)^T$$

$g_{31}(x_3,z) = \left(0,0,1,0,0\right)^T$ and $g_{32}(x_3,z) = \left(1,2x_{34},0,0,1\right)^T$

Put $\nu^1 = g_{31}$ and $\nu^2 = g_{32}$. we look for a change of coordinates $x_2 = \varphi_3(x_3,z)$ that rectifies the distribution $\Delta_3 = \mathrm{span}\left\{\nu^1,\nu^2\right\}$, i.e., such that $(\varphi)_*\Delta = span\left\{\partial_{x_{32}},\partial_{x_{35}}\right\}$. Apply Theorem 2.5 to $\nu^2 = g_{32}(x_3,z)$ with $n = 5$ and $\sigma_5 = 1$. Since $M_{\nu^2}^{s-1}\nu_1^2 = 0$ and $M_{\nu^2}^{s-1}\nu_2^2 = 0$ for all $s \geq 1$. It follows easily that the change of coordinates is defined by

$$x_2 = \varphi_3(x_3,z) \triangleq \begin{cases} x_{21} = x_{31} + \sum_{s=1}^{\infty}\frac{(-1)^s x_{35}^s}{s!}M_{\sigma_5\nu^2}^{s-1}(\sigma_5\nu_1^2)(x_3,z) \\ \quad = x_{31} - x_{35}\nu_1^2(x_3,z) = x_{31} - x_{35} \\ x_{22} = x_{32} + \sum_{s=1}^{\infty}\frac{(-1)^s x_{35}^s}{s!}M_{\sigma_5\nu^2}^{s-1}(\sigma_5\nu_2^2)(x_3,z) \\ \quad = x_{32} - x_{35}\nu_2^2(x_3,z) = x_{32} + 2x_{35}x_{34} \\ x_{23} = x_{33} \\ x_{24} = x_{34} \\ x_{25} = x_{35} \end{cases}$$

System Σ_3 is brought under this change of coordinates into

$$\Sigma_2 : \dot{\mathrm{x}}_2 = \mathrm{f}_2(\mathrm{x}_2,z) + g_{21}(\mathrm{x}_2,z)u_1 + g_{22}(\mathrm{x}_2,z)u_2$$

$$\mathrm{f}_2(\mathrm{x}_2,z) = \left(z, \mathrm{x}_{23} - \mathrm{x}_{21}\mathrm{x}_{24} - \mathrm{x}_{24}^3, z\,\mathrm{x}_{24} + \mathrm{x}_{21}\mathrm{x}_{25} + 3\mathrm{x}_{24}^2\mathrm{x}_{25}, \mathrm{x}_{25}, 0\right)^T$$

$$\mathrm{g}_{21}(\mathrm{x}_2,z) = (0,0,1,0,0)^T \text{ and } \mathrm{g}_{22}(\mathrm{x}_2,z) = (0,0,0,0,1)^T$$

For $k = 2$ with $\mathrm{x}_{23} \triangleq \mathrm{x}_{23}^1, \mathrm{x}_{23}^2 \triangleq \mathrm{x}_{25}$ are satisfied and equivalent to

$$(SL_{k+1}) \Rightarrow \begin{cases} \dfrac{\partial^2 \mathrm{f}_2}{\partial x_{23}\partial x_{23}} = 0 \\ \dfrac{\partial^2 \mathrm{f}_2}{\partial x_{25}\partial x_{25}} = 0 \\ \dfrac{\partial^2 \mathrm{f}_2}{\partial x_{23}\partial x_{25}} = 0 \end{cases}$$

Thus the vector field f_2 decomposes uniquely as

$$\mathrm{f}_2(\mathrm{x}_2,z) = \left(z, -\mathrm{x}_{21}\mathrm{x}_{24} - \mathrm{x}_{24}^3, z\,\mathrm{x}_{24}, 0, 0\right)^T + \mathrm{x}_{23}\left(0,1,0,0,0\right)^T + \mathrm{x}_{25}\left(0,0,\mathrm{x}_{21} + 3\mathrm{x}_{24}^2, 1, 0\right)^T$$

Now we look for a change of coordinates $\mathrm{x}_1 = \varphi_2(\mathrm{x}_2,z)$ that rectifies the distribution $\breve{\Delta}_2 = \mathrm{span}\{\breve{g}_{21}(\mathrm{x}_2,z), \breve{g}_{22}(\mathrm{x}_2,z)\} = \mathrm{span}\{\breve{\nu}^1, \breve{\nu}^2\}$. Apply Theorem 2.5 to $\breve{\upsilon}^2 \triangleq \breve{g}_{22}(\mathrm{x}_2,z)$ with $n = 5$ and $\sigma_4 = 1$. We get $M_{\sigma_4\breve{\upsilon}^2}(\sigma_4\breve{\upsilon}_3^2)(\mathrm{x}_2,z) = 6\mathrm{x}_{24}$, $M_{\sigma_4\breve{\upsilon}^2}^2(\sigma_4\breve{\upsilon}_3^2)(\mathrm{x}_2,z) = 6$, $M_{\sigma_4\breve{\upsilon}^2}^{s-1}(\sigma_4\breve{\upsilon}_3^2)(\mathrm{x}_2,z) = 0$ for all $s \geq 4$. We deduce the change of coordinates as follows

$$\mathrm{x}_1 = \varphi_2(\mathrm{x}_2,z) \triangleq \begin{cases} \mathrm{x}_{11} = \mathrm{x}_{21} + \displaystyle\sum_{s=1}^{\infty} \frac{(-1)^s \mathrm{x}_{24}^s}{s!} M_{\sigma_4\breve{\nu}^2}^{s-1}(\sigma_4\breve{\nu}_1^2)(\mathrm{x}_2,z) \\ \quad\;\; = \mathrm{x}_{21} \\ \mathrm{x}_{12} = \mathrm{x}_{22} \\ \mathrm{x}_{13} = \mathrm{x}_{23} + \displaystyle\sum_{s=1}^{\infty} \frac{(-1)^s \mathrm{x}_{24}^s}{s!} M_{\sigma_4\breve{\nu}^2}^{s-1}(\sigma_4\breve{\nu}_1^2)(\mathrm{x}_2,z) \\ \quad\;\; = \mathrm{x}_{23} - \mathrm{x}_{24}\upsilon_3 + \dfrac{\mathrm{x}_{24}^2}{2}(6\mathrm{x}_{24}) - \dfrac{\mathrm{x}_{24}^3}{6}(6) \\ \quad\;\; = \mathrm{x}_{23} - \mathrm{x}_{24}\mathrm{x}_{21} - 2\mathrm{x}_{24}^3 \\ \mathrm{x}_{14} = \mathrm{x}_{24} \\ \mathrm{x}_{15} = \mathrm{x}_{25} \end{cases}$$

The change of coordinates transforms Σ_2 into

$$\Sigma_1 : \dot{x}_1 = A\,x_1 + B_1 u_1 + B_2 u_2 \triangleq \begin{cases} \dot{x}_{11} = z \\ \dot{x}_{12} = x_{13} \\ \dot{x}_{13} = u_1 \\ \dot{x}_{14} = x_{15} \\ \dot{x}_{15} = u_2 \\ 0 = x_{13} + x_{11}x_{14} + x_{14}^3 - z \end{cases}$$

A linearizing coordinates is obtained by taking a composition of the different coordinates $x_1 = \varphi_2 \circ \varphi_3(x_3, z)$

$$x_1 = \varphi_2 \circ \varphi_3(x_3, z) \triangleq \begin{cases} x_{11} = x_{31} - x_{35} \\ x_{12} = x_{32} + 2x_{35}x_{34} \\ x_{13} = x_{33} - x_{31}x_{34} + x_{34}x_{35} + x_{34}^2 \\ x_{14} = x_{34} \\ x_{15} = x_{35} \end{cases} \qquad \square$$

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