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Property T for $C(X) \rtimes_\sigma \mathbb{Z}$

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Abstract

In this note we investigate Property T for crossed product C^*-algebra $C(X) \rtimes_\sigma \mathbb{Z}$. Let μ be a Borel measure on X. We show that every Hilbert bimodule induced from μ has almost central vectors. We derive several important corollaries.

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1 Introduction

Property T was first introduced by Kazhdan [6] to study the lattice structure of groups. Bekka extended the notion of Property T to C^*-algebras [1] following the approach of Connes [3]. He showed that a discrete group G has Property T if and only if its group C^*-algebra $C^*(G)$ has Property T. It has been since studied by several authors in [2, 4, 5, 7]. In particular, Brown showed that a nuclear C^*-algebra with a *faithful* tracial state must be finite dimensional [2].

The goal of this note is to investigate Property T for crossed product C^*-algebra $C(X) \rtimes_\sigma \mathbb{Z}$. The class of C^*-algebras $C(X) \rtimes_\sigma \mathbb{Z}$ include many important examples including the rotation algebras and Bunce-Deddens algebras. Note that $C(X) \rtimes_\sigma \mathbb{Z}$ is a nuclear C^*-algebra and has a tracial state. However, the tracial state on $C(X) \rtimes_\sigma \mathbb{Z}$ is not necessarily faithful so we can not use Brown's theorem [2]. Our main result is Theorem 2.3, where we show that a certain class of Hilbert bimodules corresponding to Borel measures on X always have almost central unit vectors. As a corollary, we show that $C(X) \rtimes_\sigma \mathbb{Z}$ does not have Property T. In addition, we obtain that the pair $(C(X) \rtimes_\sigma \mathbb{Z}, C(X))$ does not have Property T.

2 Background and Results

Let X be a compact Hausdorff space and α a homeomorphism of X. We can define the action of $\mathbb{Z}$ on X by $x \cdot n = \alpha^n(x)$ for all $x \in X$ and $n \in \mathbb{Z}$. The pair $(X, \mathbb{Z})$ is called a dynamical system. We can further define the action of $\mathbb{Z}$ on $C(X)$ by $(f_n)(x) = f(x \cdot n)$ for all $f \in C(X)$, $x \in X$ and $n \in \mathbb{Z}$. We denote by $C(X) \rtimes_\sigma \mathbb{Z}$ the C^*-algebra associated to the dynamical system $(X, \mathbb{Z})$. Recall that $C(X) \rtimes_\sigma \mathbb{Z}$ is the closure of the linear span of the set $\{fV_n\}$, where $f \in C(X)$ and V_n is the unitary representing $n \in \mathbb{Z}$.

Definition 2.1. *A Hilbert bimodule over a C^*-algebra A is a Hilbert space $\mathcal{H}$ carrying a pair of commuting representations, one of A and one of its opposite algebra A^o. We denote the action by*

$$\xi \mapsto a\xi b$$

for all $\xi \in \mathcal{H}$, $a \in A$ and $b \in A^o$.

Let μ be a finite Borel measure on X and form the Hilbert space $\mathcal{H}_\mu = L^2(X \times \mathbb{Z}, \mu \times \lambda)$, where λ is the counting measure on $\mathbb{Z}$. Define the left action of $C(X) \rtimes_\sigma \mathbb{Z}$ on $\mathcal{H}_\mu \otimes \mathcal{H}_\mu$ by

$$((fV_n)\xi)((x,i),(y,j)) = f(y \cdot j)\xi((x,i),(y,j+n)$$

and the right action by

$$(\xi(fV_n))((x,i),(y,j)) = f(x \cdot (i-n))\xi((x,i-n),(y,j))$$

for all $f \in C(X), \xi \in \mathcal{H}_\mu \otimes \mathcal{H}_\mu$, $x, y \in X$ and $n, i, j \in \mathbb{Z}$. We call $\mathcal{H}_\mu \otimes \mathcal{H}_\mu$ the Hilbert bimodule induced from the measure μ.

Definition 2.2. *A C^*-algebra A has Property T if every bimodule with almost central vectors has a central vector; i.e. if $\mathcal{H}$ is a bimodule and there exist unit vectors $\xi_i \in \mathcal{H}$ such that $\| a\xi_i - \xi_i a \| \to 0$ for all $a \in A$, then there exists a unit vector $\xi \in \mathcal{H}$ such that $a\xi = \xi a$ for all $a \in A$.*

We are now ready to state our main result.

Theorem 2.3. *Let* $(X, \mathbb{Z})$ *be a dynamical system, where* X *is a compact Hausdorff second countable space. Suppose that* μ *is a finite Borel measure on* X *and* $\mathcal{H}_\mu \otimes \mathcal{H}_\mu$ *is the Hilbert bimodule induced from* μ. *Then* $\mathcal{H}_\mu \otimes \mathcal{H}_\mu$ *has almost central unit vectors.*

Proof. Let μ be a finite Borel measure on X. Since X is compact there exists a point $x_0 \in X$ such that $\mu(W) > 0$ for all open sets W containing x_0. It follows from the hypothesis that X is a metrizable space with metric d. Let $B(x,r)$ denote the ball centered at $x \in X$ with radius $r > 0$. For each $n \geq 1$ define $E_n = \bigcap_{1 \leq i \leq n} (B(x_0 \cdot i, 1/n)) \cdot (-i)$ and $D_n = \{(i,i) \in \mathbb{Z} \times \mathbb{Z} : 1 \leq i \leq n\}$. Define a sequence of unit vectors in $\mathcal{H}_\mu \otimes \mathcal{H}_\mu$ by

$$\xi_n((x,i),(y,j)) = [\frac{1}{n} \frac{1}{\mu^{(2)}(E_n^{(2)})}]^{1/2} \chi_{E_n \times E_n}(x,y) \chi_{D_n}(i,j).$$

Let $f \in C(X)$ and $\epsilon > 0$ be given. Since X is compact then f is uniformly continuous on X. There exists N such that $|f(x) - f(y)| < \epsilon$ for all $x, y \in X$ with $d(x,y) < 1/N$. For each $n \geq 2N$ and $1 \leq i \leq n$ we have

$$d(x \cdot i, y \cdot i) < 1/N$$

for all $(x,y) \in E_n \times E_n$. It follows that $|f(x \cdot i) - f(y \cdot i)| < \epsilon$ for all $1 \leq i \leq n$ and $(x,y) \in E_n \times E_n$. Then we have

$$\begin{aligned}
&||(fV_m)\xi_n - \xi_n(fV_m)||^2 \\
&\quad = \frac{1}{n} \frac{1}{\mu^{(2)}(E_n^{(2)})} (\sum_{-m \leq j < 0} \int_{E_n \times E_n} |f(y \cdot j)|^2 \\
&\quad + \sum_{0 \leq j \leq n-m} \int_{E_n \times E_n} |f(x \cdot j) - f(y \cdot j)|^2 \\
&\quad + \sum_{n-m < j \leq n} \int_{E_n \times E_n} |f(x \cdot j)|^2) \\
&\quad < \frac{2mM^2}{n} + \frac{\epsilon^2 (n-m)}{n}
\end{aligned}$$

for all $n \geq 2N$, where $M = \sup_x |f(x)|$. It follows $||(fV_m)\xi_n - \xi_n(fV_m)|| \to 0$ for all $f \in C(X)$ and $m \in \mathbb{Z}$. Since the linear span of fV_m is dense in $C(X) \rtimes_\sigma \mathbb{Z}$ then $||a\xi_n - \xi_n a|| \to 0$ for all $a \in C(X) \rtimes_\sigma \mathbb{Z}$. It follows that $(\xi_n)_{n \geq 1}$ is a sequence of almost central vectors in $\mathcal{H}_\mu \otimes \mathcal{H}_\mu$. □

The above theorem has important implications about Property T for $C(X) \rtimes_\sigma \mathbb{Z}$.

Corollary 2.4. *Let $(X, \mathbb{Z})$ be a dynamical system, where X is a compact Hausdorff second countable space. Then $C(X) \rtimes_\sigma \mathbb{Z}$ does not have Property T.*

Proof. Suppose for contradiction that $C(X) \rtimes_\sigma \mathbb{Z}$ has Property T. Let $x_0 \in X$ and define a measure μ_0 on X by $\mu_0(E) = \chi_{x_0}(E)$. Then the Hilbert bimodule $\mathcal{H}_{\mu_0} \otimes \mathcal{H}_{\mu_0}$ induced by μ_0 is isomorphic to $L^2(\mathbb{Z} \times \mathbb{Z})$. By Theorem 2.3 there exists a sequence of almost central unit vectors in $L^2(\mathbb{Z} \times \mathbb{Z})$. Then there is a unit vector ξ such that $a\xi = \xi a$ for all $a \in C(X) \rtimes_\sigma \mathbb{Z}$. In particular, we have $\xi(i, j+m) = \xi(i-m, j)$ for all $i, j, m \in \mathbb{Z}$. It follows that $\xi(i, j) = \xi(i+m, j+m)$ for all $i, j, m \in \mathbb{Z}$. Choose $i_0, j_0 \in \mathbb{Z}$ such that $\xi(i_0, j_0) \neq 0$. Then

$$||\xi||^2 = \sum_{i,j \in \mathbb{Z}} |\xi(i, j)|^2 \geq \sum_{m \in \mathbb{Z}} |\xi(i_0 + m, j_0 + m)|^2 = \sum_{m \in \mathbb{Z}} |\xi(i_0, j_0)|^2 > \infty.$$

It follows that $C(X) \rtimes_\sigma \mathbb{Z}$ does not have Property T. □

Definition 2.5. *Let A be a C^*-algebra and B a subalgebra of A. Then the pair (A, B) has Property T if every A-bimodule with almost central vectors has a B-central vector.*

The next corollary follows directly from Theorem 2.3.

Corollary 2.6. *Let $(X, \mathbb{Z})$ be a dynamical system, where X is a compact Hausdorff second countable space. If X is infinite then $(C(X) \rtimes_\sigma \mathbb{Z}, C(X))$ does not have Property T.*

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