Int. J. Contemp. Math. Sciences, Vol. 8, 2013, no. 15, 699 - 704
HIKARI Ltd, www.m-hikari.com
http://dx.doi.org/10.12988/ijcms.2013.3679

Computing Generators of Second Homotopy Module Using Tietze Transformation Methods

Yanita

Department of Mathematics, Faculty of Mathematics and Natural Science
Universitas Andalas, Kampus Unand Limau Manis, Padang, 25163 Indonesia
yanitao@yahoo.com

Abdul Ghafur Ahmad

School of Mathematical Sciences
Faculty of Science and Technology
Universiti Kebangsaan Malaysia, Selangor 43600, Malaysia
ghafur@ukm.my

Abstract

This paper discusses the relationship of second homotopy module for two different presentations defining a similar group. These two presentations can be transformed to each other using Tietze transformation. This relationship was determined by considering the generators of second homotopy module for both presentations.

Mathematics Subject Classification : 14F35, 14H30, 20F05, 20M05

Keywords: Second homotopy module, Tietze transformation, Generator

1 Introduction

Let $\mathcal{P} = \langle \boldsymbol{x} : \boldsymbol{r} \rangle$ be a presentation for a group *G*. Then we have the *first fundamental group* $\pi_1(\mathcal{P})$ over $\mathcal{P} = \langle \boldsymbol{x} : \boldsymbol{r} \rangle$. The elements of $\pi_1(\mathcal{P})$ are equivalent classes of *words* $[W]$. Moreover, we can have a *picture* $\mathbb{P}$ over $\mathcal{P}$. A *picture* $\mathbb{P}$ over $\mathcal{P}$ is an object consist of disjoint arcs labelled by element of $\boldsymbol{x}$, discs labelled by element of $\boldsymbol{r}$, and a boundary disc with a basepoint.

A picture $\mathbb{P}$ over $\mathcal{P}$ is a spherical picture if all arcs in $\mathcal{P}$ do not touch the boundary disc. Then we have the *second homotopy module* $\pi_2(\mathcal{P})$. The elements of $\pi_2(\mathcal{P})$ are equivalent classes of spherical picture $[\mathbb{P}]$.

Let a group G defined by two group presentation, say $\mathcal{P}_1$ and $\mathcal{P}_2$. There are some alternations one can make to presentation $\mathcal{P}_2$ which result in presentation of a group isomorphic to the original $\mathcal{P}_1$ (see [1] and [5]). These are called *Tietze transformations.* Tietze transformation are simply the obvious ways of transforming a finite presentation $\langle \boldsymbol{x}:\boldsymbol{r}\rangle$.

Tietze transformation are useful in special cases for showing that two given presentations define isomorphic group, and, in particular, for simplifying a given presentation. We describe this transformations as follows.

Let $\mathcal{P}_1 = \langle \boldsymbol{x}:\boldsymbol{r}\rangle$ dan $\mathcal{P}_2 = \langle \boldsymbol{y}:\boldsymbol{s}\rangle$ be two presentations of the group G. Then there are the following Tietze transformations which may be performed upon the group presentations:

(T1) If the word *S* is derivable from $\{\boldsymbol{r}\}$, then add *S* to the list of relators.

(T2) If the word *S* is derivable from $\{\boldsymbol{r}\}\backslash S$, remove *S* from the list relators.

(T3) If *R* is word in the $\boldsymbol{x}$, and *y* is some symbol not in the generating set, add *y* to the generating set and add word $y^{-1}R$ to the relator set.

(T4) If there is a relator of the form $y^{-1}R \in \{\boldsymbol{r}\}$, $y \in \{\boldsymbol{x}\}$ with *y* not appearing in *R*, delete this relator and delete *y* from the generating set, replacing all order occurences of *y* in the relator words with R.

The problem of $\pi_2(\mathcal{P})$ is to compute its generator (see [4]). Suppose that $\boldsymbol{P}$ is set of spherical pictures over $\mathcal{P}$. If all spherical pictures $\mathbb{P}$ are equivalent to the empty picture (relative to $\boldsymbol{P}$) then we say that $\boldsymbol{P}$ generates $\pi_2(\mathcal{P})$. In this paper we are going to determine the relationship between generators of $\pi_2(\mathcal{P}_1)$ and $\pi_2(\mathcal{P}_2)$ if $\mathcal{P}_1$ and $\mathcal{P}_2$ define the same group.

We are going to prove:

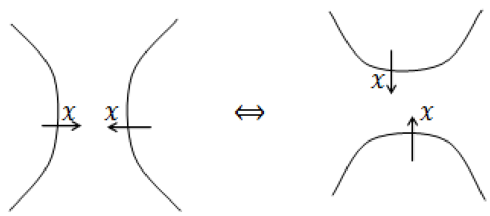
Theorem 1. *Let* $\mathcal{P}_1 = \langle \boldsymbol{x}:\boldsymbol{r}\rangle$ *and* $\mathcal{P}_2 = \langle \boldsymbol{x}:\boldsymbol{r}, T\rangle$ *be a presentation define a group* G, *where* T *is a cyclically reduced word define and* $T\sim_{\boldsymbol{r}}1$ (*relative to* $\boldsymbol{r}$). *If* $\pi_2(\mathcal{P}_1)$ *is generated by* $\boldsymbol{P}_1$ *then* $\pi_2(\mathcal{P}_2)$ *is generated by* $\boldsymbol{P}_1 \cup \{[\mathbb{P}_T]\}$, *where* $\mathbb{P}_T$ *is spherical picture having a T-disc joining to a picture* $\mathbb{P}$ *over* $\mathcal{P}_1$.

Theorem 2. *Let* $\mathcal{P}_1 = \langle \boldsymbol{x}:\boldsymbol{r}\rangle$ *and* $\mathcal{P}_2 = \langle \boldsymbol{x}, y:\boldsymbol{r}, y = S\rangle$ *be a presentation define a group* G, *where S a word on* $\boldsymbol{x}$. *Then* $\pi_2(\mathcal{P}_1)$ *has same generator with* $\pi_2(\mathcal{P}_2)$.

Proof of these theorem by using operations on picture and van Kampen's Lemma and will be given on section 3.

2. Picture and Operation on Picture

A picture $\mathbb{P}$ in $\mathcal{P} = \langle \boldsymbol{x}: \boldsymbol{r} \rangle$ is an object consist of disjoint arcs labeled by element of $\boldsymbol{x}$, discs labeled by element of $\boldsymbol{r}$ and a boundary disc with a basepoint (see [4] and [2]). A picture $\mathbb{P}$ in $\mathcal{P} = \langle \boldsymbol{x}: \boldsymbol{r} \rangle$ is a *spherical picture* if all arcs in $\mathbb{P}$ do not touch the boundary disc. Certain basic operation can be applied to a picture (spherical picture) $\mathbb{P}$ as follows: deletion and insertion *floating circle*, deletion and insertion *floating semicircle*, deletion and insertion *folding pair* and bridge move (see [3]), as depicts below.



Two spherical pictures $\mathbb{P}_1$ and $\mathbb{P}_2$ are said to be *equivalent* if either: (a) both are spherical and one can be transformed to the other by a finite number of operation deletion and insertion floating circle, deletion and insertion folding pair and bridge move; or (b) both are not spherical and one can be transformed to the other by a finite number of operation deletion and insertion floating circle, deletion and insertion semicircle, deletion and insertion folding pair and bridge move.

The equivalent class containing the spherical picture $\mathbb{P}$ is denoted by $[\mathbb{P}]$. The equivalent class containing the empty picture (null) is denoted by [4]. The mirror image for the spherical picture $\mathbb{P}$ is denoted by $-\mathbb{P}$. The addition $\mathbb{P}_1 + \mathbb{P}_2$ is defined by drawing $\mathbb{P}_1$ and $\mathbb{P}_2$.

Set of equivalent classes of spherical picture with binary operation $[\mathbb{P}_1] + [\mathbb{P}_2] = [\mathbb{P}_1 + \mathbb{P}_2]$ form a abelian group under this operation and this abelian group is right $\mathbb{Z}G$-module, where the action is given by $[\mathbb{P}]\overline{W} = [\mathbb{P}W]$ ($\overline{W}$ denotes the element of G represented by W). This module is called the *second homotopy module* of $\mathcal{P}$, denoted by $\pi_2(\mathcal{P})$.

A set $\boldsymbol{P}$ of spherical pictures over $\mathcal{P}$ will be called a *generating set of pictures* if $\{[\mathbb{P}]:\mathbb{P} \in \boldsymbol{P}\}$ generates the $\mathbb{Z}G$-module $\pi_2(\mathcal{P})$ (see [6]) . It follow [4], that $\boldsymbol{P}$ is generating set if and only if every spherical picture over $\mathcal{P}$ can be transformed to empty picture by operations: bridge moves, insertion/deletion of floating circles, insertion/deletion of folding pairs, insertion/deletion of pictures from $\pm\boldsymbol{P}$.

Consider a collection $\boldsymbol{S}$ of spherical pictures. Now, we define two extended operation on pictures as follows :

1). (Deletion of an $\boldsymbol{S}$-picture) If there is a simple closed path in a picture such that the part of the picture enclosed by the simple closed path is a copy of a spherical picture.

2). (Insertion of an $\boldsymbol{S}$-picture) The opposite of 1).

Two pictures will be said to be *equivalent* (relative $\boldsymbol{S}$) if either: a). the pictures are both spherical and one can be transformed to the other by a finite number of operation deletion and insertion floating circle, deletion and insertion folding pair, bridge move, and deletion and insertion $\boldsymbol{S}$-picture; or b). the picture are not both spherical and one can be transformed to the other by a finite number of operations deletion and insertion floating circle, deletion and insertion floating semicircle, deletion and insertion folding pair, bridge move and deletion and insertion $\boldsymbol{S}$-picture (see [3]).

3. Proof of Theorem 1.1 and Theorem 1.2

Proof of Theorem 1.1

Suppose that $\mathcal{P}_1 = \langle \boldsymbol{x}: \boldsymbol{r} \rangle$ is generated by $\boldsymbol{P_1}$. Consider that:

$$\mathcal{P}_1 = \langle \boldsymbol{x}: \boldsymbol{r} \rangle \xrightarrow{\text{T1}} \mathcal{P}_2 = \langle \boldsymbol{x}: \boldsymbol{r}, T \rangle \qquad (*)$$

is a one of operation Tietze transformation. From (*) we know that T is a relator which is add on $\mathcal{P}_2$ and $T \sim_{\boldsymbol{r}} 1$. Based on van Kampen Lemma, there is a picture $\mathbb{Q}$ over $\mathcal{P}_1$ where $W(\mathbb{Q}) = T$. Then picture

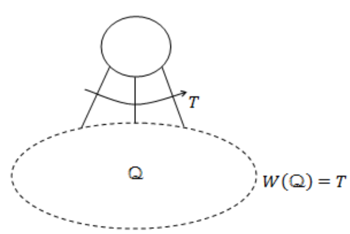


Figure 1. Spherical picture $W(\mathbb{Q}) = T$

is a spherical picture.

Since $\mathbb{Q}$ has T-disc, then it could not be got $\mathbb{Q}$ of picture in $\mathcal{P}_1$. Therefore, $\mathbb{Q}$ is one of generator of $\mathcal{P}_2$. From this, we have generator of $\mathcal{P}_2$ is generator of $\mathcal{P}_1$ and picture $\mathbb{Q}$.

Let $\mathbb{P}$ spherical picture in $\mathcal{P}_2$. We consider two case, i. e. 1). $\mathbb{P}$ has no T-disc, and 2). $\mathbb{P}$ has T-disc.

If $\mathbb{P}$ has no T-*disc*, then $\mathbb{P}$ is picture in $\mathcal{P}_1$. So $\mathcal{P}_1 \sim 1$ (relative $\mathcal{P}_1$). If $\mathbb{P}$ has T-disc,

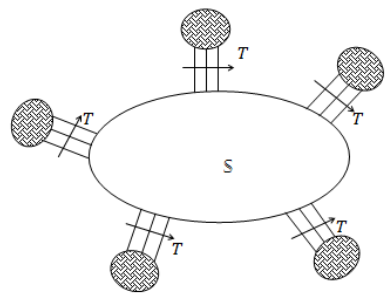


Figure 2. Spherical Picture $\mathbb{P}$ has T-disc and picture $\mathbb{S}$ has no T-disc.

then we may put the picture on Figure 1. on left side Figure 2. We apply bridge move operation to delete the inverses pair T-disc. The operation is applied until there is no T-disc in $\mathbb{P}$. So we deduce that $\pi_2(\mathcal{P}_2)$ is generated by $\boldsymbol{P} \cup \{[\mathbb{P}_T]\}$, where $\mathbb{P}_T$ is spherical picture having a T-disc joining to a picture $\mathbb{P}_T$ over $\mathcal{P}_1$.■

Proof of Theorem 1.2

Suppose that $\mathcal{P}_1 = \langle \boldsymbol{x} : \boldsymbol{r} \rangle$ is generated by $\boldsymbol{P}$. Consider that

$$\mathcal{P}_1 = \langle \boldsymbol{x} : \boldsymbol{r} \rangle \xrightarrow{T3} \mathcal{P}_2 = \langle \boldsymbol{x}, y : \boldsymbol{r}, y = S \rangle$$

is one of Tietze transformation operations. Recall that if $\mathcal{P}_1 = \langle \boldsymbol{x} : \boldsymbol{r} \rangle$ with generator $\boldsymbol{P}$ is spherical picture with labeled $\boldsymbol{r}$. By using (T3) operation is added a new generator in $\mathcal{P}_1$, say y, where y is labeled by S, so we have a new presentation, that is $\mathcal{P}_2 = \langle \boldsymbol{x}, y : \boldsymbol{r}, y = S \rangle$.

Suppose that $\boldsymbol{Q}$ is generator of $\pi_2(\mathcal{P}_2)$, but it isn't generator of $\pi_2(\mathcal{P}_1)$. So $\boldsymbol{Q}$ must have disc yS^{-1}. Since spherical picture arc y is related to a disc which is inverses pair, so we can use bridge move operation. We use this operation until there are no disc S. Therefore, generator of $\mathcal{P}_2$ is labeled by $\boldsymbol{r}$, thus we have generator of $\pi_2(\mathcal{P}_2)$ is $\boldsymbol{P}$.■

Corrolari 1. *Let $\mathcal{P}_1 = \langle \boldsymbol{x} : \boldsymbol{r}, T \rangle$ and $\mathcal{P}_2 = \langle \boldsymbol{x} : \boldsymbol{r} \rangle$ be a presentation define a group G, where T is a cyclically reduced word define and $T \sim_{\boldsymbol{r}} 1$ (relative to $\boldsymbol{r}$). Let $\pi_2(\mathcal{P}_1)$ is generated by $\boldsymbol{P}$ then $\pi_2(\mathcal{P}_2)$ is generated by all disc are labeled by T changed with a picture in $\langle \boldsymbol{x} : \boldsymbol{r} \rangle$ is labeled T.*

Corrolari 2. *Let $\mathcal{P}_1 = \langle \boldsymbol{x}, y : \boldsymbol{r}, y = S \rangle$ and $\mathcal{P}_2 = \langle \boldsymbol{x} : \boldsymbol{r} \rangle$ be a presentation define a group G, where S a word on $\boldsymbol{x}$. Let $\pi_2(\mathcal{P}_1)$ is generated by $\boldsymbol{P}$ then $\pi_2(\mathcal{P}_2)$ is generated by same pictures in $\boldsymbol{P}$ with arc y changed by arc S.*

References

[1] D.L. Johnson, Presentation of Group, Second Edition, London Mathematical Society, Student Text 15. Cambridge University Press, 1997.

[2] J. Huebschmann: Aspherical 2-complexes and an unsettled problem of J. H. C . Whitehead, Math. Ann. 258 (1981), 17-37.

[3] S.J. Pride, Identities among relations of groups presentations, in Group theory from geometrical viewpoint – Trieste 1990, World Scientific Publishing Co. Pte. Ltd., Singapore, (1991) 687-717.

[4] W.A. Bogley, , Pride, S.J.: Calculating generator of π_2, in wo-dimensional homotopy and combinatorial group theory (Hog-Angeloni, C., Metzler, W., Sieradski, A.J., eds.), *CUP*, (1993) 157-188.

[5]. W. Magnus, A. Karras, and D. Solitar, Combinatorial Group Theory : Presentations of Groups in Terms of Generator and Relations. Dover Publications, Inc. New York, 1976.

[6] Y.G. Baik, J. Harlander, S.J. Pride, , The geometry of group extensions, J. Group Theory 1 (1998) 395-416.

Received: June 20, 2013