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Ordering Generalized Trapezoidal Fuzzy Numbers

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Abstract

This paper describes a ranking method for ordering fuzzy numbers based on Area, Mode, Spreads and Weights of generalized trapezoidal fuzzy numbers. The area used in this method is obtained from the generalized trapezoidal fuzzy number, first by splitting the generalized trapezoidal fuzzy numbers into three plane figures and then calculating the centroids of each plane figure followed by the incentre of these centroids and then finding the area of this incentre from the original point. In this paper, we also apply mode and spreads in those cases where the discrimination is not possible. This method is simple in evaluation and can rank various types of trapezoidal fuzzy numbers and also crisp numbers which are considered to be a special case of fuzzy numbers.

Keywords: Ranking function; incentre ; centroid points; Generalized trapezoidal fuzzy numbers

1. Introduction

Ranking fuzzy numbers is an important tool in decision making. In fuzzy decision analysis, fuzzy quantities are used to describe the performance of alternatives in modeling a real-world problem. Most of the ranking procedures proposed so far in literature cannot discriminate fuzzy quantities and some are counter-intuitive. As fuzzy numbers are represented by possibility distributions, they may overlap with each other and hence it is not possible to order them. It is true that fuzzy numbers are frequently partial order and cannot be compared like real numbers which can be linearly ordered. In order to rank fuzzy quantities, each fuzzy quantity is converted into a real number and compared by defining a ranking function from the set of fuzzy numbers to a set of real numbers which assigns a real number to each fuzzy number where a natural order exists. Usually by reducing the whole of any analysis to a single number, much of the information is lost and hence an attempt is to be made to minimize this loss. Various ranking procedures have been developed since 1976 when the theory of fuzzy sets was first introduced by Zadeh [1]. Ranking fuzzy numbers was first proposed by Jain [2] for decision making in fuzzy situations by representing the ill-defined quantity as a fuzzy set. Since then, various procedures to rank fuzzy quantities are proposed by various researchers. Bortolan and Degani [12] reviewed some of these ranking methods [2-11] for ranking fuzzy subsets. Chen [13] presented ranking fuzzy numbers with maximizing set and minimizing set. Dubois and Prade [16] presented the mean value of a fuzzy number. Lee and Li [18] presented a comparison of fuzzy numbers based on the probability measure of fuzzy events. Delgado, Verdegay and Vila [19] presented a procedure for ranking fuzzy numbers. Campos and Munoz [20] presented a subjective approach for ranking fuzzy numbers. Kim and Park [21] presented a method of ranking fuzzy numbers with index of optimism. Yuan [22] presented a criterion for evaluating fuzzy ranking methods. Heilpern [23] presented the expected value of a fuzzy number. Saade and Schwarzlander [24] presented ordering fuzzy sets over the real line. Liou and Wang [25] presented ranking fuzzy numbers with integral value. Choobineh and Li [26] presented an index for ordering fuzzy numbers. Since then several methods have been proposed by various researchers which include ranking fuzzy numbers using area compensation, distance method, decomposition principle and signed distance [27, 28, 29]. Wang and Kerre [30, 31] classified all the above ranking procedures into three classes. The first class consists of ranking procedures based on fuzzy mean and spread [6, 7, 8, 9, 20, 25, 26, 27] and second class consists ranking procedures based on fuzzy scoring [2, 4, 10, 13, 17, 21] whereas the third class consists of methods based on preference relations [3, 5, 11, 14, 15, 19, 22, 24] and concluded that the ordering procedures associated with first class are relatively reasonable for the ordering of fuzzy numbers specially the ranking procedure presented by Adamo [7] which satisfies all the reasonable properties for the ordering of fuzzy quantities. The methods presented in the second class are not doing well and the methods [14, 15, 22, 24] which belong to class three are reasonable. Later on, ranking fuzzy numbers by preference ratio [32], left and right dominance [33], area between the centroid point and original point [34],], sign distance [36], distance minimization [37] came into existence. Later in 2007, Garcia and Lamata [38] modified the index of Liou and Wang [25] for ranking fuzzy numbers, by stating that the index of optimism is not alone sufficient to discriminate fuzzy numbers and proposed an index of modality to rank fuzzy numbers. Most of the methods presented above cannot discriminate fuzzy numbers and some methods do not agree with human intuition whereas, some methods cannot rank crisp numbers which are a special case of fuzzy numbers.

In this paper a new method is proposed which is based on Incentre of Centroids to rank fuzzy quantities. In a trapezoidal fuzzy number, first the trapezoid is split into three parts where the first, second and third parts are a triangle, a rectangle and a triangle respectively. Then the centroids of these three parts are calculated followed by the calculation of the centroid of these centroids. Finally, a ranking function is defined which is the Euclidean distance between the centroid point and the original point to rank fuzzy numbers. Most of the ranking procedures proposed in literature use Centroid of trapezoid as reference point, as the Centroid is a balancing point of the trapezoid. But the Incentre of centroids can be considered a much more balancing point than the centroid. Further, this method uses an index of optimism to reflect the decision maker's optimistic attitude and also uses an index of modality that represents the neutrality of the decision maker.

The work is organized as follows: Section 2 briefly introduces the basic concepts and ranking of fuzzy numbers. Section 3 presents the proposed new method. In Section 4 the proposed method has been explained with examples which describe the advantages and the efficiency of the method which ranks generalized fuzzy numbers, images of fuzzy numbers and even crisp numbers. In Section 5, the method demonstrates its robustness by comparing with other methods like Liou and Wang, Yager and others where the methods cannot discriminate fuzzy quantities and do not agree with human intuition. Finally, the conclusions of the work are presented in Section 6.

2. Fuzzy concepts and ranking of fuzzy numbers

2.1 Fuzzy concepts

Definition 1. Let U be a universe set. A fuzzy set $\tilde{A}$ of U is defined by a membership function $f_{\tilde{A}} : U \to [0,1]$, where $f_{\tilde{A}}(x)$ is the degree of x in $\tilde{A}$, $\forall\, x \in U$.

Definition 2. A fuzzy set $\tilde{A}$ of universe set U is normal if and only if $\sup_{x\in U} f_{\tilde{A}}(x) = 1$

Definition 3. A fuzzy set $\tilde{A}$ of universe set U is convex if and only if

$$f_{\tilde{A}}(\lambda x + (1-\lambda)y) \geq \min\left(f_{\tilde{A}}(x),\ f_{\tilde{A}}(y)\right), \forall x, y \in U \ \textit{and}\ \lambda \in [0,1].$$

Definition 4. A fuzzy set $\tilde{A}$ of universe set U is a fuzzy number iff $\tilde{A}$ is normal and convex on U.

Definition5. A real fuzzy number $\tilde{A}$ is described as any fuzzy subset of the real line $\tilde{A}$ with membership function $f_{\tilde{A}}(x)$ possessing the following properties:

(i) $f_{\tilde{A}}(x)$ is a continuous mapping from $\Re$ to the closed interval $[0, w], 0 < w \le 1$

(ii) $f_{\tilde{A}}(x)$=0, for all $x \in (-\infty, a]$

(iii) $f_{\tilde{A}}(x)$ is strictly increasing on $[a,b]$

(iv) $f_{\tilde{A}}(x)$ =1, for all $x \in [b,c]$

(v) $f_{\tilde{A}}(x)$ is strictly decreasing on $[c,d]$

(vi) $f_{\tilde{A}}(x)$ =0, for all $x \in (d,\infty]$, where *a, b, c, d* are real numbers

Definition.6.

The membership function of the real fuzzy number $\tilde{A}$ is given by

$$f_{\tilde{A}}(x) = \begin{cases} f_{\tilde{A}}^{L}, & a \le x \le b, \\ w, & b \le x \le c, \\ f_{\tilde{A}}^{R}, & c \le x \le d, \\ 0, & otherwise, \end{cases} \qquad (1)$$

where $0 < w \le 1$ is a constant, *a, b, c, d* are real numbers and $f_{\tilde{A}}^{L} : [a,b] \to [0,w]$, $f_{\tilde{A}}^{R} : [c,d] \to [0,w]$ are two strictly monotonic and continuous functions from $\Re$ to the closed interval $[0, w]$. It is customary to write a fuzzy number as $\tilde{A} = (a,b,c,d;w)$. If $w = 1$, then $\tilde{A} = (a,b,c,d;1)$ is a normalized fuzzy number, otherwise $\tilde{A}$ is said to be a generalized or non-normal fuzzy number

If the membership function $f_{\tilde{A}}(x)$ is piecewise linear, then $\tilde{A}$ is said to be a trapezoidal fuzzy number. The membership function of a trapezoidal fuzzy number is given by:

$$f_{\tilde{A}}(x)=\begin{cases}\dfrac{w(x-a)}{b-a}, & a\le x\le b,\\ w, & b\le x\le c,\\ \dfrac{w(x-d)}{c-d}, & c\le x\le d,\\ 0, & otherwise,\end{cases} \qquad (2)$$

If w=1, then $\tilde{A}=(a,b,c,d;1)$ is a normalized trapezoidal fuzzy number and $\tilde{A}$ is a generalized or non normal trapezoidal fuzzy number if $0<w<1$.

The image of $\tilde{A}=(a,b,c,d;w)$. is given by $-\tilde{A}=(-d,-c,-b,-a;w)$..
As a particular case if $b=c$, the trapezoidal fuzzy number reduces to a triangular fuzzy number given by $\tilde{A}=(a,b,d;w)$. The value of '*b*' corresponds with the mode or core and [*a*, *d*] with the support. If w=1, then $\tilde{A}=(a,b,d)$. is a normalized triangular fuzzy number $\tilde{A}$ is a generalized or non normal triangular fuzzy number if $0<w<1$.
As $f^{L}_{\tilde{A}}:[a,b]\to[0,w]$ and $f^{R}_{\tilde{A}}:[c,d]\to[0,w]$ are strictly monotonic and continuous functions, their inverse functions $g^{L}_{\tilde{A}}:[0,w]\to[a,b]$ and $g^{R}_{\tilde{A}}:[0,w]\to[c,d]$ are also continuous and strictly monotonic. Hence $g^{L}_{\tilde{A}}$ and $g^{R}_{\tilde{A}}$ are integrable on $[0,w]$.

2.2 Ranking of fuzzy numbers

2.2.1 Cheng's Ranking method

Cheng [28] ranked fuzzy numbers with the distance method using the Euclidean distance between the Centroid point and original point. For a generalized fuzzy number $\tilde{A}=(a,b,c,d;w)$.the centroid is given by:

$$(\bar{x}_0,\bar{y}_0)=\left(\frac{w\left(d^2-2c^2+2b^2-a^2+dc-ab\right)+3\left(c^2-b^2\right)}{3w\left(d-c+b-a\right)+6\left(c-b\right)},\frac{w}{3}\left(1+\frac{\left(b+c\right)-\left(a+d\right)\left(1-w\right)}{\left(b+c-a-d\right)+2\left(a+d\right)w}\right)\right)$$

and the ranking function $\tilde{A}_i > \tilde{A}_j$ associated with $\tilde{A}$ as $R(\tilde{A}) = \sqrt{\bar{x}_0^2 + \bar{y}_0^2}$

Let $\tilde{A}_i$ and $\tilde{A}_j$ two fuzzy numbers, (i) If $R(\tilde{A}_i) > R(\tilde{A}_j)$ then $\tilde{A}_i > \tilde{A}_j$.

(ii) If $R(\tilde{A}_i) < R(\tilde{A}_j)$ then $\tilde{A}_i < \tilde{A}_j$ (iii) If $R(\tilde{A}_i) = R(\tilde{A}_j)$ then $\tilde{A}_i = \tilde{A}_j$

He further improved Lee and Li's method by proposing the index of coefficient of variation (*CV*) as $CV = \frac{\sigma}{|\mu|}$ where σ is standard error and μ is mean, $\mu \neq 0$ and $\sigma > 0$, , the fuzzy number with smaller *CV* is ranked higher.

2.2.2 Wang et al. Ranking method

Wang *et al*. [35] found that the centroid formulae proposed by Cheng [28] are incorrect and have led to some misapplications such as by Chu and Tsao [34]. They presented the correct centroid formulae, for a generalized fuzzy number $\tilde{A} = (a,b,c,d;w)$. as:

$$(\bar{x}_0, \bar{y}_0) = \left(a + b + c + d - \frac{dc - ab}{(d+c) - (a+b)}, \frac{w}{3}\left(1 + \frac{c - b}{(d+c) - (a+b)} \right) \right)$$

and the ranking function associated with $\tilde{A}$ as $R(\tilde{A}) = \sqrt{\bar{x}_0^2 + \bar{y}_0^2}$

Let $\tilde{A}_i$ and $\tilde{A}_j$ two fuzzy numbers, (i) If $R(\tilde{A}_i) > R(\tilde{A}_j)$ then $\tilde{A}_i > \tilde{A}_j$

(ii) If $R(\tilde{A}_i) < R(\tilde{A}_j)$ then $\tilde{A}_i < \tilde{A}_j$ (iii) If $R(\tilde{A}_i) = R(\tilde{A}_j)$ then $\tilde{A}_i = \tilde{A}_j$

2.2.3 Liou and Wang's Ranking method

Liou and Wang [25] ranked fuzzy numbers with total integral value. For a fuzzy number defined by definition 6 the total integral value is defined as $I_T^\alpha(\tilde{A}) = \alpha I_R(\tilde{A}) + (1-\alpha) I_L(\tilde{A})$ where $I_R(\tilde{A}) = \int_0^1 g_{\tilde{A}}^R(y)dy$ and $I_L(\tilde{A}) = \int_0^1 g_{\tilde{A}}^L(y)dy$ are the right and left integral values of $\tilde{A}$, respectively and $\alpha \in [0,1]$ is the index of optimism which represents the degree of optimism of a decision maker. If $\alpha = 0$ the total integral value represents a pessimistic decision maker's view point which is equal to left integral value. If $\alpha = 1$, the total integral value represents an optimistic decision maker's view point and is equal to the right integral value and when $\alpha = 0.5$, the total integral value represents an moderate decision maker's view point and is equal to the mean of right and left integral values. For a decision maker, the larger the value of α is, the higher is the degree of optimism.

2.2.4 Garcia and Lamata's Ranking method

Garcia and Lamata [38] modified the index of Liou and Wang [25] for ranking fuzzy numbers. This method use an index of optimism to reflect the decision maker's optimistic attitude, which is not enough to discriminate fuzzy numbers, but rather it also uses an index of modality that represents the neutrality of the decision maker. For a fuzzy number defined by definition 6, Garcia and Lamata [38] proposed an index associated with the ranking as the convex combination: $I_{\beta,\alpha}\left(\tilde{A}\right) = \beta S_M\left(\tilde{A}\right) + (1-\beta) I_T^{\alpha}\left(\tilde{A}\right)$, where $S_M\left(\tilde{A}\right)$ is the area of the core of the fuzzy number which is equal to '*b*' for a triangular fuzzy number defined by $\tilde{A} = (a,b,d;w)$. and the average value of the plateau in case of a trapezoidal fuzzy number given by $\tilde{A} = (a,b,c,d;w)$., $\beta \in [0,1]$ is the index of modality that represents the importance of central value against the extreme values, $\alpha \in [0,1]$ is the degree of optimism of the decision maker and $I_T^{\alpha}\left(\tilde{A}\right)$ has its own meaning as defined in ranking method 2.2.3.

3. Proposed ranking Method

The Centroid of a trapezoid is considered as the balancing point of the trapezoid (Fig.1). Divide the trapezoid into three plane figures. These three plane figures are a triangle (APB), a rectangle (BPQC), and a triangle (CQD), respectively. Let the Centroids of the three plane figures be G_1, G_2 and G_3 respectively. The Incenter of these Centroids G_1, G_2 and G_3 is taken as the point of reference to define the ranking of generalized trapezoidal fuzzy numbers. The reason for selecting this point as a point of reference is that each Centroid point are balancing points of each individual plane figure, and the Incentre of these Centroid points is a much more balancing point for a generalized trapezoidal fuzzy number . Therefore, this point would be a better reference point than the Centroid point of the trapezoid.

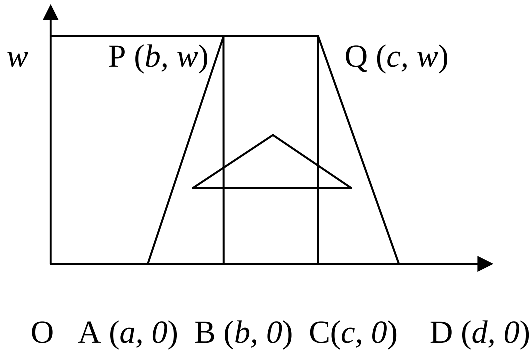

Fig. 1. Trapezoidal fuzzy number

Consider a generalized trapezoidal fuzzy number $\tilde{A} = (a,b,c,d;w)$. (Fig.1). The Centroids of the three plane figures are $G_1 = \left(\frac{a+2b}{3}, \frac{w}{3}\right)$, $G_2 = \left(\frac{b+c}{2}, \frac{w}{2}\right)$ and $G_3 = \left(\frac{2c+d}{3}, \frac{w}{3}\right)$ respectively. Equation of the line $\overleftrightarrow{G_1G_3}$ is $y = \frac{w}{3}$ and G_2 does not lie on the line $\overleftrightarrow{G_1G_3}$. Therefore, G_1 G_2 and G_3 are non-collinear and they form a triangle.

We define the Incentre $I_{\tilde{A}}(\bar{x}_0, \bar{y}_0)$ of the triangle with vertices G_1, G_2 and G_3 of the generalized trapezoidal fuzzy number $\tilde{A} = (a,b,c,d;w)$ as

$$I_{\tilde{A}}(\bar{x}_0, \bar{y}_0) = \left(\frac{\alpha\left(\frac{a+2b}{3}\right) + \beta\left(\frac{b+c}{2}\right) + \gamma\left(\frac{2c+d}{3}\right)}{\alpha+\beta+\gamma}, \frac{\alpha\left(\frac{w}{3}\right) + \beta\left(\frac{w}{2}\right) + \gamma\left(\frac{w}{3}\right)}{\alpha+\beta+\gamma} \right) \quad (3)$$

where

$$\alpha = \frac{\sqrt{(c-3b+2d)^2 + w^2}}{6} \qquad \beta = \frac{\sqrt{(2c+d-a-2b)^2}}{3} \qquad \gamma = \frac{\sqrt{(3c-2a-b)^2 + w^2}}{6}$$

As a special case, for triangular fuzzy number $\tilde{A} = (a,b,d;w)$. i.e., $c = b$ the incentre of centroids is given by

$$I_{\tilde{A}}\left(\bar{x}_0,\bar{y}_0\right)=\left(\frac{x\left(\frac{a+2b}{3}\right)+yb+z\left(\frac{2b+d}{3}\right)}{x+y+z},\frac{x\left(\frac{w}{3}\right)+y\left(\frac{w}{2}\right)+z\left(\frac{w}{3}\right)}{x+y+z}\right) \qquad (4)$$

Where

$$x=\frac{\sqrt{(2d-2b)^2+w^2}}{6} \qquad y=\frac{\sqrt{(d-a)^2}}{3} \qquad z=\frac{\sqrt{(2b-2a)^2+w^2}}{6}$$

The ranking function of the generalized trapezoidal fuzzy number $\tilde{A}=(a,b,c,d;w)$. which maps the set of all fuzzy numbers to a set of real numbers is defined as:

$$R(\tilde{A})=x_0\times y_0=\left(\frac{x\left(\frac{a+2b}{3}\right)+yb+z\left(\frac{2b+d}{3}\right)}{x+y+z}\times\frac{x\left(\frac{w}{3}\right)+y\left(\frac{w}{2}\right)+z\left(\frac{w}{3}\right)}{x+y+z}\right) \qquad (5)$$

This is the Area between the incenter of the centroids $I_{\tilde{A}}(\bar{x}_0,\bar{y}_0)$ as defined in Eq.(3) and the original point.

The Mode (m) of the generalized trapezoidal fuzzy number $\tilde{A}=(a,b,c,d;w)$. is defined as:

$$m=\frac{1}{2}\int_0^w (b+c)dx=\frac{w}{2}(b+c) \qquad (6)$$

The Spread(s) of the generalized trapezoidal fuzzy number $\tilde{A}=(a,b,c,d;w)$. is defined as:

$$s=\int_0^w (d-a)dx=w(d-a) \qquad (7)$$

The Left spread (ls)of the generalized trapezoidal fuzzy number $\tilde{A}=(a,b,c,d;w)$. is defined as:

$$ls=\int_0^w (b-a)dx=w(b-a) \qquad (8)$$

The Right spread (rs) of the generalized trapezoidal fuzzy number $\tilde{A}=(a,b,c,d;w)$. is defined as:

$$rs=\int_0^w (d-c)dx=w(d-c) \qquad (9)$$

Using the above definitions we now define the ranking procedure of two generalized trapezoidal fuzzy numbers.

Let $\tilde{A} = (a_1, b_1, c_1, d_1; w_1)$ and $\tilde{B} = (a_2, b_2, c_2, d_2; w_2)$ be two generalized trapezoidal fuzzy numbers. The working procedure to compare $\tilde{A}$ *and* $\tilde{B}$ is as follows:

Step 1: Find $R\left(\tilde{A}\right) and\, R\left(\tilde{B}\right)$

Case (i) If $R\left(\tilde{A}\right) > R\left(\tilde{B}\right)$ then $\tilde{A} > \tilde{B}$

Case (ii) If $R\left(\tilde{A}\right) < R\left(\tilde{B}\right)$ then $\tilde{A} < \tilde{B}$

Case (iii) If $R\left(\tilde{A}\right) = R\left(\tilde{B}\right)$ comparison is not possible, then go to step 2.

Step 2: Find $m\left(\tilde{A}\right)$ *and* $m\left(\tilde{B}\right)$

Case (i) If $m\left(\tilde{A}\right) > m\left(\tilde{B}\right)$ then $\tilde{A} > \tilde{B}$

Case (ii) If $m\left(\tilde{A}\right) < m\left(\tilde{B}\right)$ then $\tilde{A} < \tilde{B}$

Case (iii) If $m\left(\tilde{A}\right) = m\left(\tilde{B}\right)$ comparison is not possible, then go to step 3.

Step 3: Find $s\left(\tilde{A}\right)$ *and* $s\left(\tilde{B}\right)$

Case (i) If $s\left(\tilde{A}\right) > s\left(\tilde{B}\right)$ then $\tilde{A} < \tilde{B}$

Case (ii) If $s\left(\tilde{A}\right) < s\left(\tilde{B}\right)$ then $\tilde{A} > \tilde{B}$

Case (iii) If $s\left(\tilde{A}\right) = s\left(\tilde{B}\right)$ comparison is not possible, then go to step 4.

Step 4: Find $ls\left(\tilde{A}\right)$ *and* $ls\left(\tilde{B}\right)$

Case (i) If $ls\left(\tilde{A}\right) > ls\left(\tilde{B}\right)$ then $\tilde{A} > \tilde{B}$

Case (ii) If $ls\left(\tilde{A}\right) < ls\left(\tilde{B}\right)$ then $\tilde{A} < \tilde{B}$

Case (iii) If $ls\left(\tilde{A}\right) = ls\left(\tilde{B}\right)$ comparison is not possible, then go to step 5.

Step 5: Examine $w_1\, and\, w_2$

Case (i) If $w_1 > w_2$ then $\tilde{A} > \tilde{B}$

Case (ii) If $w_1 < w_2$ then $\tilde{A} < \tilde{B}$

Case (iii) If $w_1 = w_2$ then $\tilde{A} \approx \tilde{B}$

4. Numerical Examples

In this section, the proposed method is first explained by ranking some fuzzy numbers.

Example 4.1

Let $\tilde{A} = (3,5,7;1)$ and $\tilde{B} = \left(4,5,\frac{51}{8};1\right)$

Step 1:

Then $I_{\tilde{A}}(\bar{x}_0, \bar{y}_0) = (5.0001, 0.3877)$ and $I_{\tilde{B}}(\bar{x}_0, \bar{y}_0) = (5.0047, 0.4131)$

Therefore, $R(\tilde{A}) = 1.9385$ and $R(\tilde{B}) = 2.0674$

Since $R(\tilde{A}) < R(\tilde{B}) \Rightarrow \tilde{A} < \tilde{B}$

Example 4.2

Let $\tilde{A} = (0,1,2;1)$ and $\tilde{B} = \left(\frac{1}{5},1,\frac{7}{4};1\right)$

Step 1:

Then $I_{\tilde{A}}(\bar{x}_0, \bar{y}_0) = (0.9998, 0.4120)$ and $I_{\tilde{B}}(\bar{x}_0, \bar{y}_0) = (1.6933, 0.4094)$

Therefore, $R(\tilde{A}) = 0.4119$ and $R(\tilde{B}) = 0.6932$

Since $R(\tilde{A}) < R(\tilde{B}) \Rightarrow \tilde{A} < \tilde{B}$

Example 4.3

Let $\tilde{A} = (0,1,2;1) \Rightarrow -\tilde{A} = (-2,-1,0;1)$ and $\tilde{B} = \left(\frac{1}{5},1,\frac{7}{4};1\right) \Rightarrow -\tilde{B} = \left(-\frac{7}{4},-1,-\frac{1}{5};1\right)$

Step 1:

$I_{-\tilde{A}}(\bar{x}_0, \bar{y}_0) = (-0.9998, 0.4120)$ and $I_{-\tilde{B}}(\bar{x}_0, \bar{y}_0) = (-1.6933, 0.4094)$

Therefore, $R(-\tilde{A}) = -0.4119$ and $R(-\tilde{B}) = -0.6932$

Since $R(-\tilde{A}) > R(-\tilde{B}) \Rightarrow -\tilde{A} > -\tilde{B}$

From examples 4.2 and 4.3 we see that the proposed method can rank fuzzy numbers and their images as it is proved that $\tilde{A} < \tilde{B} \Rightarrow -\tilde{A} > -\tilde{B}$.

Example 4.4

Let $\tilde{A} = (0.1, 0.3, 0.5;1)$ and $\tilde{B} = (0.2, 0.3, 0.4;1)$

Step 1:

Then $I_{\tilde{A}}(\bar{x}_0, \bar{y}_0) = (0.2996, 0.3782)$ and $I_{\tilde{B}}(\bar{x}_0, \bar{y}_0) = (0.2995, 0.3604)$

Therefore, $R(\tilde{A}) = 0.1133$ and $R(\tilde{B}) = 0.1079$

Since $R(\tilde{A}) > R(\tilde{B}) \Rightarrow \tilde{A} > \tilde{B}$

Example 4.5

Let $\tilde{A} = (0.1, 0.3, 0.5; 0.8)$ and $\tilde{B} = (0.1, 0.3, 0.5; 1)$

Ste*p* 1:

Then $I_{\tilde{A}}(\bar{x}_0, \bar{y}_0) = (0.2995, 0.3077)$ and $I_{\tilde{B}}(\bar{x}_0, \bar{y}_0) = (0.2996, 0.3783)$

Therefore, $R(\tilde{A}) = 0.0921$ and $R(\tilde{B}) = 0.1133$

Since $R(\tilde{A}) < R(\tilde{B}) \Rightarrow \tilde{A} < \tilde{B}$

From example 4.5 it is clear that the proposed method can rank fuzzy numbers with different height and same spreads.

Example 4.6

Let $\tilde{A} = (0.1, 0.2, 0.4, 0.5; 1)$ and $\tilde{B} = (0.1, 0.3, 0.5; 1)$

Ste*p* 1:

Then $I_{\tilde{A}}(\bar{x}_0, \bar{y}_0) = (0.2997, 0.3973)$ and $I_{\tilde{B}}(\bar{x}_0, \bar{y}_0) = (0.2996, 0.3783)$

Therefore, $R(\tilde{A}) = 0.1190$ and $R(\tilde{B}) = 0.1133$

Since $R(\tilde{A}) > R(\tilde{B}) \Rightarrow \tilde{A} > \tilde{B}$

5. Results and discussion

In this section the advantages of the proposed method is shown by comparing with other existing methods in literature, where the methods cannot discriminate fuzzy numbers. The results are shown in Table 1 and Table 2.

Example 5.1

Consider two fuzzy numbers $\tilde{A} = (1, 4, 5)$ and $\tilde{B} = (2, 3, 6)$

By Liou and Wang Method [25],it is clear that the two fuzzy numbers are equal for all the decision makers as

$I_T^{\alpha}(\tilde{A}) = 4.5\alpha + (1-\alpha)2.5$ and

$I_T^{\alpha}(\tilde{B}) = 4.5\alpha + (1-\alpha)2.5$

Which is not even true by intuition.

By using our method we have

$I_{\tilde{A}}(\bar{x}_0, \bar{y}_0) = (3.9872, 0.4150)$ and $I_{\tilde{B}}(\bar{x}_0, \bar{y}_0) = (3.0127, 0.4150)$

Therefore, $R(\tilde{A}) = 1.6546$ and $R(\tilde{B}) = 1.2502 \Rightarrow \tilde{A} > \tilde{B}$

Since $R(\tilde{A}) > R(\tilde{B}) \Rightarrow \tilde{A} > \tilde{B}$

Example 5.2

Let $\tilde{A} = (0.1, 0.3, 0.5; 1)$, $\tilde{B} = (1, 1, 1, 1; 1)$

Cheng [28] ranked fuzzy numbers with the distance method using the Euclidean distance between the Centroid point and original point. Where as Chu and Tsao [34] proposed a ranking function which is the area between the centroid point and original point .Their centroid formulae are given by

$$(\bar{x}_0,\bar{y}_0)=\left(\frac{w(d^2-2c^2+2b^2-a^2+dc-ab)+3(c^2-b^2)}{3w(d-c+b-a)+6(c-b)},\frac{w}{3}\left(1+\frac{(b+c)-(a+d)(1-w)}{(b+c-a-d)+2(a+d)w}\right)\right)$$

$$(\bar{x}_0,\bar{y}_0)=\left(\frac{w(d^2-2c^2+2b^2-a^2+dc-ab)+3(c^2-b^2)}{3w(d-c+b-a)+6(c-b)},\frac{w}{3}\left(1+\frac{(b+c)}{a+b+c+d}\right)\right)$$

Both these centroid formulae cannot rank crisp numbers which are a special case of fuzzy numbers as it can be seen from the above formulae that the denominator in the first coordinate of their centroid formulae is zero, and hence centroid of crisp numbers are undefined for their formulae. By using our method, we have

$I_{\tilde{A}}(\bar{x}_0,\bar{y}_0)=(0.2996,0.3783)$ and $I_{\tilde{B}}(\bar{x}_0,\bar{y}_0)=(1,0.3331)$

Therefore, $R(\tilde{A})=0.1133$ and $R(\tilde{B})=0.3331$

Since $R(\tilde{A})<R(\tilde{B})\Rightarrow\tilde{A}<\tilde{B}$

From this example it is proved that the proposed method can rank crisp numbers whereas, other methods failed to do so.

Example 5.3

Consider four fuzzy numbers

$\tilde{A}_1=(0.1,0.2,0.3;1),\tilde{A}_2=(0.2,0.5,0.8;1),\tilde{A}_3=(0.3,0.4,0.9;1),\tilde{A}_4=(0.6,0.7,0.8;1)$

Which were ranked earlier by Yager[8], Fortemps and Roubens[27], Liou and Wang[25], and Chen and Lu [33] as shown in Table1.

Table 1. Comparison of various ranking methods

Method		$\tilde{A}$	$\tilde{A}_2$	$\tilde{A}_3$	$\tilde{A}_4$	**Ranking order**
Yager [8]		0.20	0.50	0.50	0.70	$\tilde{A}_4 > \tilde{A}_2 \approx \tilde{A}_3 > \tilde{A}_1$
Fortemps & Roubens[27]		0.20	0.50	0.50	0.70	$\tilde{A}_4 > \tilde{A}_2 \approx \tilde{A}_3 > \tilde{A}_1$
Liou & Wang[25]	$\alpha = 1$	0.25	0.65	0.65	0.75	$\tilde{A}_4 > \tilde{A}_2 \approx \tilde{A}_3 > \tilde{A}_1$
	$\alpha = 0.5$	0.20	0.50	0.50	0.70	$\tilde{A}_4 > \tilde{A}_2 \approx \tilde{A}_3 > \tilde{A}_1$
	$\alpha = 0$	0.15	0.35	0.35	0.65	$\tilde{A}_4 > \tilde{A}_2 \approx \tilde{A}_3 > \tilde{A}_1$
Chen [13]	$\beta = 1$	-0.20	0.00	0.00	-0.20	$\tilde{A}_2 \approx \tilde{A}_3 > \tilde{A}_1 \approx \tilde{A}_4$
	$\beta = 0.5$	-0.20	0.00	0.00	-0.20	$\tilde{A}_2 \approx \tilde{A}_3 > \tilde{A}_1 \approx \tilde{A}_4$
	$\beta = 0$	-0.20	0.00	0.00	-0.20	$\tilde{A}_2 \approx \tilde{A}_3 > \tilde{A}_1 \approx \tilde{A}_4$
Proposed method		0.0720	0.1948	0.1683	0.2522	$\tilde{A}_4 > \tilde{A}_2 > \tilde{A}_3 > \tilde{A}_1$

It can be seen from Table1 that none of the methods discriminates fuzzy numbers. Yager [8] and Fortemps and Roubens [27] methods failed to discriminate the fuzzy numbers $\tilde{A}_2$ and $\tilde{A}_3$, Whereas the methods of Liou and Wang[24], and Chen and Lu [33]

Cannot discriminate the fuzzy numbers $\tilde{A}_2$, $\tilde{A}_3$ and $\tilde{A}_1$, $\tilde{A}_4$

By using our method, we have

$I_{\tilde{A}1}(\bar{x}_0, \bar{y}_0) = (0.1998, 0.3605)$, $I_{\tilde{A}2}(\bar{x}_0, \bar{y}_0) = (0.4999, 0.3898)$

$I_{\tilde{A}3}(\bar{x}_0, \bar{y}_0) = (0.4337, 0.3882)$, $I_{\tilde{A}4}(\bar{x}_0, \bar{y}_0) = (0.6997, 0.3605)$

Therefore, $R(\tilde{A}_1) = 0.0720$, $R(\tilde{A}_2) = 0.1948$, $R(\tilde{A}_3) = 0.1683$, $R(\tilde{A}_4) = 0.2522$

$$\Rightarrow \tilde{A}_4 > \tilde{A}_2 > \tilde{A}_3 > \tilde{A}_1$$

Example 5.4

In this we consider seven sets of fuzzy numbers available in literature and the comparative study is presented in Table 2.

Set 1: $\tilde{A} = (0.2, 0.4, 0.6, 0.8; 0.35)$ and $\tilde{B} = (0.1, 0.2, 0.3, 0.4; 0.7)$

Set 2: $\tilde{A} = (0.1, 0.2, 0.4, 0.5; 1)$ and $\tilde{B} = (0.1, 0.3, 0.3, 0.5; 1)$

Set 3: $\tilde{A} = (0.1, 0.2, 0.4, 0.5; 1)$ and $\tilde{B} = (1, 1, 1, 1; 1)$

Set 4: $\tilde{A} = (-0.5, -0.3, -0.3 - 0.1; 1)$, and $\tilde{B} = (0.1, 0.3, 0.3, 0.5; 1)$

Set 5: $\tilde{A} = (0.3, 0.5, 0.5, 1; 1)$, and $\tilde{B} = (0.1, 0.6, 0.6, 0.8; 1)$

Set 6: $\tilde{A} = (0, 0.4, 0.6, 0.8; 1)$, $\tilde{B} = (0.2, 0.5, 0.5, 0.9; 1)$ and $\tilde{C} = (0.1, 0.6, 0.7, 0.8; 1)$,

Set 7: $\tilde{A} = (0.1, 0.2, 0.4, 0.5; 1)$, and $\tilde{B} = (-2, 0, 0, 2; 1)$

Table 2. A Comparison of the ranking results for different approaches

Methods	Set 1	Set 2	Set 3	Set 4	Set 5	Set 6	Set 7
Cheng [13]	$\tilde{A} < \tilde{B}$	$\tilde{A} \approx \tilde{B}$	**	$\tilde{A} \approx \tilde{B}$	$\tilde{A} > \tilde{B}$	$\tilde{A} < \tilde{B} < \tilde{C}$	**
Chu and Tsao [34]	$\tilde{A} < \tilde{B}$	$\tilde{A} \approx \tilde{B}$	**	$\tilde{A} < \tilde{B}$	$\tilde{A} > \tilde{B}$	$\tilde{A} < \tilde{B} < \tilde{C}$	**
Chen and Chen (2007)	$\tilde{A} < \tilde{B}$	$\tilde{A} < \tilde{B}$	$\tilde{A} < \tilde{B}$	$\tilde{A} < \tilde{B}$	$\tilde{A} > \tilde{B}$	$\tilde{A} < \tilde{C} < \tilde{B}$	$\tilde{A} > \tilde{B}$
Abbasbandy and Hajjari (2009)	Not Comparable	$\tilde{A} \approx \tilde{B}$	$\tilde{A} < \tilde{B}$	$\tilde{A} \approx \tilde{B}$	$\tilde{A} < \tilde{B}$	$\tilde{A} < \tilde{B} < \tilde{C}$	$\tilde{A} > \tilde{B}$
Chen and Chen (2009)	$\tilde{A} < \tilde{B}$	$\tilde{A} < \tilde{B}$	$\tilde{A} < \tilde{B}$	$\tilde{A} < \tilde{B}$	$\tilde{A} > \tilde{B}$	$\tilde{A} < \tilde{B} < \tilde{C}$	$\tilde{A} > \tilde{B}$
Kumar et al.[23]	$\tilde{A} < \tilde{B}$	$\tilde{A} < \tilde{B}$	$\tilde{A} < \tilde{B}$	$\tilde{A} < \tilde{B}$	$\tilde{A} > \tilde{B}$	$\tilde{A} < \tilde{B} < \tilde{C}$	$\tilde{A} > \tilde{B}$
Proposed method	$\tilde{A} < \tilde{B}$	$\tilde{A} > \tilde{B}$	$\tilde{A} < \tilde{B}$	$\tilde{A} < \tilde{B}$	$\tilde{A} < \tilde{B}$	$\tilde{A} < \tilde{B} < \tilde{C}$	$\tilde{A} > \tilde{B}$

**Not Comparable

6. Conclusions and future work

This paper proposes a method that ranks fuzzy numbers which is simple and concrete. This method ranks trapezoidal as well as triangular fuzzy numbers and their images. This method also ranks crisp numbers which are special case of fuzzy numbers whereas some methods proposed in literature cannot rank crisp numbers. This method which is simple and easier in calculation not only gives satisfactory results to well defined problems, but also gives a correct ranking order to problems. Comparative examples are used to illustrate the advantages of the proposed method. Application of this ranking procedure in various decision making problems such as, fuzzy risk analysis and in fuzzy optimization like network analysis is left as future work.

References

1. L.A. Zadeh, Fuzzy sets, Information and control 8 (1965) 338-353.
2. R. Jain, Decision making in the presence of fuzzy variables, IEEE Trans. on Sys., Man and Cybernetics 6 (1976) 698-703.
3. S.M. Bass, H. Kwakernaak, Rating and ranking of multiple-aspect alternatives using Fuzzy sets, Automatica 13 (1977) 47-58.
4. R. Jain, A procedure for multi aspect decision making using fuzzy sets. Int. J. Syst. Sci. 8 (1978) 1-7.
5. J. F. Baldwin, N.C.F Guild, Comparison of fuzzy sets on the same decision space, Fuzzy Sets and Systems 2 (1979) 213-233.
6. R.R. Yager, On choosing between fuzzy subsets, Kybernetes 9 (1980) 151-154.
7. J.M. Adamo, Fuzzy decision trees, Fuzzy Sets and Systems 4 (1980) 207-219
8. R. R. Yager, A procedure for ordering fuzzy subsets of the unit interval, Information Sciences 24 (1981) 143-161.
9. W. Chang, Ranking of fuzzy utilities with triangular membership functions, Proceedings of International Conference on Policy Analysis and Systems, 1981, pp. 263-272.

10.E. Kerre, The use of fuzzy set theory in electrocardiological diagnostics, in: M.M.Gupta, E. Sanchez (Eds.), Approximate Reasoning in Decision-Analysis, North- Holland, Amsterdam, 1982, pp. 277-282.

11.D. Dubois and H. Prade, Ranking fuzzy numbers in the setting of possibility theory, Information Sciences 30 (1983) 183-224.

12. G. Bortolan, R. Degani, A review of some methods for ranking fuzzy subsets, Fuzzy Sets and Systems 15 (1985) 1-19.

13.S.H. Chen, Ranking fuzzy numbers with maximizing set and minimizing set, Fuzzy Sets and Systems 17 (1985) 113-129.

14.W. Kolodziejczyk, Orlovsky, Concept of decision-making with fuzzy preference relation – Further results, Fuzzy Sets and Systems 19 (1986) 11-20.

15.K. Nakamura, Preference relations on a set of fuzzy utilities as a basis for decision making. Fuzzy Sets and Systems 20 (1986) 147-162.

16. D. Dubois, H. Prade, The mean value of a fuzzy number, Fuzzy Sets and Systems 24 (1987) 279-300.

17.X. Wang, A class of approaches to ordering alternatives, MSc Thesis, Taiyuan University Technology, 1987 (in Chinese).

18.E. S. Lee, R.-J. Li, Comparison of fuzzy numbers based on the probability measure of fuzzy events, Comput. Math. Applic. 15(10) (1988) 887-896.

19.M. Delgado, J.L. Verdegay, M.A. Vila, A procedure for ranking fuzzy numbers, Fuzzy Sets and Systems 26 (1988) 49-62.

20.L. Campos, A. Munoz, A subjective approach for ranking fuzzy numbers, FuzzySets and Systems 29 (1989) 145-153.

21.K. Kim, K. S. Park, Ranking fuzzy numbers with index of optimism, Fuzzy and Systems 35 (1990) 143-150.

22.Y. Yuan, Criteria for evaluating fuzzy ranking methods, Fuzzy Sets and Systems 44 (1991) 139-157.

23. S. Heilpern, The expected value of a fuzzy number, Fuzzy Sets and Systems 47 (1992) 81-86.

24. J. J. Saade, H. Schwarzlander, Ordering fuzzy sets over the real line: An approach based on decision making under uncertainty, Fuzzy Sets and Systems 50 (1992) 237-246.

25. T. S. Liou, M. J. Wang, Ranking fuzzy numbers with integral value, Fuzzy Sets and Systems 50 (1992) 247-255.

26. F. Choobineh H. Li, An index for ordering fuzzy numbers, Fuzzy Sets and Systems 54 (1993) 287-294.
27. P.Fortemps, M.Roubens, Ranking and defuzzification methods based on area compensation, Fuzzy Sets and Systems 82 (1996) 319-330.

28. C. H. Cheng, A new approach for ranking fuzzy numbers by distance method, Fuzzy Sets and Systems 95 (1998) 307-317.

29. J.S. Yao. K. Wu, Ranking fuzzy numbers based on decomposition principle and signed distance, Fuzzy Sets and Systems 116 (2000) 275-288.

30. X. Wang, E. E. Kerre, Reasonable properties for the ordering of fuzzy quantities (I), Fuzzy Sets and Systems 118 (2001) 375-385.

31. X. Wang, E. E. Kerre, Reasonable properties for the ordering of fuzzy quantities (II), Fuzzy Sets and Systems 118 (2001) 387-405.

32. M. Modarres, S. S.-Nezhad, Ranking fuzzy numbers by preference ratio, Fuzzy Sets and Systems 118 (2001) 429-436.

33. L.H. Chen, H.-W. Lu, An approximate approach for ranking fuzzy numbers based on left and right dominance, Computers and Mathematics with Applications 41 (2001) 1589-1602.

34. T.C. Chu, C.T.Tsao, Ranking fuzzy numbers with an area between the Centroid point and original point, Computers and Mathematics with Applications 43 (2002) 111-117.

35. Y.M. Wang, J. B. Yang, D.- L. Xu, K.- S. Chin, On the centroids of fuzzy numbers, Fuzzy Sets and Systems 157 (2006) 919-926.

36. S.Abbasbandy, B. Asady, Ranking of fuzzy numbers by sign distance, Information Sciences 176 (2006) 2405-2416.

37. B. Asady, A. Zendehnam, Ranking fuzzy numbers by distance minimization, Applied Mathematical Modelling 31 (2007) 2589-2598.

38. M. S. Garcia, M. T. Lamata, A modification of the index of liou and wang for ranking fuzzy number, International Journal of Uncertainty, Fuzziness and knowledge Based systems 15(4) (2007) 411-424.

39. Y.-J. Wang, H.-S. Lee, The revised method of ranking fuzzy numbers with an area between the centroid and original points, Computers and Mathematics with Applications 55 (2008) 2033-2042.

40. C.C. Chen, H.C. Tang, Ranking nonnormal p-norm trapezoidal fuzzy numbers with integral value, Computers and Mathematics with Applications 56 (2008) 2340-2346.

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