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Conditions for Hypercyclicity Criterion of a Tuple

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Abstract

In this paper we give necessary and sufficient conditions for a tuple satisfies the Hypercyclicity Criterion.

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1 Introduction

Let $\mathcal{T} = (T_1, T_2)$ be a pair of commuting continuous linear operators acting on an infinite dimensional Banach space X. We will let

$$\mathcal{F} = \{{T_1}^{k_1}{T_2}^{k_2} : k_i \geq 0, i = 1, 2\}.$$

For $x \in X$, the orbit of x under the tuple $\mathcal{T}$ is the set

$$Orb(\mathcal{T}, x) = \{Sx : S \in \mathcal{F}\}.$$

A vector x is called a hypercyclic vector for $\mathcal{T}$ if $Orb(\mathcal{T}, x)$ is dense in X and in this case the tuple $\mathcal{T}$ is called hypercyclic. Also, by $\mathcal{T}_d$ we will refer to the set

$$\mathcal{T}_d = \{S \oplus S : S \in \mathcal{F}\}.$$

We say that $\mathcal{T}_d$ is hypercyclic provided there exist $x_1, x_2 \in X$ such that

$$\{W(x_1 \oplus x_2) : W \in \mathcal{T}_d\}$$

is dense in $X \oplus X$.

It is interesting to know that many continuous linear mappings can actually be hypercyclic. The first example of a hypercyclic operator appeared in the space of entire functions, by Birkhoff [2] in 1929. He showed the hypercyclicity of the translation operator. Also, in 1952, Maclane [9] proved the hypercyclicity of the differentiation operator. An example of a hypercyclic operator on Banach spaces was constructed by Rolewicz [10] in 1969. He showed that if B is the backward shift on ℓ^p, then λB is hypercyclic if and only if $|\lambda| > 1$.

A nice criterion namely Hypercyclicity Criterion, was developed independently by Kitai [8] and, Gethner and Shapiro [7]. This criterion has been used to show that hypercyclic operators arise within the classes of composition operators [4], weighted shifts [14], adjoints of multiplication operators [5], and adjoints of subnormal and hyponormal operators [3]. The extension of this criterion for tuples has been given in [6], and we want to give necessary and sufficient conditions for a tuple on a separable Banach space satisfies this Criterion. Although the techniques work for any n-tuple of operators, but for simplicity we prove our results only for the case $n = 2$. For some topics we refer to [1–13].

2 Main Results

First consider the following extension of Hypercyclicity Criterion for tuples that has been given in ([6]).

Theorem 2.1 *(The Hypercyclicity Criterion for tuples) Suppose X is a separable infinite dimensional Banach space and $\mathcal{T} = (T_1, T_2)$ is a pair of continuous linear mappings on X. If there exist two dense subsets Y and Z in X, and a pair of strictly increasing sequences $\{m_j\}$ and $\{n_j\}$ such that :*
1. $T_1^{m_j} T_2^{n_j} \to 0$ on Y as $j \to \infty$,
2. There exists a sequence of function $\{S_j : Z \to X\}$ such that for every $z \in Z$, $S_j z \to 0$, and

$$T_1^{m_j} T_2^{n_j} S_j z \to \ z,$$

then T is a hypercyclic tuple.

Following theorem states the Hypercyclicity Criterion in terms of open sets.

Theorem 2.2 *Let X be a separable infinite dimensional Banach space and $\mathcal{T} = (T_1, T_2)$ be a pair of operators T_1, T_2. Then the followings conditions are equivalent:*

$i)$ $\mathcal{T}$ satisfies the Hypercyclicity Criterion.

ii) $\mathcal{T}$ is hypercyclic and for each nonempty open subset U and each neighborhood W of zero,

$$T_1^m T_2^n U \cap W \neq \phi$$

and

$$T_1^{-m} T_2^{-n} U \cap W \neq \phi$$

for some integers m, n.

iii) For each pair U and V of non-void open subsets of X, and each neighborhood W of zero,

$$T_1^m T_2^n U \cap W \neq \phi$$

and

$$T_1^m T_2^n W \cap V \neq \phi$$

for some integers m, n.

Proof. It is easy to see that (i) implies (ii). Suppose that T satisfies the condition (ii), U and V are nonempty open subsets of X and W is a neighborhood of zero. Since $\mathcal{T}$ is hypercyclic, we have

$$U \cap T_1^{-m} T_2^{-n} V \neq \phi$$

for some positive integers m, n. Now let G be a neighborhood of zero that is contained in

$$W \cap T_1^{-m} T_2^{-n} W.$$

By condition (ii), there exist some positive integer i, j such that

$$T_1^{-i} T_2^{-j} G \cap (U \cap T_1^{-m} T_2^{-n} V) \neq \phi$$

and

$$G \cap T_1^{-i} T_2^{-j} (U \cap T_1^{-m} T_2^{-n} V) \neq \phi.$$

But

$$T_1^{-i} T_2^{-j} G \cap (U \cap T_1^{-m} T_2^{-n} V)$$

is a subset of

$$T_1^{-i} T_2^{-j} W \cap U,$$

hence

$$T_1^{-i} T_2^{-j} W \cap U \neq \phi.$$

Also

$$G \cap T_1^{-i} T_2^{-j} (U \cap T_1^{-m} T_2^{-n} V)$$

is a subset of

$$T_1^{-m} T_2^{-n} W \cap T_1^{-i} T_2^{-j} (T_1^{-m} T_2^{-n} V) = T_1^{-m} T_2^{-n} (W \cap T_1^{-i} T_2^{-j} V),$$

thus

$$T_1^{-i}T_2^{-j}V \cap W \neq \phi$$

which satisfies the condition (iii). Now we prove that (iii) implies (i). It is sufficient to prove that $\mathcal{T}_d$ is hypercyclic. For this consider four arbitrary open subsets U_i and V_i for $i = 1, 2$. There exist open subsets $\hat{U}_i$ and $\hat{V}_i$ for $i = 1, 2$, and a neighborhood $W_\circ$ of zero such that :

$$\hat{U}_i + W_\circ \subseteq U_i; \quad \hat{V}_i + W_\circ \subseteq V_i; \quad i = 1, 2.$$

Note that condition (iii) implies that $\mathcal{T}$ is hypercyclic. Hence there exist integers p_1, q_1, p_2, q_2 such that:

$$G_1 = \hat{U}_1 \cap T_1^{-p_1}T_2^{-q_1}\hat{V}_1 \neq \phi$$

and

$$G_2 = \hat{U}_2 \cap T_1^{-p_2}T_2^{-q_2}\hat{V}_2 \neq \phi.$$

Put

$$W = W_\circ \cap T_1^{-p_1}T_2^{-q_1}W_\circ \cap T_1^{-p_2}T_2^{-q_2}W_\circ.$$

Now by condition (iii) there are integers m, n such that:

$$T_1^m T_2^n G_1 \cap W \neq \phi$$

and

$$T_1^m T_2^n W \cap G_2 \neq \phi.$$

Choose the vectors $x_\circ$ and $y_\circ$ in X such that

$$x_\circ \in \hat{U}_1, \; T_1^{p_1}T_2^{q_1}x_\circ \in \hat{V}_1 \; T_1^m T_2^n x_\circ \in W,$$

and

$$y_\circ \in W, \; T_1^m T_2^n y_\circ \in \hat{U}_2, \; T_1^{m+p_2}T_2^{n+q_2}y_\circ \in \hat{V}_2.$$

Put $x = x_\circ + y_\circ$ and

$$y = T_1^{p_1}T_2^{q_1}x_\circ + T_1^{p_2}T_2^{q_2}y_\circ.$$

Then $x \oplus y \in U_1 \oplus V_1$ and

$$(T_1^m T_2^n \oplus T_1^m T_2^n)(x \oplus y) \in U_2 \oplus V_2.$$

So $\mathcal{T}_d^{(2)}$ is hypercyclic and the proof is complete.

Definition 2.3 *We say that a pair $\mathcal{T} = (T_1, T_2)$ is semi-hereditarily hypercyclic with respect to a pair of nonnegative increasing sequences $(\{m_k\}, \{n_r\})$ of integers provided for all pair of subsequences $(\{m_{k_i}\}, \{n_{r_j}\})$ of $(\{m_k\}, \{n_r\})$, the sequence*

$$\{T_1{}^{m_{k_i}}T_2{}^{n_{r_j}} : i, j \geq 1\}$$

is hypercyclic.

The following theorem gives a necessary and sufficient condition (in terms of open sets) for Tuples being hereditarily hypercyclic with respect to a a pair of nonnegative increasing sequences of integers.

Theorem 2.4 *Let $\mathcal{T} = (T_1, T_2)$ be a pair of operators acting on a separable infinite dimensional Banach space X. Then the followings are equivalent:*

$i)$ $\mathcal{T}$ is hereditarily hypercyclic with respect to the pair $(\{m_k\}, \{n_k\})$.

$ii)$ For each pair U, V of non-void open subsets of X, there is $N \geq 1$ such that

$$T_1^{m_k} T_2^{n_k} U \cap V \neq \phi$$

for any $k \geq N$.

Proof. $(i) \rightarrow (ii)$: Suppose that the condition (ii) does not hold. So there exist some pair U, V of nonempty open sets such that

$$T_1^{m_{k_j}} T_2^{n_{k_j}} U \cap V = \phi$$

for a pair of subsequences $(\{m_{k_j}\}, \{n_{k_j}\})$ of $(\{m_k\}, \{n_k\})$. This is a contradiction since $\{T_1^{m_{k_j}} T_2^{n_{k_j}}\}$ is hypercyclic. The assertion $(ii) \rightarrow (i)$ is clear.

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