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Cyclic Direct Sum of Tuples

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Abstract

In this paper we characterize conditions for the direct sum of a tuple of operators to be cyclic.

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1 Introduction

Let $\mathcal{T} = (T_1, T_2)$ be a pair of commuting continuous linear operators acting on an infinite dimensional Banach space X. We will let

$$\mathcal{F} = \{{T_1}^{k_1}{T_2}^{k_2} : k_i \geq 0, i = 1, 2\}.$$

For $x \in X$, the orbit of x under the tuple $\mathcal{T}$ is the set

$$Orb(\mathcal{T}, x) = \{Sx : S \in \mathcal{F}\}.$$

A vector x is called a hypercyclic vector for $\mathcal{T}$ if $Orb(\mathcal{T}, x)$ is dense in X and in this case the tuple $\mathcal{T}$ is called hypercyclic. Also, by $\mathcal{T}_d$ we will refer to the set

$$\mathcal{T}_d = \{S \oplus S : S \in \mathcal{F}\}.$$

We say that $\mathcal{T}_d$ is hypercyclic provided there exist $x_1, x_2 \in X$ such that

$$\{W(x_1 \oplus x_2) : W \in \mathcal{T}_d\}$$

is dense in $X \oplus X$. Similarly, A vector x is called a cyclic vector for $\mathcal{T}$ if the linear span of the orbit $Orb(\mathcal{T}, x)$ is dense in X and in this case the tuple $\mathcal{T}$ is called cyclic. By a polynomial $p(.,.)$ we will mean

$$p(z, w) = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} z^i w^j.$$

Let

$$\mathcal{P} = \{p(T_1, T_2) : p(.,.) \text{ is a polynomial}\},$$

and note that the linear span of the orbit $Orb(\mathcal{T}, x)$ is equal to $\{Sx : S \in \mathcal{P}\}$. Thus the tuple $\mathcal{T}$ is cyclic if $\{Sx : S \in \mathcal{P}\}$ is dense in X. Also, let

$$\mathcal{P}(\mathcal{T})_d = \{S \oplus S : S \in \mathcal{P}\}.$$

We say that $\mathcal{P}(\mathcal{T})_d$ is cyclic if there exist $x_1, x_2 \in X$ such that

$$\{W(x_1 \oplus x_2) : W \in \mathcal{P}(\mathcal{T})_d\}$$

is dense in $X \oplus X$.

Hypercyclic operators arise within the class of composition operators ([4]), weighted shifts ([12]), adjoints of multiplication operators ([5]), and adjoints of subnormal and hyponormal operators ([2]), and hereditarily operators ([1]), and topologically mixing operators ([7, 11]). Here, we want to extend some properties of hypercyclic and cyclic operators to a pair of commuting operators, and although the techniques work for any n-tuple of operators but for simplicity we prove our results only for the case $n = 2$. For some other topics we refer to [1–16].

2 Main Results

In this section we characterize some conditions for the direct sum of a pair of operators to be cyclic. We will denote the collection of hypercyclic and cyclic vectors of a pair $\mathcal{T}$ by $HC(\mathcal{T})$ and $C(\mathcal{T})$, respectively.

Lemma 2.1 *Let X be a separable infinite dimensional Banach space and $\mathcal{T} = (T_1, T_2)$ be a cyclic pair of operators T_1 and T_2. If T_i^* has no eigenvalues for $i = 1, 2$, then $C(\mathcal{T})$ is dense.*

Proof. Note that if x is a cyclic vector for $\mathcal{T}$, then for all polynomials $p(.,.)$ we have:

$$\begin{aligned} span(Orb(\mathcal{T}, p(T_1, T_2)x)) &= \{q(T_1, T_2)p(T_1, T_2)x : q(.,.) \text{ is a polynomial}\} \\ &= p(T_1, T_2)\{q(T_1, T_2)x : q(.,.) \text{ is a polynomial}\}. \end{aligned}$$

Since T_i^* has no eigenvalues for $i = 1, 2$, thus $p(T_1, T_2)$ has dense range and so the linear span of $Orb(\mathcal{T}, p(T_1, T_2)x)$ is dense in X. Hence,

$$p(T_1, T_2)x \in C(\mathcal{T})$$

for all polynomials $p(.,.)$ and so

$$span(Orb(\mathcal{T}, x)) \subset C(\mathcal{T}).$$

This implies that $C(\mathcal{T})$ is dense in X.

Theorem 2.2 *Let X be a separable infinite dimensional Banach space and $\mathcal{T} = (T_1, T_2)$ be the pair of operators T_1 and T_2 and T_i^* has no eigenvalues for $i = 1, 2$. If $\mathcal{T}$ is cyclic, then for any nonempty open subsets U, V of X, there exists a polynomial $q(.,.)$ such that the set $q(T_1, T_2)(U) \cap V$ is nonempty.*

Proof. Since $C(\mathcal{T})$ is dense, thus the set $U \cap C(\mathcal{T})$ is nonempty. Choose $x \in U \cap C(\mathcal{T})$. Since x is a cyclic vector,

$$\{p(T_1, T_2)x : p(.,.) \textit{ is a polynomial}\}$$

is dense in X and so

$$V \cap \{p(T_1, T_2)x : p(.,.) \textit{ is a polynomial}\}$$

is nonempty. Thus there exists a polynomial $q(.,.)$ such that

$$q(T_1, T_2)x \in V \cap C(\mathcal{T}),$$

from which we can conclude that $q(T_1, T_2)(U) \cap V$ is nonempty.

Theorem 2.3 *Let X be a separable infinite dimensional Banach space and $\mathcal{T} = (T_1, T_2)$ be the pair of operators T_1 and T_2 and T_i^* has no eigenvalues for $i = 1, 2$. If $\mathcal{P}(\mathcal{T})_d$ is cyclic, then for any nonempty open subsets U_1, U_2, $V1$, V_2 of X, there exists a polynomial $q(.,.)$ such that the sets $q(T_1, T_2)(U_1) \cap V_1$ and $q(T_1, T_2)(U_2) \cap V_2$ are nonempty.*

Proof. By a similar method used in Lemma 2.1, one can see that $\mathcal{P}(\mathcal{T})_d$ is dense in $X \oplus X$. Now, by the technique used in the proof of Theorem 2.2, the proof is complete.

Theorem 2.4 *Let X be a separable infinite dimensional Banach space and $\mathcal{T} = (T_1, T_2)$ be a hypercyclic pair of operators T_1, T_2, and T_i^* has no eigenvalues for $i = 1, 2$. If for every nonempty open subsets U, V of X and every neighborhood W of 0, there exists a polynomial $q(.,.)$ such that*

$$q(T_1, T_2)(U) \cap W \neq \emptyset$$

and

$$q(T_1, T_2)(W) \cap V \neq \emptyset,$$

then $\mathcal{T}_d$ is hypercyclic.

Proof. It is sufficient to show that for every nonempty open subsets U, V of X and every neighborhood W of 0, there exist integers m and n such that

$$T_1^m T_2^n(U) \cap W \neq \emptyset$$

and

$$T_1^m T_2^n (W) \cap V \neq \emptyset.$$

So let U, V be any nonempty open subsets of X and W be a neighborhood of 0. Let $q(.,.)$ be the polynomial such that

$$q(T_1, T_2)(U) \cap W \neq \emptyset$$

and

$$q(T_1, T_2)(W) \cap V \neq \emptyset.$$

Since $\mathcal{T}$ is hypercyclic, thus $HC(\mathcal{T}$ is dense in X and so

$$HC(\mathcal{T}) \cap (q(T_1, T_2)^{-1}(W) \cap U) \neq \emptyset.$$

Choose

$$x \in HC(\mathcal{T}) \cap (q(T_1, T_2)^{-1}(W) \cap U),$$

thus $x \in U$ and $q(T_1, T_2)x \in W$. Since

$$q(T_1, T_2)^{-1} V \cap W \neq \emptyset$$

and it is also open, we get

$$(q(T_1, T_2)^{-1} V \cap W) \cap Orb(\mathcal{T}, x) \neq \emptyset.$$

Hence there exist integers m and n such that

$$T_1^m T_2^n x \in q(T_1, T_2)^{-1} V \cap W.$$

Since $x \in U$ and $T_1^m T_2^n x \in W$, so

$$T_1^m T_2^n (U) \cap W \neq \emptyset.$$

Also,

$$q(T_1, T_2) T_1^m T_2^n x \in V$$

and

$$q(T_1, T_2)x \in W,$$

thus

$$T_1^m T_2^n (W) \cap V \neq \emptyset.$$

Now the proof is complete.

Corollary 2.5 *Let X be a separable infinite dimensional Banach space and $\mathcal{T} = (T_1, T_2)$ be a hypercyclic pair of operators T_1, T_2, and T_i^* has no eigenvalues for $i = 1, 2$. Then for every nonempty open subsets U, V of X and every neighborhood W of 0, there exists a polynomial $q(.,.)$ such that*

$$q(T_1, T_2)(U) \cap W \neq \emptyset$$

and

$$q(T_1, T_2)(W) \cap V \neq \emptyset,$$

if and only if $\mathcal{P}(\mathcal{T})_d$ is cyclic.

Corollary 2.6 *Let X be a separable infinite dimensional Banach space and $\mathcal{T} = (T_1, T_2)$ be a hypercyclic pair of operators T_1, T_2, and T_i^* has no eigenvalues for $i = 1, 2$. Then $\mathcal{T}_d$ is hypercyclic if and only if $\mathcal{P}(\mathcal{T})_d$ is cyclic.*

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