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On the Norms of Toeplitz Matrices Involving k-Fibonacci and k-Lucas Numbers

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Abstract

In this paper, we give upper and lower bounds for the spectral norms of Toeplitz matrices $A = [F_{k,i-j}]_{i,j=1}^n$ and $B = [L_{k,i-j}]_{i,j=1}^n$, where $F_{k,n}$ and $L_{k,n}$ are the k-Fibonacci and k-Lucas numbers, then obtain some bounds for the spectral norms of Hadamard and Kronecker products of these matrices.

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1 Introduction and Preliminaries

For $n > 0$, let k be any positive real number, then the k-Fibonacci sequence $\{F_{k,n}\}_{n\in N}$ and the k-Lucas sequence $\{L_{k,n}\}_{n\in N}$[1-3] are defined respectively by the following equations:

$$F_{k,n+1} = kF_{k,n} + F_{k,n-1}, \quad F_{k,0} = 0, \quad F_{k,1} = 1 \tag{1}$$

$$L_{k,n+1} = kL_{k,n} + L_{k,n-1}, \quad L_{k,0} = 2, \quad L_{k,1} = k \tag{2}$$

The rules (1) and (2) can be used to extend the sequences backwards, thus

$$F_{k,-1} = F_{k,1} - kF_{k,0}, \quad F_{k,-2} = F_{k,0} - kF_{k,-1}, \quad \cdots$$
$$L_{k,-1} = L_{k,1} - kL_{k,0}, \quad L_{k,-2} = L_{k,0} - kL_{k,-1}, \quad \cdots$$

These produce

n	0	1	2	3	4	5	$\cdots$
$F_{k,n}$	0	1	k	k^2+1	k^3+2k	k^4+3k^2+1	$\cdots$
$F_{k,-n}$	0	1	$-k$	k^2+1	$-k^3-2k$	k^4+3k^2+1	$\cdots$
$L_{k,n}$	2	k	k^2+2	k^3+3k	k^4+4k^2+2	k^5+5k^3+5k	$\cdots$
$L_{k,-n}$	2	$-k$	k^2+2	$-k^3-3k$	k^4+4k^2+2	$-k^5-5k^3-5k$	$\cdots$

and generally, we have

$$F_{k,-n} = (-1)^{n+1} F_{k,n}, \quad L_{k,-n} = (-1)^n L_{k,n}. \tag{3}$$

Recently, there have been several papers on the norms of some special matrices [4-9]. For example, Akbulak and Bozkurt [4] have found lower and upper bounds for the spectral norms of Toeplitz matrices $A = [F_{i-j}]_{i,j=1}^n$ and $B = [L_{i-j}]_{i,j=1}^n$. Solak [6,7] has defined $A = [a_{ij}]$ and $B = [b_{ij}]$ as $n \times n$ circulant matrices, where $a_{ij} \equiv F_{(mod(j-i,n))}$ and $b_{ij} \equiv L_{(mod(j-i,n))}$, then he has given some bounds for the A and B matrices concerned with the spectral and Euclidean norms. Shen and Cen [8] have generalized these results. In addition [9], they also have established upper and lower bounds for the spectral norms of r-circulant matrices $\mathcal{A} = C_r(F_{k,0}, F_{k,1}, \cdots, F_{k,n-1})$ and $\mathcal{B} = C_r(L_{k,0}, L_{k,1}, \cdots, L_{k,n-1})$, then have obtained some bounds for the spectral norms of Hadamard and Kronecker products of these matrices.

In this paper, let $A = [F_{k,i-j}]_{i,j=1}^n$ and $B = [L_{k,i-j}]_{i,j=1}^n$ be Toeplitz matrices, Afterwards, we give upper and lower bounds for the spectral norms of matrices A and B. In addition, we also obtain some bounds for the spectral norms of Hadamard and Kronecker products of these matrices.

Now we give some preliminaries related to our study. Let $\{t_n\}_{n=-\infty}^{\infty}$ be a doubly infinite sequence, A matrix $T = [t_{ij}] \in M_n$ is called a Toeplitz matrix if it is of the form $t_{ij} = t_{i-j}$ for $i, j = 1, 2, \cdots, n$.

For any $A = [a_{ij}] \in M_{m,n}$. The well-known Frobenius (or Euclidean) norm of matrix A is

$$\|A\|_F = \Big(\sum_{i=1}^m \sum_{j=1}^n |a_{ij}|^2 \Big)^{\frac{1}{2}}$$

and also the spectral norm of matrix A is

$$\|A\|_2 = \sqrt{\max_{1 \le i \le n} \lambda_i(A^H A)}$$

where $\lambda_i(A^H A)$ is eigenvalue of $A^H A$ and A^H is conjugate transpose of matrix A. Then the following inequality holds:

$$\frac{1}{\sqrt{n}} \|A\|_F \le \|A\|_2 \le \|A\|_F \tag{4}$$

Lemma 1[10] *For any $A, B \in M_{m,n}$, we have*

$$\|A \circ B\|_2 \le \|A\|_2 \|B\|_2$$

where $A \circ B$ is the Hadamard product of A and B.

Lemma 2[10] *Let $A \in M_{m,n}$, $B \in M_{p,q}$ be given, then we have*

$$\|A \otimes B\|_2 = \|A\|_2 \|B\|_2$$

where $A \otimes B$ is the Kronecker product of A and B.

Lemma 3 *For the sequences $\{F_{k,n}\}_{n\in N}$ and $\{L_{k,n}\}_{n\in N}$, then we have the following formulas*

$$\sum_{i=0}^{n-1} F_{k,i}F_{k,i+1} = \begin{cases} \frac{F_{k,n}^2-1}{k}, & n\ odd \\ \frac{F_{k,n}^2}{k}, & n\ even \end{cases}, \tag{5}$$

$$\sum_{i=0}^{n-1} L_{k,i}L_{k,i+1} = \begin{cases} \frac{L_{k,n}^2}{k}+k, & n\ odd \\ \frac{L_{k,n}^2-4}{k}, & n\ even \end{cases}. \tag{6}$$

Proof: Since

$$\begin{aligned} F_{k,i}F_{k,i+1} + F_{k,i+1}F_{k,i+2} &= F_{k,i+1}(F_{k,i+2}+F_{k,i}) \\ &= \frac{1}{k}(F_{k,i+2}^2 - F_{k,i}^2), \quad i = 0, 1, \cdots, n-2. \end{aligned}$$

hence, we obtain

$$\sum_{i=0}^{n-1} F_{k,i}F_{k,i+1} = \begin{cases} \frac{F_{k,n}^2-1}{k}, & n\ odd \\ \frac{F_{k,n}^2}{k}, & n\ even \end{cases}.$$

Similarly, we can verify (6). Thus, the proof is completed.

2 Main Results

Theorem 1 *Let $A = [F_{k,i-j}]_{i,j=1}^n$ be Toeplitz matrix, then we have*

$$\begin{cases} \frac{1}{k}\sqrt{\frac{2(F_{k,n}^2-1)}{n}}, & n\ odd \\ \frac{1}{k}\sqrt{\frac{2F_{k,n}^2}{n}}, & n\ even \end{cases} \le \|A\|_2 \le \frac{2(F_{k,n}+F_{k,n-1}-1)}{k}.$$

Proof: The matrix A is of the form

$$A = \begin{pmatrix} F_{k,0} & F_{k,-1} & F_{k,-2} & \cdots & F_{k,1-n} \\ F_{k,1} & F_{k,0} & F_{k,-1} & \cdots & F_{k,2-n} \\ F_{k,2} & F_{k,1} & F_{k,0} & \cdots & F_{k,3-n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ F_{k,n-1} & F_{k,n-2} & F_{k,n-3} & \cdots & F_{k,0} \end{pmatrix}$$

then we have

$$\begin{aligned} \|A\|_F^2 &= nF_{k,0}^2 + \sum_{i=1}^{n-1}(n-i)F_{k,i}^2 + \sum_{i=1}^{n-1}(n-i)F_{k,-i}^2 = 2\sum_{i=1}^{n-1}\sum_{j=1}^{i} F_{k,j}^2 \\ &= \frac{2}{k}\sum_{i=1}^{n-1} F_{k,i}F_{k,i+1} = \begin{cases} \frac{2(F_{k,n}^2-1)}{k^2}, & n\ odd \\ \frac{2F_{k,n}^2}{k^2}, & n\ even \end{cases}, \end{aligned}$$

hence from (4), we obtain

$$\|A\|_2 \geq \frac{1}{\sqrt{n}}\|A\|_F = \begin{cases} \frac{1}{k}\sqrt{\frac{2(F_{k,n}^2-1)}{n}}, & n\ odd \\ \frac{1}{k}\sqrt{\frac{2F_{k,n}^2}{n}}, & n\ even \end{cases}.$$

On the other hand, let matrices $\mathcal{P} \in M_n$ and $\mathcal{Q} \in M_n$ be as

$$\mathcal{P} = \begin{pmatrix} 0 & 0 \\ I_{n-1} & 0 \end{pmatrix},\ \mathcal{Q} = \begin{pmatrix} 0 & I_{n-1} \\ 0 & 0 \end{pmatrix}$$

where I_{n-1} is the $(n-1)\times(n-1)$ identity matrix, then we have

$$A = \sum_{i=0}^{n-1} F_{k,i}\mathcal{P}^i + \sum_{i=1}^{n-1} F_{k,-i}\mathcal{Q}^i$$

From the properties of matrix norm, we obtain

$$\|A\|_2 \leq \|\sum_{i=0}^{n-1} F_{k,i}\mathcal{P}^i\|_2 + \|\sum_{i=1}^{n-1} F_{k,-i}\mathcal{Q}^i\|_2 \leq \sum_{i=0}^{n-1} F_{k,i}\|\mathcal{P}^i\|_2 + \sum_{i=1}^{n-1} F_{k,i}\|\mathcal{Q}^i\|_2$$

since

$$\|\mathcal{P}^i\|_2 = \sqrt{\max_{1\leq k\leq n} \lambda_k[(\mathcal{P}^i)^H\mathcal{P}^i]} = 1,$$

$$\|\mathcal{Q}^i\|_2 = \sqrt{\max_{1\leq k\leq n} \lambda_k[(\mathcal{Q}^i)^H\mathcal{Q}^i]} = 1, \quad i = 0, 1, \cdots, n-1.$$

hence

$$\|A\|_2 \leq 2\sum_{i=1}^{n-1} F_{k,i} = \frac{2(F_{k,n} + F_{k,n-1} - 1)}{k}.$$

Thus, the proof is completed.

Theorem 2 *Let $B = [L_{k,i-j}]_{i,j=1}^n$ be Toeplitz matrix, then we have*

$$\begin{cases} \frac{1}{k}\sqrt{\frac{2(L_{k,n}^2+k^2)}{n}}, & n\ odd \\ \frac{1}{k}\sqrt{\frac{2(L_{k,n}^2-4)}{n}}, & n\ even \end{cases} \leq \|B\|_2 \leq \frac{2(L_{k,n} + L_{k,n-1} - 2)}{k}.$$

Proof: The matrix B is of the form

$$B = \begin{pmatrix} L_{k,0} & L_{k,-1} & L_{k,-2} & \cdots & L_{k,1-n} \\ L_{k,1} & L_{k,0} & L_{k,-1} & \cdots & L_{k,2-n} \\ L_{k,2} & L_{k,1} & L_{k,0} & \cdots & L_{k,3-n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ L_{k,n-1} & L_{k,n-2} & L_{k,n-3} & \cdots & L_{k,0} \end{pmatrix}$$

then we have

$$\begin{aligned}\|B\|_F^2 &= nL_{k,0}^2+\sum_{i=1}^{n-1}(n-i)L_{k,i}^2+\sum_{i=1}^{n-1}(n-i)L_{k,-i}^2=4n+2\sum_{i=1}^{n-1}\sum_{j=1}^{i}L_{k,j}^2\\ &= 4n+2\sum_{i=1}^{n-1}\left(\frac{L_{k,i}L_{k,i+1}}{k}-2\right)=\frac{2}{k}\sum_{i=0}^{n-1}L_{k,i}L_{k,i+1}\\ &= \begin{cases}\frac{2(L_{k,n}^2+k^2)}{k^2}, & n \text{ odd}\\ \frac{2(L_{k,n}^2-4)}{k^2}, & n \text{ even}\end{cases},\end{aligned}$$

hence from (4), we obtain

$$\|B\|_2 \geq \frac{1}{\sqrt{n}}\|B\|_F = \begin{cases}\frac{1}{k}\sqrt{\frac{2(L_{k,n}^2+k^2)}{n}}, & n \text{ odd}\\ \frac{1}{k}\sqrt{\frac{2(L_{k,n}^2-4)}{n}}, & n \text{ even}\end{cases}.$$

On the other hand, let matrices $\mathcal{P}\in M_n$ and $\mathcal{Q}\in M_n$ be as

$$\mathcal{P}=\begin{pmatrix}0 & 0\\ I_{n-1} & 0\end{pmatrix},\ \mathcal{Q}=\begin{pmatrix}0 & I_{n-1}\\ 0 & 0\end{pmatrix}$$

where I_{n-1} is the $(n-1)\times(n-1)$ identity matrix, then we have

$$B=\sum_{i=0}^{n-1}L_{k,i}\mathcal{P}^i+\sum_{i=1}^{n-1}L_{k,-i}\mathcal{Q}^i$$

Since $\|\mathcal{P}^i\|_2=\|\mathcal{Q}^i\|_2=1$ for $i=0,1,\cdots,n-1$, hence

$$\begin{aligned}\|B\|_2 &\leq \|\sum_{i=0}^{n-1}L_{k,i}\mathcal{P}^i\|_2+\|\sum_{i=1}^{n-1}L_{k,-i}\mathcal{Q}^i\|_2\leq\sum_{i=0}^{n-1}L_{k,i}\|\mathcal{P}^i\|_2+\sum_{i=1}^{n-1}L_{k,i}\|\mathcal{Q}^i\|_2\\ &= 2\sum_{i=0}^{n-1}L_{k,i}-2=\frac{2(L_{k,n}+L_{k,n-1}-2)}{k}.\end{aligned}$$

Thus, the proof is completed.

Considering the results of Theorem 1 and Theorem 2, then we obtain the following important results.

Corollary 1 *Let $A=[F_{k,i-j}]_{i,j=1}^n$ and $B=[L_{k,i-j}]_{i,j=1}^n$ be Toeplitz matrices, then we have*

$$\|A\circ B\|_2 \leq \frac{4(F_{k,n}+F_{k,n-1}-1)(L_{k,n}+L_{k,n-1}-2)}{k^2}.$$

Proof: Since $\|A\circ B\|_2\leq\|A\|_2\|B\|_2$, the proof is trivial by Theorems 1 and 2.

Corollary 2 *Let $A = [F_{k,i-j}]_{i,j=1}^n$ and $B = [L_{k,i-j}]_{i,j=1}^n$ be Toeplitz matrices, then we have*

$$\|A \otimes B\|_2 \leq \frac{4(F_{k,n} + F_{k,n-1} - 1)(L_{k,n} + L_{k,n-1} - 2)}{k^2},$$

and

$$\|A \otimes B\|_2 \geq \begin{cases} \frac{2}{nk^2}\sqrt{(F_{k,n}^2 - 1)(L_{k,n}^2 + k^2)}, & n \text{ odd} \\ \frac{2F_{k,n}}{nk^2}\sqrt{L_{k,n}^2 - 4}, & n \text{ even} \end{cases}.$$

Proof: Since $\|A \otimes B\|_2 = \|A\|_2 \|B\|_2$, the proof is trivial by Theorems 1 and 2.

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