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Linear Star Decomposition of Lobster

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Abstract

Let G = (V, E) be a simple connected graph with p vertices and q edges. If $G_1, G_2, \ldots, G_n$ are connected edge disjoint subgraphs of G with $E(G)=E(G_1)\cup E(G_2)\cup \ldots \cup E(G_n)$, then $(G_1, G_2, \ldots, G_n)$ is said to be a decomposition of G. A decomposition $(G_1, G_2, \ldots, G_n)$ of G is said to be continuous monotonic decomposition(CMD) if each G_i is connected and $|E(G_i)|=i$, for every i=1, 2, 3, …,n. In this paper, we introduced the new concept Linear Star Decomposition. A decomposition $(G_1, G_2, \ldots, G_n)$of G is said to be a Linear Decomposition (LD) or Arithmetic decomposition if $|E(G_i)| = a+ (i-1)d$, for every i=1, 2, 3, …, n and a, d$\in$Z. Clearly $q = \frac{n}{2}\left[2a+(n-1)d\right]$. If a=1 and d=1, then $q = \frac{n(n+1)}{2}$. That is, LD is a CMD. In this paper, we study the graphs when a=1 and d=2. If d=2, then $q = n^2$. That is, the number of edges of G is a perfect square. Also we obtained the bound for diam (L) where L is a Lobster, which is the graph, discussed in this paper and discussed several theorems based on diam (L).

Mathematics Subject Classification: 05C99

Keywords: Decomposition of Graph, Continuous Monotonic Decomposition, Linear Decomposition(LD) or Arithmetic Decomposition, Linear Star Decomposition(LSD)

1. Introduction

All basic terminologies from Graph Theory are used in this paper in the sense of J.A.Bondy and U.S.R.Murty [1] and Frank Harary [3]. By a graph we mean a finite, undirected graph without loops or multiple edges.

Definition 1.1: Let G = (V, E) be a simple connected graph with p vertices and q edges. If $G_1, G_2, \ldots, G_n$ are connected edge disjoint subgraphs of G with $E(G)=E(G_1) \cup E(G_2) \cup \ldots \cup E(G_n)$, then $(G_1, G_2, \ldots, G_n)$ is said to be a **Decomposition** of G.

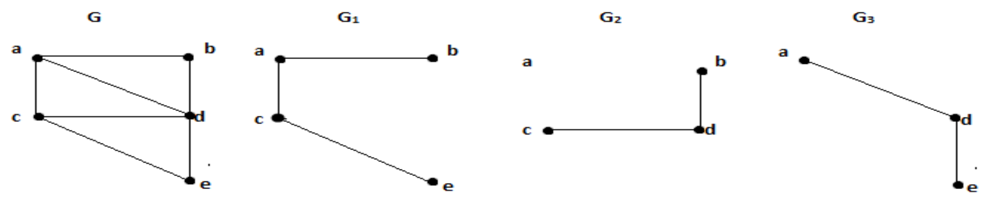

Figure (1): Decomposition (G_1, G_2, G_3) of G

N.Gnanadhas and J.Paulraj Joseph discussed on Continuous Monotonic Decomposition (CMD) of graphs [4]. This paper deals with Linear Star Decomposition for a very particular class of tree namely Lobster.

Definition 1.2: A Decomposition (G_1, G_2, …, G_n) of G is said to be **Continuous Monotonic Decomposition**(CMD) if $|E(G_i)|=i$, for every i=1, 2, 3, …, n. Clearly $q = \frac{n(n+1)}{2}$

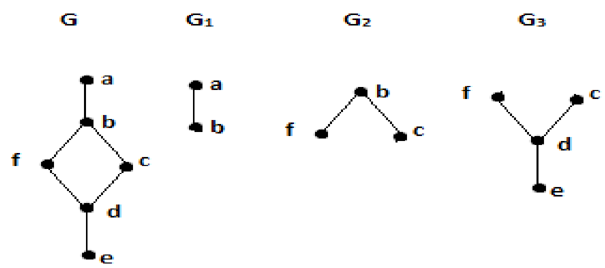

Figure (2): Continuous Monotonic Decomposition (G_1, G_2, G_3) of G

Definition 1.3: Let v be a vertex in a graph G. The eccentricity e (v) of v is defined by e (v) =max {d (u, v) /v∈V (G)}. The radius r (G) = min {e (v)/v∈V (G)}. v is called a central point if e (v) = r (G) and the set of all central points is called the **centre** of G.

Definition 1.4: The **diameter** of G denoted by diam (G) is the maximum distance between two vertices of G.

Definition 1.5: A vertex which is adjacent to k pendant vertices in a graph G is called a k-**support**. A 1-support is simply called a support.

We also recall the following definitions.

Definition 1.6: **Caterpillar** is a tree in which the removal of pendant vertices results in a path.

Definition 1.7: Lobster is a tree in which the removal of pendant vertices results in a caterpillar.

Definition1.8: In a Lobster L,the vertex with degree at least3is called a **junction** of L.

Definition1.9: An edge **e = uv** such that u is adjacent to a junction and v is adjacent to another junction is said to be a **junction-neighbor**.

2. Linear Decomposition of Graphs

The concept of Linear Decomposition or Arithmetic Decomposition was introduced by E. Ebin Raja Merly and N. Gnanadhas [2]

Definition 2.1: A Decomposition (G_1, G_2, …, G_n) of G is said to be a **Linear Decomposition** (LD) or Arithmetic decomposition if $|E(G_i)| = a + (i-1)d$, for every i=1, 2, 3…, n, and a, d∈Z. Clearly $q = \frac{n}{2}[2a + (n-1)d]$

If a=1 and d=1, then $q = \frac{n(n+1)}{2}$. That is, LD is a CMD. If a = 1 and d = 2 then, $q = n^2$. That is, the number of edges of G is a perfect square. Here after we consider the edge disjoint sub graphs of G as $G_1, G_3, G_5, \ldots, G_{(2n-1)}$.

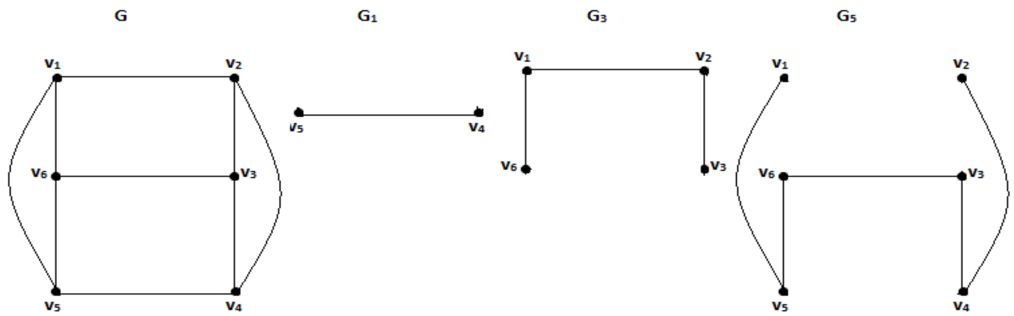

Figure (3): Linear Decomposition (G_1, G_3, G_5) of G

Definition 2.2: The Shell Graph C (m, k) is obtained from the cycle C_m with k cords sharing a common end point.

Result 2.3: Shell graph C (m, k) admits LD if q = m+k.

Remark 2.4: C (m, k) admits LD only when k ≤ m-3. The following figure illustrates C (6, 3), which admits LD.

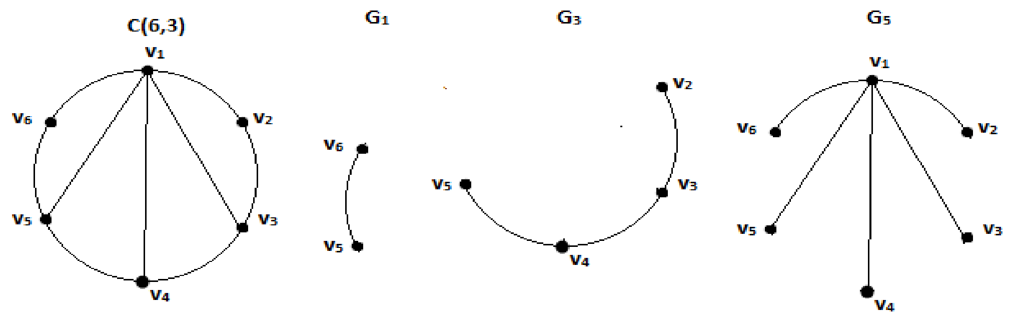

Figure (4) Linear Decomposition (G_1, G_3, G_5) of C (6, 3)

3. Linear Star Decomposition of Lobster

In this section, we characterize a tree namely Lobster using the concept diameter of Lobster.

Definition3.1: A linear decomposition (G_1, G_3, G_5, …, $G_{(2n-1)}$) in which each $G_{(2i-1)}$ is a star is said to be a **Linear Star Decomposition** (LSD).

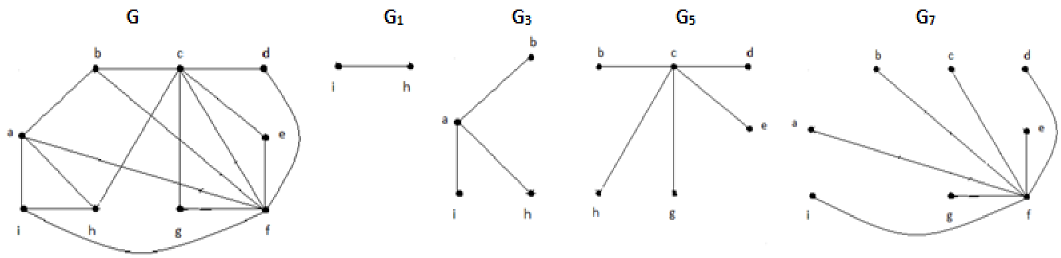

Figure (5): Linear Star Decomposition (G_1, G_3, G_5, G_7) of G

Result 3.2: If a Lobster L admits LSD then the distance between any two nearest junction is less than or equal to 3.

Proof: Suppose L admits LSD. To prove the distance between any two nearest junction is ≤ 3. Let u_i and u_j be two nearest junction. To prove d (u_i , u_j) ≤ 3. Suppose not. Then d (u_i , u_j) ≥ 4. Therefore the u_i - u_j path P contains a $u_{(i-1)} – u_{(j-1)}$ subpath P′ of length atleast 3. But P′ cannot be linearly decomposed in to stars. Therefore L cannot be linearly decomposed into stars, which is a contradiction. ▌

The following theorem is essential to the proof of the succeeding theorems. Here after P denotes the longest path in the Lobster L.

Notation 3.3: Let L be a Lobster with the longest path P. Let N_1 denotes the set of vertices which are at a distance one from P .Let $n_1 = |N_1|$. Let N_2 denotes the set of pendant vertices of L which are at a distance two from P. Let $n_2 = |N_2|$.

Theorem 3.4: If a Lobster L admits LSD ($S_1, S_3, S_5, …, S_{(2n-1)}$), then diam(L) ≤2n-1.

Proof: diam(L) ≤ diam(S_1) + diam(S_3) + diam(S_5) + … + diam($S_{(2n-1)}$)

= 1+ [(2+2+2+… +2) (n-1) times] =2n-1.

That is, diam (L) ≤ 2n-1. ▌

Theorem 3.5: Let L be a Lobster with diam (L) =2n-1 and $q=n^2$. Then L admits LSD ($S_1, S_3, S_5, …, S_{(2n-1)}$) with S_1 is in the first or the last edge of P if and only if (i) L is a caterpillar. (ii) There are (n-1) non-adjacent junction supports in L whose degrees are 3, 5, 7, …, (2n-1). (iii) There is at most one junction-neighbor in L.

Proof: Suppose L admits LSD ($S_1, S_3, S_5, …, S_{(2n-1)}$). Since diam (L) =2n-1, the centres of each stars lie in P is illustrated in Figure (6)

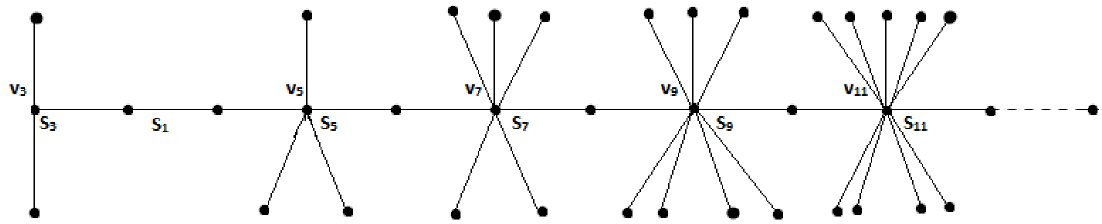

Figure (6)

Therefore, L is a caterpillar. Let $v_3, v_5, \ldots, v_{(2n-1)}$ be the centres of $S_3, S_5, \ldots, S_{(2n-1)}$. Therefore, they are junctions. Also since diam (L) = 2n-1, all the centres are distinct and are supports. Given that S_1 is in between $S_3, S_5, \ldots, S_{(2n-1)}$. Therefore, the origin and terminus of S_1 are not supports. Hence there are (n-1) non-adjacent junction supports whose degrees are 3, 5, 7, …, (2n-1). To prove (iii). Suppose there are two distinct junction-neighbors e_1 and e_2 such that $e_1 = x_1y_1$ and $e_2 = x_2y_2$ as shown in Figure (7).

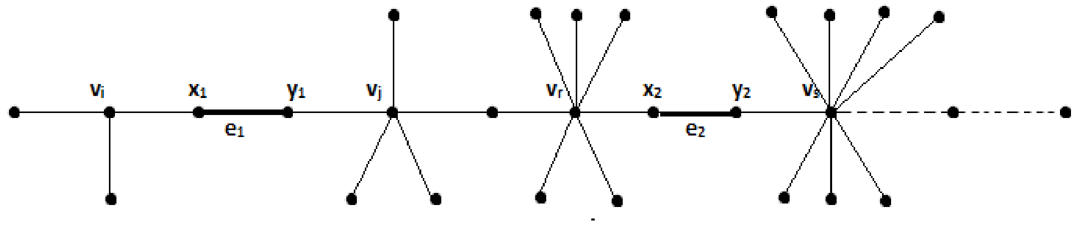

Figure (7)

Then there exists pair of junction supports v_i, v_j and v_r, v_s such that d $(v_i, v_j) = 3$ and

$(v_r, v_s) = 3$. Therefore $< E(L)-E(S_3 \cup S_5 \cup \ldots\ldots \cup S_{(2n-1)}) > = 2K_2 = 2S_1$, which is contradiction. Hence there is at most one junction-neighbor in L.

Conversely, assume (i), (ii) and (iii). To prove L admits LSD $(S_1, S_3, S_5, \ldots, S_{(2n-1)})$.

By (iii), S_1 must be in P. Also since diam (L) =2n-1, S_1 is in between $S_3, S_5, \ldots, S_{(2n-1)}$ By (ii) $S_3, S_5, \ldots, S_{(2n-1)}$ exist in L. Hence L admits LSD. ▌

Remark 3.6: In the above theorem, if S_1 is not in between S_3, S_5, …, $S_{(2n-1)}$ then there is no junction-neighbor in L.

Theorem 3.7: Let L be a Lobster with diam (L) =2n-2 and $q=n^2$. Then L admits LSD (S_1, S_3, S_5, …, $S_{(2n-1)}$) with S_1 is not in P if and only if (i) L –e is a caterpillar, where e is the edge of S_1. (ii) There are (n-1) non-adjacent junctions in L whose degrees are 3, 5, 7, …, (2n-1). (iii) There is no junction-neighbor in L.

Proof: Suppose L admits LSD (S_1, S_3, S_5, …, $S_{(2n-1)}$). Let e = u_1u_2. Since S_1 is not in P, u_1 and u_2 not lie on P. Without loss of generality, we may assume that u_1 is at a distance one from P. Therefore, u_2 is at a distance 2 from P (see figure (8)). Since diam (L) = 2n-2, all the centers of S_3, S_5, …, $S_{(2n-1)}$ must lie in P. Therefore, there is no other vertex in L which is at a distance 2 from P. Hence $|N_2|$ = 1 and L- e is a caterpillar. Since S_1 is not in P and diam (L) = 2n-1, there are (n-1) non-adjacent junctions in L whose degrees are 3, 5, 7, …, (2n-1) .To prove there is no junction neighbor in L.

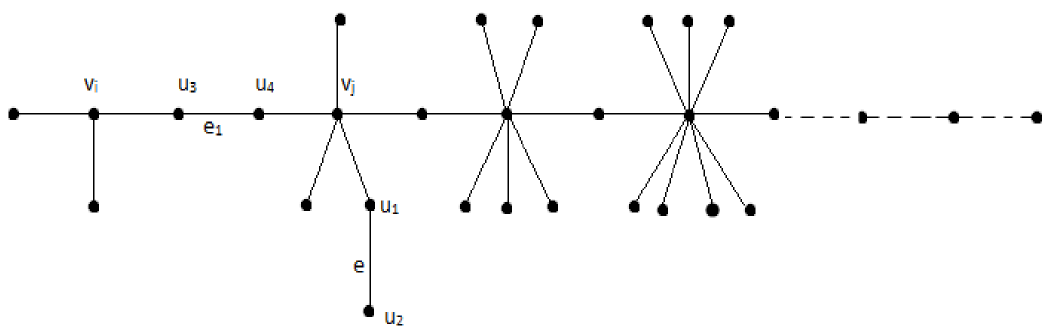

Figure (8)

Suppose there is an edge $e_1=u_3u_4$ in P which is a junction-neighbor as illustrated in figure (8). Then there exist two junctions v_i and v_j such that d (v_i, v_j) =3. Since u_3 and u_4 are not supports, e_1 is not in any of S_3, S_5, …, $S_{(2n-1)}$. Therefore

$<E(L)-E(S_3 \cup S_5 \cup \ldots \cup S_{(2n-1)})> = 2K_2 = 2S_1$, which is a contradiction to $q=n^2$. Hence there is no junction neighbor in L.

Conversely assume (i), (ii) and (iii). Since L-e is a caterpillar, we take e as S_1, which is not in P. By (iii), S_3, S_5, …, $S_{(2n-1)}$ exists in L. Hence L admits LSD (S_1, S_3, S_5, …, $S_{(2n-1)}$) such that S_1 is not in P. ▌

Theorem 3.8: Let L be a Lobster with diam (L) = 2n-3, $q=n^2$, (n-2) distinct supports with no junction neighbour in P of L and $N_2 \neq \emptyset$. Then L admits LSD (S_1, S_3, S_5, …, $S_{(2n-1)}$) if and only if(i) No vertex of exactly one star S_{2i-1}, $i \geq 2$ is in P (ii) All the vertices of N_2 are adjacent to exactly one vertex of N_1.

Proof: Assume that L admits LSD (S_1, S_3, S_5, …, $S_{(2n-1)}$).

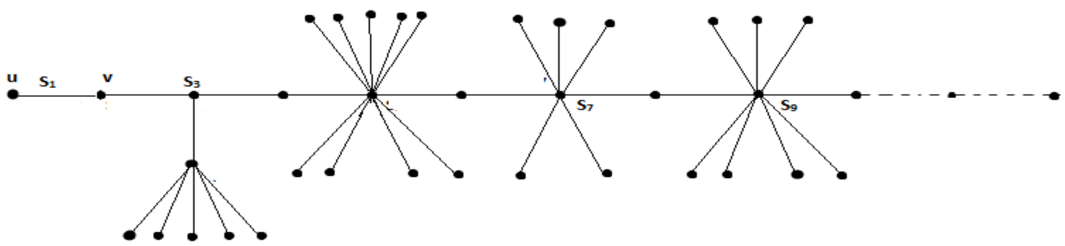

Figure (9)

Since there are (n-2) non-adjacent junctions in L and diam (L) = 2n-3, S_1 must be in P as illustrated in figure (9). Suppose at least one vertex of each stars S_{2i-1}, (i = 2, 3, …, n) is in P. Then there exist (n-1) junction supports in L, not all of them are distinct, which is a contradiction. Hence for exactly one star S_{2i-1}, $i \geq 2$, no vertex is in P. Here $|N_2| \neq 1$. Suppose $|N_2| = 3$. Therefore, no vertex of S_5 is in P. To prove all the vertices of N_2 are adjacent to exactly one vertex of N_1. Suppose the vertices of N_2 are adjacent to two distinct vertices u_i and u_{i+1} of N_1.

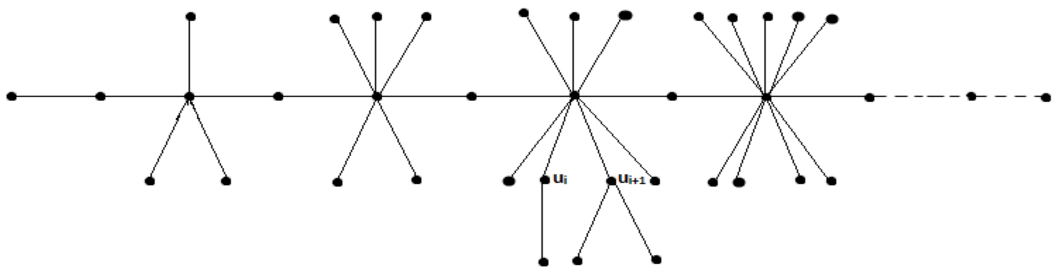

Figure (10)

From figure (10) it is clear that there are two stars S_i and $S_{(i+1)}$, i = 2, 3, 4, …, (n-1) such that u_i and u_{i+1} are centres of S_i and S_{i+1}. Since L admits LSD, the existence of S_i or S_{i+1} is not possible. Therefore, our assumption is wrong. Hence all the vertices of N_2 are adjacent to exactly one vertex of N_1. If $|N_2| = 5$, then by the above concept, no vertex of S_5 is in P and vertices of N_2 are adjacent to exactly one vertex of N_1.

Continuing in this way, if $|N_2| = 2n-1$, then no vertex of $S_{(2n-1)}$ is in P and vertices of N_2 are adjacent to exactly one vertex of N_1.Hence $|N_2| \geq 2i-1$, $i \geq 2$ and all the vertices of N_2 are adjacent to exactly one vertex of N_1.

Conversely, assume (i), (ii) and (iii).To prove L admits LSD. Since diam (L) = 2n-3 and $N_2 \neq \emptyset$, S_1 exists in L and there exists at least one star $S_{(2i-1)}$ (say) such that the centre of $S_{(2i-1)}$ is not in P. Since there are (n-2) distinct junction supports with no junction neighbour in P, (n-2) stars $S_3, S_5, \ldots, S_{2i-3}, S_{2i+1}, \ldots, S_{2n-1}$ exist in L. Since $q=n^2$ and all the vertices of N_2 are adjacent to exactly one vertex of N_1, S_{2i-1} also exists in L. Hence L admits Linear Star Decomposition. ▌

Corollary 3.9: If $N_2 = \emptyset$ in the above theorem, then the centres of any two stars is the same. This case is illustrated in figure (11).

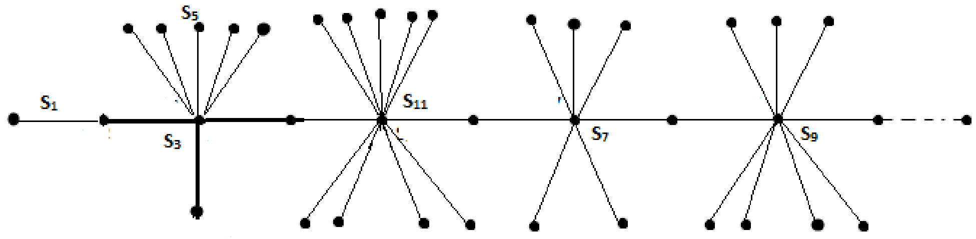

Figure (11)

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