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βθ- Monads in General Topology

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Abstract

The major purpose of this paper is to introduce and discuss a new type of monads, named βθ-monad, which plays an important role in our approach to β-closed spaces, β-T_2 spaces, and ……..etc., Also we investigate some properties and relationship between this monad with other types of monads.

Keywords: Nonstandard topology, monads, β-open set, semipre-θ-open set, β-closed spaces

Introduction

In 1966 A.Robinson [10], introduced the notion of point monads, and has been proved to be a useful device for characterizing and studying some topological concepts. In 1976 R.A.Herrmann[5] introduced two new types of monads, The θ and α monads in general topology. Which are capable of similarly characterizing the various topological concepts associated with quasi-H-closed, nearly compact , Urysohn spaces,…..etc., In 1983 Abd.El-Monsef et al.[1], introduced the notion of β-open sets in topological space, In 1986 D.Andrijevic [8] defined and investigated semipre-open sets which are equivalent to β-open sets, in this work, we apply the notion of β-open sets in topological space to present and study a new type of monads called βθ-monad.

Basic Backgrounds in General Topology

Throughout this work, (X,τ) or (simply X) denote a standard topological space on which no separation axioms are assumed unless explicitly stated, we recall the following definitions, notational conventions and characterizations.

The closure (resp., Interior) of a subset A of a space X is denoted by ClA(resp.,IntA). A subset A of a space X is said to be β-open(semipre-open) [1](resp., preopen [4],regular open [4]) iff A⊂ClIntClA (resp., A⊂IntClA, A=IntClA).

A subset F of a space X is said to be β-closed (resp., preclosed, regular closed) if X/F is β-open(resp., preopen,regular open)set.

A subset A of a space X is said to be θ-open[4](resp., sp-θ-open[8]) if for each x∈A, there is an open(resp., semipre-open) subset G of X such that x∈G⊂ ClG⊂A (resp., x∈G⊂ βClG⊂A).

The intersection of all β-closed (resp., pre-closed, θ-closed,sp-θ-closed) sets containing A is called β-closure (resp., preclosure, θ-closure, sp-θ-closure) and denoted by βClA(resp., pClA,$Cl_\theta A$, $spCl_\theta A$).

The union of all β-open(resp., preopen)sets contained in A is called β-interior (resp.,preinterior) and denoted by βIntA (resp.,pIntA).

A subset A of a space X is said to be sp-regular open [2] if it is β-open and β-closed . Equivalently A= βIntβClA.

The family of all β-open(resp., preopen, regular open,sp-regular, θ-open, sp-θ-open)sets of a space X is denoted by βO(X)(resp., PO(X),RO(X),SPRO(X), θO(X), SPθO(X)).

Definition 2.1[4]: A space X is extremely disconnected if the closure of every open set is open set.

Definition 2.2 [8]: A point x∈X is said to be semipre-θ accumulation point of a subset A of a space X if βClG∩A≠ϕ for every G∈βO(X).

The set of all semipre-θ-accumulation points of A is called sp-θ-closure of A and is denoted by $spCl_{\theta}A$, if A= $spCl_{\theta}A$ then A is sp-θ-closed

Definition 2.3[2]: A space X is said to be β-closed space if every β-open cover of X has a finite subfamily whose β-closure cover X.

Definition 2.4[4]: A space X is said to be p-closed space if every pre-open cover of X has a finite subfamily whose pre-closure cover X.

Definition2.5 [1]: A space X is said to be β-T_2 iff for each $x,y\in X$, such that $x\neq y$, there exists $G,H\in\beta O(X)$, such that $x\in G$, $y\in H$, $G\cap H=\phi$.

Theorem 2.1 [8]: A space X is extremely disconnected iff PO(X)= βO(X).

Theorem 2.2[8]: A space X is said to be β-T_2 iff for each $x,y\in X$, such that $x\neq y$, there exists $G,H\in\beta O(X)$, such that $x\in G$, $y\in H$, $\beta ClG\cap\beta ClH=\phi$.

Theorem 2.3[8]: Let (X,τ) be a topological space, and A let be a subset of X, then $A\in \beta O(X)$ iff $\beta ClG\in SPRO(X)$.

Theorem 2.4 [4]: Let (X,τ) be a topological space, and let G be a subset of X, then

i) $\beta ClG=G\cup IntClIntG$.

ii) $pClG=G\cup ClIntG$

Theorem 2.5 [4,8]: Let (X,τ) be a topological space, and A be a subset of X, then

i) $\beta ClA\subseteq ClA$., $\beta ClA\subseteq pClA$.

Basic Backgrounds in Nonstandard analysis

In this section, we use Nelson's nonstandard analysis construction, based on theory called internal set theory IST , The axioms of IST is the axiom of Zermelo-Frankel with choice (briefly ZFC) together with three axioms which are the transfer axiom, the idealization axiom, and the standardization axiom, for more see[9].

Recall that for a topological space (X,τ), The monad μ(p), α-monad μ α (p), and θ-monad μθ(p) at a point p are defined as follows[6]

$\mu(p)=\cap\{{}^*G;\ p\in G\in\tau\}$, $\mu_{\alpha}(p)=\cap\{{}^*(IntClG);\ p\in G\in\tau\}$, $\mu_{\theta}(p)=\cap\{{}^*ClG;\ p\in G\in\tau\}$.

Definition 3.1 [7] : Let (X,τ) be a standard topological space, then the Pθ- Monad at the point $a\in X$ is defined as follows

$\mu p\theta(a)=\cap\{A;\ A\in \overline{GP(a)}\ \}$. Where $\overline{GP(a)}=\{pClA\ ;\ A\in GP(a)\}$, and $GP(a)=\{A;\ a\in A\in PO(X)\}$

Definition 3.2[3]: A relation r is called concurrent in U, if $r\in U$, and if a1 , a2 ,…,an$\in$dom(r), then there is an element b such that $(ai ,b)\in r$, for i=1,2,…,n

Theorem 3.1 [3]: Concurrence relation

Let r be a standard concurrent relation in a standard U, then there is an element $b\in U$, such t hat $(a,b)\in r$, for each $a\in$dom(r).

Theorem 3.2[5]: A space X is quasi -H-closed space iff $X=\cup\{\mu_\theta(a);\ a\in X\}$.

Theorem 3.3[7]: A space X is p-closed space iff $X=\cup\{\mu_{p\theta}(a);\ a\in X\}$.

Theorem 3.4[11]:'Ballon Principle

Let (X,τ) be a standard topological space, and $x\in X$, m(x) be the monad of x, if $m(x)\subseteq B$, for some internal subset B of X, then there exists an open set G such that $m(x)\subseteq^* G\subseteq B$.

4. βθ-Monads

Definition 4.1: Let (X,τ) be a standard topological space, then the βθ- Monad at the point $a\in X$ is defined as follows

$\mu\beta\theta(a)=\cap\{A;\ A\in \overline{G\beta(a)}\ \}$. Where $\overline{G\beta(a)}=\{\ \beta ClA\ ;\ A\in G\beta(a)\}$, and $G\beta(a)=\{A;\ a\in A\in \beta O(X)\}$

Proposition 4.1: Let (X,τ) be a standard topological space, and $a\in X$, then

$\mu_{\beta\theta}(a)=\cap\{\beta ClG;\ G\in G\beta(a)\ \}$.

Proof: Follows directly from their definitions.

Proposition 4.2: Let (X,τ) be a standard topological space, then for each $a\in X$,

$\mu_{\beta\theta}(a)\subseteq\mu_\theta(a)$, $\mu_{\beta\theta}(a)\subseteq\mu_{p\theta}(a)$.

Proof: Follows from Theorem 2.5 and their definitions.

Remark 4.1: the equality of Proposition 4.2 is not true in general, as shown in the following example.

Example 4.1: Let X={a,b,c}, τ={ϕ,{a},{b},{a,b},X}, then the family of all β-open sets are βO(X)= {ϕ,{a},{b},{a,b},{a,c},{b,c},X}, and PO(X)= τ, then $\overline{G\beta(a)}$={{a},{a,c},X } , $\mu_{\beta\theta}(a)$={a}, $\overline{GP(a)}$={a,c},X }, $\mu_{p\theta}(a)$= $\mu_{\theta}(a)$ ={a,c}.

Proposition 4.3: If X is extremely disconnected, then $\mu_{\beta\theta}(a) = \mu_{p\theta}(a)$, for each a∈X.

Proof: Follows from Theorem 2.1 and Proposition 4.2.

Proposition 4.4: Let (X,τ) be a standard topological space, and a∈X , if every β-open set is regular closed set, then $\mu_{\beta\theta}(a)= \cap\{G;\ G\in G\beta(a)\ \}$.

Proof: By Theorem 2.6 βClG=G∪IntClIntG, since G∈ βO(X), then G is regular closed set, therefore βClG=G, by Proposition 4.1 we obtain $\mu_{\beta\theta}(a)= \cap\{G;\ G\in G\beta(a)\ \}$.

Proposition 4.5: Let (X,τ) be a standard topological space, and a∈X , if every β-open set is preclosed set, then $\mu_{\beta\theta}(a)= \cap\{G;\ G\in G\beta(a)\ \}$.

Proof: Similar to Proposition 4.4.

Theorem 4.1: Let (X,τ) be a standard topological space, then the following statements are true

i) For each a∈X, a∈ $\mu_{\beta\theta}(a)$.
ii) For each a∈X, b∈ $\mu_{\beta\theta}(a)$ implies that $\mu_{\beta\theta}(b) \subseteq \mu_{\beta\theta}(a)$.

Proof: i) obvious.

ii)Let x∈ $\mu_{\beta\theta}(b)$, since b∈ $\mu_{\beta\theta}(a)$, then for each standard G∈ βO(X)

(a∈ βClG implies that b∈ βClG, and b∈ βClG implies that x∈ βClG), by transfer axiom, for each G∈ βO(X)

(a∈ βClG implies that b∈ βClG, and b∈ βClG implies that x∈ βClG)

Hence $\mu_{\beta\theta}(b) \subseteq \mu_{\beta\theta}(a)$.

Proposition 4.6: Let (X,τ) be a standard topological space, and a∈X , then

$\mu_{\beta\theta}(a)= \cap\{G;\ G\in SPR(X,a)\ \}$.

Proof: Follows directly from Theorem 2. 3 and Proposition 4.1

Theorem 4.2: Let (X,τ) be a standard topological space, and let a∈X be any element, then there exists a standard β-open H such that βClH ⊆ μβθ(a)

Proof: Let r(βClG, βClH) be a binary relation define by r(βClG, βClH) equivalently βClH $\subseteq$ βClG, this relation is concurrent relation, since if G1 , G2,…, Gn $\in$ βO(X), such that βClH=$\cap$ βClGi for i=1,2,…,n.

satisfies r(βClG, βClH) for i=1,2,…,n, by Concurrence Theorem, there is βClH such that βClH $\subseteq$ βClG for each G$\in$ βO(X), therefore βClH $\subseteq$ $\mu_{\beta\theta}$(a).

Corollary 4.1: Let (X,τ) be a standard topological space, and let a$\in$X be any element, then there exists a standard sp-regular-open set H such that H $\subseteq$ $\mu_{\beta\theta}$(a).

Proof: Follows directly from Theorem 4.2 and Theorem 2.3

Theorem 4.3: Let A be a standard subset of a space X, then A is sp-θ-open set iff $\mu_{\beta\theta}$(a) $\subseteq$A, for each a$\in$A.

Proof: Assume that A is sp-θ-open set and let a$\in$A, By its definition, there exists a standard β-open G such that a$\in$G$\subseteq$ βClG $\subseteq$A, by transfer axiom βClG $\subseteq$A for each G,A and a$\in$G$\in$ βO(X),

Now $\cap$\{βClG; G$\in$Gβ(a)\}$\subseteq$ βClG$\subseteq$A, by proposition 4.1, $\mu_{\beta\theta}$(a) $\subseteq$A.

Conversely suppose that $\mu_{\beta\theta}$(a) $\subseteq$A. for each a$\in$A, Theorem 4.2 implies that there exists a standard β-open G such that βClG $\subseteq$ $\mu_{\beta\theta}$(a), then a$\in$G$\subset$ βClG$\subset$A, for each standard a, by transfer axiom a$\in$G$\subset$ βClG$\subset$A, for each a. Hence A is sp-θ-open set.

Theorem 4.4: Let A be a standard subset of a space X, then A is sp-θ-closed set iff $\mu_{\beta\theta}$(a) $\cap$ A=ϕ. for each a$\in$X/A.

Proof: Assume that A is sp-θ-closed set, by Theorem 4.3 $\mu_{\beta\theta}$(a) $\subseteq$X/A for each a$\in$X/A hence $\mu_{\beta\theta}$(a) $\cap$ A=ϕ.

Conversely assume that $\mu_{\beta\theta}$(a) $\cap$ A=ϕ, then $\mu_{\beta\theta}$(a) $\subseteq$X/A for each a$\in$X/A, also by Theorem 4.3, we get A is sp-θ-closed set.

Theorem 4.5: Let A be a standard subset of a space X, then A is sp-θ-closed set iff $\mu_{\beta\theta}$(a) $\cap$ A$\neq$ϕ, implies that a$\in$A.

Proof: Similar to Theorem 4.4.

Theorem 4.6: Let G be a standard subset of a space X, then G is sp-θ-open set iff for each a$\in$G , and b$\in$$\mu_{\beta\theta}$(a), implies that b$\in$G.

Proof: Follows directly from theorem 4.3

Corollary 4.2: Let F be a standard subset of a space X, then F is sp-θ-closed set iff for each p$\notin$F , and q$\in$$\mu_{\beta\theta}$(p), implies that q$\notin$F.

Proof: Follows from Theorem4.3 and Theorem4.6.

Theorem 4.7: Let (X,τ) be a standard topological space, and x∈X, $\mu_{\beta\theta}(p)$ be the βθ- monad at the point p∈X, if $\mu_{\beta\theta}(p) \subseteq B$, for some internal subset B of X, then there exists a standard β-open set G such that $\mu_{\beta\theta}(p) \subseteq$ βClG⊆B.

Proof: Suppose not, that βClG-B≠ϕ, for all G∈βO(X), p∈G, the family {βClG-B} has a finite intersection property, since βClG1-B∩ βClG2-B= (βClG1∩ βClG2)-B, it follows that $\mu_{\beta\theta}(p)$-B≠ϕ which is contradiction.

Theorem 4.8: Let A be a standard subset of a space X, then $spCl_\theta$ A={ a∈X; $\mu_{\beta\theta}(a) \cap A \neq \phi$ }.

Proof: Let a∈ spClθ A, then a∈F, for each sp-θ-closed superset of A, if $\mu_{\beta\theta}(a) \cap A=\phi$, then $\mu_{\beta\theta}(a) \subseteq X/A$, by Theorem 4.7, there exists a standard β-open set G such that $\mu_{\beta\theta}(a) \subseteq$ βClG⊆ X/A, therefore βClG ∩ A=ϕ , which implies that a∉$spCl_\theta$ A.

Conversely suppose that a∈X and $\mu_{\beta\theta}(a) \cap A \neq \phi$, we have to show that a∈F, for all sp-θ-closed superset of A, if a∉F, by Theorem 4.3, $\mu_{\beta\theta}(a) \subseteq X/F$, therefore $\mu_{\beta\theta}(a) \cap A=\phi$, which is contradiction.

Corollary 4.3: Let A be a standard sp-θ-closed subset of space X, then A={ a∈X; $\mu_{\beta\theta}(a) \cap A \neq \phi$ }.

Proof: Follows directly from Definition 2.2 and Theorem 4.8

Corollary 4.4: Let A be a standard sp-regular-open subset of space X, then A={ a∈X; $\mu_{\beta\theta}(a) \cap A \neq \phi$ }.

Proof: Follows directly from Theorem 4.8 and the fact that SPRO(X) ⊆SPθO(X).

5. Properties of some topological concepts by using βθ- Monads

Theorem 5.1: A standard space X is β-closed space iff X= ∪{ $\mu_{\beta\theta}(a)$; a∈X}.

Proof: Assume that X is β-closed space, and let q∈X, such that q∉$\mu_{\beta\theta}(x)$, for each standard x∈X , therefore for each x∈X, there exists a β-open set G_x such that q∉βCl G_x , then Γ={ G_x, x∈X , q∉βCl G_x , G_x ∈ β O(X)} is a β-open cover of X , since X is β-closed space, then there exists a finite subfamily {G1,G2,…,Gn} such that X= ∪{ βCl Gi , for i=1,2,…,n}, this means, for each standard x∈X, implies that x∈βCl Gi for some i, by transfer axioms for each x∈X, therefore x∈βCl Gi ,for some I, which is contradiction .

Conversely suppose that X is not β-closed space, and let ρ be a β-open cover of X such that it has no finite subfamily whose β-closure cover X, let {G1,G2,…,Gn}⊆

$\beta O(X)$, define a relation r such that r(βCl G,x) iff $x \notin \beta$Cl G, then it is clear that r is concurrent relation, by Theorem3.1, there is $y \in X$, with $y \in \beta$Cl G, If $x \in X$ such that $x \in \beta$Cl G, but $y \notin \beta$Cl G, for each standard G $\in \beta$ O(X), therefore $y \notin \mu_{\beta\theta}(x)$, which is contradiction.

Theorem 5.2: A space X is β-T_2 iff $\mu_{\beta\theta}(x) \cap \mu_{\beta\theta}(y)=\phi$, for each $x,y \in X$, such that $x \neq y$.

Proof: Assume that X is β-T_2 , and let $x,y \in X$, such that $x \neq y$, then there exists $G,H \in \beta O(X)$, $x \in G, y \in H$, an $G \cap H=\phi$. By Theorem 2.2 $\beta ClG \cap \beta ClH=\phi$, by proposition 4.1, $\mu_{\beta\theta}(x) \subseteq \beta ClG$ and $\mu_{\beta\theta}(y) \subseteq \beta ClH$. Hence $\mu_{\beta\theta}(x) \cap \mu_{\beta\theta}(y)=\phi$.

Conversely assume that the condition valid, by Theorem 4.2, there exists $G,H \in \beta O(X)$, such that $\beta ClG \subseteq \mu_{\beta\theta}(x)$ and $\beta ClH \subseteq \mu_{\beta\theta}(y)$, which implies that X is β-T2 space.

Theorem 5.3: A space X is β-T_2 ,β-closed space iff $\{ \mu_{\beta\theta}(a); a \in X\}$ is a partition for X

Proof: Follows from Theorem5.1 and Theorem 5.2.

Theorem 5.4: Let (X,τ) be a standard topological space, then the following statements are true

i) Every β-closed space is quasi –H-closed space.
ii) Every β-closed space is p-closed space.

Proof: Follows from Theorem 3.2, Theorem 3.3, and Proposition 4.2.

Theorem 5.5: Let (X,τ) be a standard topological space, if X is extremely disconnected, then X is β-closed space iff p-closed space.

Proof: Follows from Theorem2.1 and Theorem 5.4.

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