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T-Rough (Prime, Primary) Ideal and T-Rough Fuzzy (Prime, Primary) Ideal on Commutative Rings

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Abstract

The purpose of this paper is to introduce and discuss the concept of T-rough (prime, primary) ideal and T-rough fuzzy (prime, primary) ideal in a commutative ring . Our main aim in this paper is, generalization of theorems which have been proved in [6, 7, 11]. At first, T-rough sets introduced by Davvaz in [6]. By using the paper, we define a concept of T-rough ideal , T-rough quotient ideal and T-rough fuzzy ideal in a commutative ring and study on a set-valued homomorphism for rings. Rough sets were originally proposed in the presence of an equivalence relation. An equivalence relation is sometimes difficult to be obtained in rearward problems due to the vagueness and incompleteness of human knowledge. From this point of view, we compare relation between a rough set and a T-rough set and prove some theorems.

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1 Introduction

The notion of rough sets has been introduced by Pawlak in his papers[15-20] and it soon involved a natural question concerning possible connection between rough sets and algebraic systems. The algebraic approach to rough sets have been given and studied by Iwinski in [9], Bonikowaski [5]. Biswas [1], Nanda [14], Biswas and Nanda [2,3] introduced the notion of rough set and rough subgroups. Kuroki [12] introduced the notion of rough ideals in a semigroups. Davvaz [7] introduced the notion of rough subring with respect to an ideal of a ring. Dubois and Prade[8,9] combined fuzzy sets and rough sets in a fruitful way by defining rough fuzzy sets and fuzzy rough sets. Davvaz [6] introduced T- rough set and T- rough homomorphism on a group. Rough set theory is an extension of set theory, in which a subset of a universe is described by a pair of ordinary sets called the lower and upper approximations. The lower approximation of a given set is the union of all the equivalence classes which are subsets of the set, and upper approximation is the union of all the equivalence classes which have a non-empty intersection with the set. It is well known that a partition induces an equivalence relation on a set and vice versa. The properties of rough sets can thus be examined via either partition or equivalence classes. Rough sets are a suitable mathematical model of vague concepts, i.e., concepts without sharp boundaries. In this paper, we discussed T-rough ideal , T-rough prime ideal T-rough fuzzy ideal and T-rough fuzzy prime ideal of rings based on definitions in [6] and generalized some theorems have been proved in[6, 7, 11]. The purpose of this paper is to introduce and discuss the concept of T-rough ideal and T-rough fuzzy ideal in a commutative ring.

2 Preliminary Notes

The following definitions and preliminaries are required in the sequel of our work and hence presented in brief. Suppose that U is a non-empty set. A partition or classification of U is a family Θ of non-empty subsets of U such that each element of U is contained in exactly one element of Θ. Recall that an equivalence relation θ on a set U is a reflexive, symmetric, and transitive binary relation on U. Each partition θ induces an equivalence relation Θ on U by setting

$$x\theta y \Leftrightarrow x \textit{ and } y \textit{ are in the same class of } \Theta.$$

Conversely, each equivalence relation θ on U induces a partition Θ of U whose classes have the form

$$[x]_\theta = \{y \in U \mid x\theta y\}.$$

The following notation will be used. Given a non-empty universe U, by P(U) we will denote the power-set on U. If θ is an equivalence relation on U, then for every $x \in U, [x]_\theta$ denotes the equivalence class of θ determined by x. For any $X \subseteq U$, we write X^c to denote the complement of X in U, that is the set $U \setminus X$.

Definition 2.1. *A pair (U, θ) where $U \neq \emptyset$ and θ is an equivalence relation on U is called an approximation space.*

Definition 2.2. *For an approximation space (U, θ) by a rough approximation in (U, θ) we mean a mapping $Apr : P(U) \longrightarrow P(U) \times P(U)$ defined by for every $X \in P(U), Apr(X) = (\underline{Apr}(X), \overline{Apr}(X))$, where*

$$\underline{Apr}(X) = \{x \in X \mid [x]_\theta \subseteq X\}, \overline{Apr}(X) = \{x \in X \mid [x]_\theta \bigcap X \neq \emptyset\}.$$

$\underline{Apr}(X)$ is called a lower rough approximation of X in (U, θ) where as $\overline{Apr}(X)$ is called upper rough approximation of X in (U, θ).

Definition 2.3. *Given an approximation space (U, θ) a pair (A, B) in $P(U)$ is called a rough set in (U, θ)if $(A, B) =(\underline{Apr}(X), \overline{Apr}(X)$ for some $X \in P(U)$.*

Definition 2.4. *A subset X of U is called definable if $\underline{Apr}(X) = \overline{Apr}(X)$. If$X \subseteq U$ is given by a predicate P and $x \in U$, then*

(1) $x \in \underline{Apr}(X)$ means that x certainly has property P,

(2) $x \in \overline{Apr}(X)$ means that x possibly has property P,

(3) $x \in U \setminus \overline{Apr}(X)$ means that x definitely does not have property P.

Definition 2.5. *Let $Apr(A) = (\underline{Apr}(A), \overline{Apr}(A))$ and $Apr(B) = (\underline{Apr}(B), \overline{Apr}(B))$ be any two rough sets in the approximation space (U, θ). Then*

(i) $Apr(A) \sqcup Apr(B) = (\underline{Apr}(A) \cup \underline{Apr}(B), \overline{Apr}(A) \cup \overline{Apr}(B))$,

(ii) $Apr(A) \sqcap Apr(B) = (\underline{Apr}(A) \cap \underline{Apr}(B), \overline{Apr}(A) \cap \overline{Apr}(B))$,

(iii) $Apr(A) \sqsubseteq Apr(B) \Longleftrightarrow Apr(A) \sqcup Apr(B) = Apr(B)$.

When $Apr(A) \sqsubseteq Apr(B)$, we say that $Apr(A)$ is a rough subset of $Apr(B)$. Thus property of rough inclusion has all the properties of set inclusion. The rough complement of $Apr(A)$ denoted by $Apr^c(A)$ is defined by

$$Apr^c(A) = (U \setminus \overline{Apr}(A), U \setminus \underline{Apr}(A).$$

Also, we can define $Apr(A) \setminus Apr(B)$ as follows:

$$\begin{aligned} Apr(A) \setminus Apr(B) &= Apr(A) \sqcap Apr^c(B) \\ &= (\underline{Apr}(A) \setminus \overline{Apr}(B), \overline{Apr}(A) \setminus \underline{Apr}(B)). \end{aligned}$$

Proposition 2.6. . (*See*[15].) *Let* U *be a nonempty set and* θ *an equivalence relation on* U. *For every subsets* $A, B \subseteq U$, *we have*

(i) $\underline{Apr}(A) \subseteq A \subseteq \overline{Apr}(A)$;

(ii) If $A \subseteq B$, *then* $\underline{Apr}(A) \subseteq \underline{Apr}(B)$ *and* $\overline{Apr}(A) \subseteq \overline{Apr}(B)$;

(iii) $\underline{Apr}(A \cap B) = (\underline{Apr}(A) \cap \underline{Apr}(B)$;

(iv) $\overline{Apr}(A) \cup \overline{Apr}(B) = \overline{Apr}(A \cup B)$.

3 T-rough ideal and T-rough prime ideal in a commutative ring

In this section, we define the concept of set-valued homomorphism and give some important examples of set-valued mapping.We show that every homomorphism of ring is a set-valued homomorphism.We also in investigate some basic properties of generalized lower and upper approximation operators in a ring. We generalized rough ideal[10] that is called T-rough ideal[4]. We applied the notion T-rough sets in a commutative ring and extended some theorems in [7].

Definition 3.1. (*See*[6].) *Let* X *and* Y *be two non-empty sets and* $B \subseteq Y$. *Let* $T : X \to P^*(Y)$ *be a set-valued mapping where* $P^*(Y)$ *denotes the set of all non-empty subsets of Y. The lower inverse and upper inverse of B under T are defined by*

$$T^+(B) = \{x \in X \mid T(x) \subseteq B\} \quad ; \quad T^{-1}(B) = \{x \in X \mid T(x) \cap B \neq \emptyset\}.$$

Definition 3.2. (*See*[6].) *Let* X *and* Y *be two non-empty sets and* $B \subseteq Y$. *Let* $T : X \to P^*(Y)$ *be a set-valued mapping where* $P^*(Y)$ *denotes the set of all non-empty subsets of Y.* $(T^+(B), T^{-1}(B))$ *is called* T*- rough set of* R.

Example 3.3. (i) *Let* (U, θ) *be an approximation space and* $T : U \to P^*(U)$ *be a set-valued mapping where* $T(x) = [x]_\theta$, *then for any* $B \subseteq U$, $T^+(B) = \underline{Apr}(B)$ *and* $T^{-1}(B) = \overline{Apr}(B)$.

(ii) *Let* $\mathbf{N}$ *be natural numbers set and* $\mathbf{Z}$ *be integer numbers set and* $T : \mathbf{N} \to P^*(\mathbf{Z})$ *where for every* $n \in \mathbf{N}$, $T(n) = \{1, 2, \ldots, n\}$. *If* $B = \{1, 2, \ldots, 10\}$, *then* $T^+(B) = \{1, 2, \ldots, 10\}$ *and* $T^{-1}(B) = \mathbf{N}$.

(iii) *Let* $\mathbf{N}$ *be natural numbers set and* $\mathbf{Z}$ *be integer numbers set and* $T : \mathbf{N} \to P^*(\mathbf{Z})$ *where for every* $n \in \mathbf{N}, T(n) = \{\pm n\}$. *If* $B = \{1, 2, \ldots, 10\}$, *then*

$T^{+}(B) = \{1, 2, \ldots, 10\} = T^{-1}(B)$.

(iv) *Let* $\mathbf{R}$ *be real numbers set and* $T : \mathbf{R} \to P^*(\mathbf{R})$ *where for every* $a \in \mathbf{R}, T(a) = [-|a|, |a|]$. *If* $A = [0, 1]$, *then* $T^{+}(A) = \{0\}$ *and* $T^{-1}(A) = [-1, 1]$.

(v) *Let* $\mathbf{Z}$ *be integer numbers set and* $T : \mathbf{Z} \to P^*(\mathbf{Z})$ *where for every* $n \in \mathbf{Z}, T(n) = n\mathbf{Z}$. *If* $A = 2\mathbf{Z}$, *then* $T^{+}(A) = 2\mathbf{Z}$ *and* $T^{-1}(A) = \mathbf{Z}$.

(vi) *Let* $f : R \to S$ *be a ring homomorphism and* A *be a non-empty subset* S. *Then for the set-valued mapping* $T : R \to P^*(S)$ *defined by* $T(r) = \{f(r)\}$, *we have* $T^{-1}(A) = \{r \in R \mid f(r) \in A\} = T^{+}(A) = f^{-1}(A)$.

Proposition 3.4. $(See[6].)$ *Let* X *and* Y *be two non-empty sets and* $A, B \subseteq Y$. *Let* $T : X \to P^*(Y)$ *be a set-valued mapping where* $P^*(Y)$ *denotes the set of all non-empty subsets of* Y. *Then the following hold:*

(1) $T^{-1}(A \cup B) = T^{-1}(A) \cup T^{-1}(B)$;

(2) $T^{+}(A \cap B) = T^{+}(A) \cap T^{+}(B)$;

(3) $A \subseteq B$ *implies* $T^{+}(A) \subseteq T^{+}(B)$ *and* $T^{-1}(A) \subseteq T^{-1}(B)$;

(4) $T^{+}(A) \cup T^{+}(B) \subseteq T^{+}(A \cup B)$ *and* $T^{-1}(A \cap B) \subseteq T^{-1}(A) \cap T^{-1}(B)$.

Using the lower inverse and upper inverse under T, we define a binary relation on subsets of Y as follows:

$$A \cong B \iff T^{-1}(A) = T^{-1}(B) \text{ and } T^{+}(A) = T^{+}(B).$$

It is an equivalence relation which induces a partition $P^*(Y)$. An equivalence class of the relation is called a T-rough set. Therefore a T-rough set of Y is a family of all subsets of Y having the same lower and upper inverse under T.

Definition 3.5. *(i) Let* R *be a ring and* $I \neq \emptyset$ *be a subset of* R. I *is called an ideal in* R, *if* $r \in R$ *and* $x, y \in I$, *then* $xr, rx, x - y \in I$.
(ii) If $U \neq \emptyset$ *be a subset of* R. U *is called a subring of* R, *if* $x, y \in U$, *then* $xy, x - y \in U$.
(iii) If A *and* B *are non-empty subsets of* R, *let* AB *denote the set of all finite sum* $\{\sum_{i=1}^{n} a_i b_i \mid a_i \in A, b_i \in B, n \in \mathbf{N}\}$.

The definition of prime ideal excludes the ideal R for both historical and technical reasons. The following is a very useful characterization of prime ideals in commutative rings.
(iv) An ideal P is called a prime ideal if $P \neq R$, and $A, B \subseteq R$ be ideals of R and $AB \subseteq P$, then $A \subseteq P$ or $B \subseteq P$. In a commutative ring R, primness is equivalent to $x, y \in R$ and $xy \in P$ implies $x \in P$ *or* $y \in P$.

Definition 3.6. *Let R and S be two commutative rings and $T : R \to P^*(S)$ be a set-valued mapping. T is called a set- valued homomorphism if*

(i) $T(x_1 + x_2) = T(x_1) + T(x_2)$ *for all* $x_1, x_2 \in R$;

(ii) $T(x_1x_2) = \{ab \mid a \in T(x_1), b \in T(x_2)\}$ *for all* $x_1, x_2 \in R$;

(iii) $T(-x) = -T(x)$ *for all* $x \in R$.

Examples 3.3(v) and (vi) are set- valued homomorphisms. So a ring homomorphism is a special case of a set- valued homomorphism. Let θ be a complete congruence relation, i.e., $[x]_\theta[y]_\theta = [xy]_\theta$ for all $x, y \in R$. Define $T : R \to P^*(R)$ by $T(x) = [x]_\theta$, then T is a set-valued homomorphism. Further, the rough sets are T- rough sets .

Lemma 3.7. *Let R and S be two commutative rings and $T : R \to P^*(S)$ be a set-valued homomorphism. If U is a subring of S and $T^+(U)$, $T^{-1}(U)$ are non-empty , then $T^+(U)$ and $T^{-1}(U)$ are subrings of R.*

Proof. Let $x, y \in T^+(U)$, by Definition 3.1, $T(x), T(y) \subseteq U$. Since U is subring of S, we have $T(x - y) = T(x) - T(y) \subseteq U$ and $T(xy) = \{ab \mid a \in T(x), b \in T(y)\} \subseteq T(x)T(y) \subseteq U$. This shows that $x - y, xy \in T^+(U)$. Moreover, Let $x, y \in T^{-1}(U)$, by Definition 3.1, $T(x) \cap U \neq \emptyset$ and $T(y) \cap U \neq \emptyset$. Suppose $a \in T(x) \cap U$ and $b \in T(y) \cap U$. Since U is a subring of S, $a - b \in U$ and $a - b \in T(x) - T(y) = T(x - y)$, thus $a - b \in T(x - y) \cap U$, hence $T(x - y) \cap U \neq \emptyset$. So $x - y \in T^{-1}(U)$. Again, $ab \in \{ab \mid a \in T(x), b \in T(y)\} = T(xy)$ and $ab \in U$. This implies that $T(xy) \cap U \neq \emptyset$. So $xy \in T^{-1}(U)$.

Lemma 3.8. *Let R and S be two commutative rings and $T : R \to P^*(S)$ be a set-valued homomorphism. If A is an ideal of S and $T^+(A) \neq \emptyset$, then $T^+(A)$ is an ideal of R.*

Proof. Let $r \in R$ and $x, y \in T^+(A)$, by Definition 3.1,$T(x), T(y) \subseteq A$. Since A is an ideal of S, so we have $T(x - y) = T(x) - T(y) \subseteq A$ and $T(rx) = \{ab \mid a \in T(r), b \in T(x)\} \subseteq T(x)T(r) \subseteq A$. This shows that $x - y, rx, xr \in T^+(A)$.

Lemma 3.9. *Let R and S be two commutative rings and $T : R \to P^*(S)$ be a set-valued homomorphism. If A is an ideal of S and $T^{-1}(A) \neq \emptyset$, then $T^{-1}(A)$ is an ideal of R.*

Proof. Let $r \in R$ and $x \in T^{-1}(A)$, by Definition 3.1, $T(x) \cap A \neq \emptyset$. Suppose $a \in T(x) \cap A$. Since A is an ideal of $S, T(r)a \subseteq A$ and $T(r)a \subseteq \{ab \mid a \in T(r), b \in T(x)\} = T(rx)$. This shows that $T(rx) \cap A \neq \emptyset$. So, $rx \in T^{-1}(A)$. Similarly, $xr \in T^{-1}(A)$.

Corollary 3.10. *Let R and S be two commutative rings and $T : R \to P^*(S)$ be a set-valued homomorphism. If A is an ideal of S and $T^{-1}(A) \neq \emptyset$ and $T^+(A) \neq \emptyset$ then $(T^+(A), T^{-1}(A))$ is a T-rough ideal of R.*

Theorem 3.11. *Let R and S be two commutative rings and $T : R \to P^*(S)$ be a set-valued homomorphism. If A is a prime ideal of S and $R \neq T^+(A) \neq \emptyset$, then $T^+(A)$ is a prime ideal of R.*

Proof. Let $x, y \in R$ and $xy \in T^+(A)$, then $\{ab \mid a \in T(x), b \in T(y)\} = T(xy) \subseteq A$. And so, for every $a \in T(x), b \in T(y), ab \in A$. Now we show that $T(x) \subseteq A$ or $T(y) \subseteq A$. Suppose $T(x)$ and $T(y)$ are not subsets of A. Then there exists $a \in T(x)$ such that $a \notin A$ and there exists $b \in T(y)$ such that $b \notin A$. Since A is a prime ideal of S, we deduce $ab \notin A$ as which is a contradiction. Hence $x \in T^+(A)$ or $y \in T^+(A)$.

Theorem 3.12. *Let R and S be two commutative rings and $T : R \to P^*(S)$ be a set-valued homomorphism. If A is a prime ideal of S and $R \neq T^{-1}(A) \neq \emptyset$, then $T^{-1}(A)$ is a prime ideal of R.*

Proof. Let $x, y \in R$ and $xy \in T^{-1}(A)$, then $T(xy) \cap A \neq \emptyset$ and $\{ab \mid a \in T(x), b \in T(y)\} = T(xy)$. So, there are $a \in T(x), b \in T(y)$ such that $ab \in A$. Since A is a prime ideal of S , we have $a \in A$ or $b \in A$. Hence $T(x) \cap A \neq \emptyset$ or $T(y) \cap A \neq \emptyset$. So $x \in T^{-1}(A)$ or $y \in T^{-1}(A)$. Therefore $T^{-1}(A)$ is a prime ideal of R.

Corollary 3.13. *Let R and S be two commutative rings and $T : R \to P^*(S)$ be a set-valued homomorphism. If A is a prime ideal of S and $R \neq T^{-1}(A) \neq \emptyset$ and $R \neq T^+(A) \neq \emptyset$, then $(T^+(A), T^{-1}(A))$ is a T-rough prime ideal of R.*

4 T-rough primary ideal in a commutative ring

Definition 4.1. *An ideal Q is called a primary ideal if $Q \neq R$, and for $a, b \in R$ and $ab \in Q$, then $a \in Q$ or $b^n \in Q$ for some $n \in \mathbf{N}$.*

Let R and S be two commutative rings and $T : R \to P^*(S)$ be a set-valued homomorphism. We call $Q \in P^*(S)$ a T-rough inverse primary ideal of R if $T^+(Q)$ a T-lower and $T^{-1}(Q)$ a T-upper rough primary ideal of R.

Every prime ideal is a primary ideal. If p is a prime number and $n \geq 2$, then in $\mathbf{Z}, < p^n >$ is a primary ideal which is not prime ideal.

Theorem 4.2. *Let R and S be two commutative rings and $T : R \to P^*(S)$ be a set-valued homomorphism. If Q is a primary ideal of S and $R \neq T^+(Q) \neq \emptyset$, then $T^+(Q)$ is a primary ideal of R.*

Proof. Let $x, y \in R$ and $xy \in T^{+}(Q)$. Then $\{ab \mid a \in T(x), b \in T(y)\} = T(xy) \subseteq Q$. And so, for every $a \in T(x), b \in T(y)$, we have $ab \in Q$. Now suppose $a \notin Q$. Since Q is a primary ideal of S , we have $b^n \in Q$ for some $n \in \mathbf{N}$. Hence $T(y^n) \subseteq Q$. It implies that $y^n \in T^{+}(Q)$. Therefore $T^{+}(Q)$ is a primary ideal of R.

Theorem 4.3. *Let R and S be two commutative rings and $T : R \to P^{*}(S)$ be a set-valued homomorphism. If Q is a primary ideal of S and $R \neq T^{-1}(Q) \neq \emptyset$, then $T^{-1}(Q)$ is a primary ideal of R.*

Proof. Let $x, y \in R$ and $xy \in T^{-1}(Q)$. Then $T(xy) \cap Q \neq \emptyset$ and $\{ab \mid a \in T(x), b \in T(y)\} = T(xy)$. So, there are $a \in T(x), b \in T(y)$ such that $ab \in Q$. Now suppose $a \notin Q$. Since Q is a primary ideal of S , we have $b^n \in Q$ for some $n \in \mathbf{N}$. Hence $b^n \in T(y^n) \cap Q \neq \emptyset$. It implies that $y^n \in T^{-1}(Q)$. Therefore $T^{-1}(Q)$ is a primary ideal of R.

Corollary 4.4. *Let R and S be two commutative rings and $T : R \to P^{*}(S)$ be a set-valued homomorphism. If A is a primary ideal of S and $R \neq T^{-1}(A) \neq \emptyset$ and $R \neq T^{+}(A) \neq \emptyset$ then $(T^{+}(A), T^{-1}(A))$ is a T-rough primary ideal of R.*

5 T- rough fuzzy ideal and T- rough fuzzy prime(primary) ideal in a commutative ring

Theory of fuzzy sets was initiated in 1967. As a natural need, Dubois and Prade [8,9] combined fuzzy sets and rough sets in a fruitful way by defining rough fuzzy sets and fuzzy rough sets. Rough fuzzy sets and fuzzy rough sets are also studied by Nakamura [13], Nanda[14], Biswas [2,3] and by Banerjee and Pal [1]. Several research directions have been suggested on fuzzy rough sets and rough fuzzy sets. In this section, we introduce the T- rough fuzzy ideal and T-fuzzy prime (primary) ideal in a commutative ring and give some properties of such ideals and then extended some theorems in which are proved in [6, 7, 11].

Definition 5.1. *Let (U, θ) be an approximation space. A subset fuzzy is a mapping μ from U to $[0, 1]$. If $x \in U$, we define ,*

$$\underline{Apr}(x) = \bigwedge_{a \in [x]_\theta} \mu(a); \ \overline{Apr}(\mu)(x) = \bigvee_{a \in [x]_\theta} \mu(a).$$

They are called , respectively , the lower and upper approximation of the fuzzy subset μ. $Apr(\mu) = (\underline{Apr}(\mu), \overline{Apr}(\mu))$ is called a rough fuzzy set respect to θ if $\underline{Apr}(\mu) \neq \overline{Apr}(\mu))$. Let μ be a subset fuzzy of U, $\lambda \in [0, 1]$. Then the sets

$$\mu_\lambda = \{x \in U \mid \mu(x) \geq \lambda\}; \ \mu^{+}_\lambda = \{x \in U \mid \mu(x) > \lambda\}$$

are called,respectively, λ-levelest and λ-strong levelest of the fuzzy set μ.

Definition 5.2. *A fuzzy subset μ of a ring R is called a fuzzy ideal [12] if*
(i) $\mu(x-y) \geq \mu(x) \wedge \mu(y)$ for all $x, y \in R$;
(ii) $\mu(xy) \geq \mu(x) \vee \mu(y)$ for all $x, y \in R$.
It is a fuzzy prime ideal if $\mu(xy) = \mu(x)$ or $\mu(xy) = \mu(y)$ and it is called a fuzzy primary ideal if $\mu(xy) = \mu(x)$ or $\mu(xy) = \mu(y^n)$ for some $n \in \mathbf{N}$.

Example 5.3. *The prime fuzzy ideals of $\mathbf{Z}$ are just fuzzy ideals μ given by*

$$\mu(n) = \begin{cases} 1, & \textit{if } p \mid n \\ \alpha, & \textit{otherwise} \end{cases}$$

where p is a prime integer or zero and $\alpha < 1$.

Example 5.4 (11). *Let $p \in \mathbf{Z}$ be a prime number, $0 < t < 1$ and n is a positive integer. Define a fuzzy ideal as follows:*

$$\mu(x) = \begin{cases} 0, & \textit{if } x \notin < p^n > \\ t, & \textit{if } x \in < p^n > \sim 0 \\ 1, & \textit{if } x = 0 \end{cases}$$

It is obvious that μ is primary but not prime.
In [7] the following theorems have been proved:

Theorem 5.5. *Let if μ be a fuzzy subset of a commutative ring R. Then μ is a fuzzy ideal(fuzzy prime ideal) of S iff $\mu_\lambda, \mu_\lambda^+$ are, if they are nonempty, ideals [prime ideals] of R for every $\lambda \in [0, 1]$.*

Theorem 5.6. *Let θ is an equivalence relation on commutative ring R and μ is a fuzzy ideal of R, then $\overline{Apr}(\mu)$ is a fuzzy ideal of R.*

Theorem 5.7. *Let θ is an complete congruence relation (A congruence θ on R is called complete if $[a]_\theta [b]_\theta = [ab]_\theta$ for any $a, b \in R$)on commutative ring R and μ is a fuzzy ideal of R, then $\underline{Apr}(\mu)$is a fuzzy ideal of R if $\underline{Apr}(\mu) \neq \emptyset$.*

Now, we prove the above theorems are hold for T-rough fuzzy (prime) ideal on a commutative ring. First of all, we define T-rough fuzzy set and T-rough fuzzy (prime) ideal.

Definition 5.8. *Let R and S be two commutative rings and $T : R \to P^*(S)$ be a set-valued homomorphism. If is μ an ideal fuzzy of S . For every $x \in R$, we define*

$$T^+(\mu)(x) = \bigwedge_{a \in T(x)} \mu(a) \quad ; \quad T^{-1}(\mu)(x) = \bigvee_{a \in T(x)} \mu(a).$$

$T^+(\mu)$ and $T^{-1}(\mu)$ are called , respectively, T-rough lower and T-rough upper fuzzy subsets on $x \in R$. $(T^+(\mu), T^{-1}(\mu))$ is said to be T-rough fuzzy set of R. If $T^+(\mu)$ and $T^{-1}(\mu)$ are fuzzy prime (primary) ideals, $(T^+(\mu), T^{-1}(\mu))$ is said to be T-rough fuzzy prime(primary) ideal of R.

Lemma 5.9. *Let R and S be two commutative rings and $T : R \to P^*(S)$ be a set-valued homomorphism. If μ is a fuzzy ideal of S, then for all $\lambda \in [0, 1]$,*

(1) $(T^+(\mu))_\lambda = T^+(\mu_\lambda)$;

(2) $(T^{-1}(\mu))_\lambda = T^{-1}(\mu_\lambda)$;

(3) $(T^+(\mu))^+_\lambda = T^+(\mu^+_\lambda)$;

(4) $(T^{-1}(\mu))^+_\lambda = T^{-1}(\mu^+_\lambda)$.

Proof. (1).

$$\begin{aligned} x \in (T^+(\mu))_\lambda &\Leftrightarrow T^+(\mu)(x) \geq \lambda \Leftrightarrow \bigwedge_{a \in T(x)} \mu(a) \geq \lambda \\ &\Leftrightarrow \mu(a) \geq \lambda \; for \; all \; a \in T(x), \\ &\Leftrightarrow T(x) \subseteq \mu_\lambda \Leftrightarrow x \in T^+(\mu_\lambda). \end{aligned}$$

(2).

$$\begin{aligned} x \in (T^{-1}(\mu))_\lambda &\Leftrightarrow T^{-1}(\mu)(x) \geq \lambda \Leftrightarrow \bigvee_{a \in T(x)} \mu(a) \geq \lambda \\ &\Leftrightarrow, \mu(a) \geq \lambda for \; some \; a \in T(x) \\ &\Leftrightarrow a \in T(x) \cap \mu_\lambda. \\ &\Leftrightarrow T(x) \cap \mu_\lambda \neq \emptyset \Leftrightarrow x \in T^{-1}(\mu_\lambda). \end{aligned}$$

(3) and (4) are similar.
The following theorems are straightforward.

Theorem 5.10. *Let R and S be two commutative rings and $T : R \to P^*(S)$ be a set-valued homomorphism. If μ is an ideal fuzzy of S, then $T^+(\mu)$ is a fuzzy ideal of R.*

Theorem 5.11. *Let R and S be two commutative rings and $T : R \to P^*(S)$ be a set-valued homomorphism. If μ is an ideal fuzzy of S, then $T^+(\mu)$ is an fuzzy ideal of R.*

If θ is a complete congruence relation on R and define $T : R \to P^*(R)$ where $T(x) = [x]_\theta$ for every $x \in R$, we obtained Theorems 5.6, 5.7.

Theorem 5.12. *If μ is an ideal fuzzy of S and $T : R \to P^*(S)$ be a set-valued homomorphism. If μ is a fuzzy prime ideal of S , then $T^+(\mu)$, $T^{-1}(\mu)$ are fuzzy prime ideal of R.*

Proof. Since μ is a fuzzy prime ideal of S, by Theorem 5.5,μ_λ and μ_λ^+ are prime ideal of S if they are non-empty. By Lemma 5.9 and Theorem 5.5, $(T^+(\mu))_\lambda = T^+(\mu_\lambda), (T^{-1}(\mu))_\lambda = T^{-1}(\mu_\lambda), (T^+(\mu))_\lambda^+ = T^+(\mu_\lambda^+)$ and $(T^{-1}(\mu))_\lambda^+ = T^{-1}(\mu_\lambda^+)$ are prime ideals, if they are non-empty. Again by Theorem 5.5, $T^+(\mu),\ T^{-1}(\mu)$ are fuzzy prime ideal.

Proposition 5.13. *Let μ be a fuzzy ideal of R.*

(1) *If μ is a fuzzy primary ideal, then μ_λ is a primary ideal for all $0 \leq \lambda \leq 1$.*

(2) *Every fuzzy prime ideal is a fuzzy primary ideal.*

Proof. It is straightforward.

6 T-rough quotient ideal in rings

Let R and S be two commutative rings and $T : R \to P^*(S)$ be a set-valued homomorphism. Suppose $R/T = \{T(x) \mid x \in R\}$. It is clear that R/T is a commutative ring.

Definition 6.1. *Let R and S be two commutative rings and $T : R \to P^*(S)$ be a set-valued homomorphism. The lower T-rough quotient and the upper T-rough quotient for $A \in P^*(S)$ are, respectively,*

$$(T^+(A))/T = \{T(x) \mid T(x) \subseteq A\};\ (T^{-1}(A))/T = \{T(x) \mid T(x) \cap A \neq \emptyset\}.$$

Lemma 6.2. *Let R and S be two commutative rings and $T : R \to P^*(S)$ be a set-valued homomorphism. If $A \in P^*(S)$ be an ideal of S and $(T^+(A))/T \neq \emptyset$, then $(T^+(A))/T$ is an ideal of R/T.*

Proof. Suppose $T(x), T(y) \in (T^+(A))/T$. Since $A \in P^*(S)$ is an ideal of S, So $T(x - y) = T(x) - T(y) \subseteq A$. Then $T(x) - T(y) \in (T^+(A))/T$. Now suppose $T(a) \in R/T$ and $T(x) \in (T^+(A))/T$, by Definition 6.1, $T(x) \subseteq A$. Since A is an ideal of S, thus $T(xa) \subseteq T(x)T(a) \subseteq A$, therefore $T(xa) \in (T^+(A))/T$.

Lemma 6.3. *Let R and S be two commutative rings and $T : R \to P^*(S)$ be a set-valued homomorphism such that $\{ab \mid a \in T(x), b \in T(y)\} = T(xy) = T(x)T(y)$. If $A \in P^*(S)$ be an ideal of S and $(T^{-1}(A))/T \neq \emptyset$, then $(T^{-1}(A))/T$ is an ideal of R/T.*

Proof. Suppose $T(x), T(y) \in (T^{-1}(A))/T$. Thus $T(x) \cap A \neq \emptyset \neq T(y) \cap A$. Hence, there exist $a \in T(x) \cap A$ and $b \in T(y) \cap A$. Since $A \in P^*(S)$ is an ideal of S, So $a - b \in T(x) - T(y) = T(x - y), a - b \in A$. Therefore $a - b \in T(x - y) \cap A$.Then $T(x) - T(y) \in (T^{-1}(A))/T$. Now suppose $T(a) \in$

R/T and $T(x) \in (T^{-1}(A))/T$, by Definition 6.1, $T(x) \cap A \neq \emptyset$. There exists $b \in T(x) \cap A$. Since A is an ideal, $T(a)b \subseteq A$ and $T(a)b \subseteq T(xa) = T(x)T(a)$. Then $T(a)b \subseteq T(xa) \cap A$. Hence $T(x)T(a) = T(xa) \in (T^{-1}(A))/T$.
The proofs the following propositions are straightforward.

Proposition 6.4. *Let R and S be two commutative rings and $T : R \to P^*(S)$ be a set-valued homomorphism. If $A \subseteq S$ be a prime ideal of S and $R/T \neq (T^+(A))/T \neq \emptyset$, then $(T^+(A))/T$ is a prime ideal of R/T.*

Proposition 6.5. *Let R and S be two commutative rings and $T : R \to P^*(S)$ be a set-valued homomorphism. If $A \subseteq S$ be a prime ideal of S and $R/T \neq (T^{-1}(A))/T \neq \emptyset$, then $(T^{-1}(A))/T$ is a prime ideal of R/T.*

Corollary 6.6. *Let θ is a complete congruence relation on commutative ring on S and A an ideal of S, then $(\underline{Apr}(A))/\theta = \{[x]_\theta \mid [x]_\theta \subseteq A\}$ and $(\overline{Apr}(A))/\theta = \{[x]_\theta \mid [x]_\theta \cap A \neq \emptyset\}$ are ideals of S/θ.*

Corollary 6.7. *Let θ is a complete congruence relation on commutative ring on S and A a prime ideal of S, then $(\underline{Apr}(A))/\theta$ and $(\overline{Apr}(A))/\theta$ are rough quotient prime ideals.*

Proposition 6.8. *Let R and S be two commutative rings and $T : R \to P^*(S)$ be a set-valued homomorphism such that $\{ab \mid a \in T(x), b \in T(y)\} = T(xy) = T(x)T(y)$. If $A \in P^*(S)$ be a primary ideal of S and $R/T \neq (T^{-1}(A))/T \neq \emptyset$ and $R/T \neq (T^+(A))/T \neq \emptyset$, then $(T^{-1}(A))/T$ and $(T^+(A))/T$ is a primary ideal of R/T.*

7 Conclusion

The rough sets theory is regarded as a generalization of the classical sets theory. A key notion in rough set is an equivalence relation. An equivalence is sometime difficult to be obtained in rearward problems due to vagueness and incompleteness of human knowledge. In the present paper, we substituted a universe set by a ring, and introduced the notions of rough prime (primary) ideals and rough fuzzy prime (primary) ideals in a commutative ring. We discussed The relations between upper(lower) rough prime and primary ideals and upper (lower) approximations of their homomorphism images. We introduced and studied the notion of T-rough prime, T-rough primary ideal, T-rough (fuzzy)ideal and T-rough(fuzzy) prime ideal of commutative rings. We extended some theorems which have been proved in [6, 7, 11]. We generalized some ideas presented by Davvaz[6,7]. Further, we introduced the notion lower and upper T-rough quotient set in a ring and studied and investigated some their interesting properties.

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