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Tangent Bundle of Hypersurface with Semi-Symmetric Metric Connection

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1 Introduction

Friedman, A. and Schouten, J. A. [1] introduced the idea of a semi-symmetric metric connection on a Riemann manifold. Later, Hayden, H. A. defined the semi-symmetric metric connection on a Riemann manifold [2]. In [4], Yano, K. studied some properties of a semi-symmetric metric connection on a Riemann manifold. Imai, T. considered a hypersurface with the semi-symmetric metric connection and obtained the Weingarten, Gauss and Codazzi-Ricci equations with respect to semi-symmetric metric connection [5]. In [7], Nakao, Z. expanded to submanifolds the study of Imai, T. in [5]. Tani, M. [3] developed the theory of hypersurfaces prolonged to tangent bundle with respect to the complete lift of metric tensor of the Riemann manifold. Yücesan, A., in [10], studied semi-Riemann submanifolds of a semi-Riemann manifold with a semi-symmetric metric connection. In this study, we consider tangent bundle of a hypersurface with semi-symmetric metric connection, especially, by following [3] and [5].

In Section 2, we shall give the neccessary notions and results which will be used in the next sections. In Section 3, we show that the complete lift of semi-symmetric metric connection on hypersurface is semi-symmetric metric connection in tangent bundle of the hypersurface and we find some certain results concerning the tangent bundle. In the last section, we obtain the structure equations with respect to semi-symmetric metric connection of tangent bundle.

2 Preliminaries

Let $\hat{\nabla}$ be a linear connection in an $m-$ dimensional Riemann manifold M. The torsion tensor $\hat{T}$ of $\hat{\nabla}$ is given by

$$\hat{T}(\hat{X},\hat{Y})=\hat{\nabla}_{\hat{X}}\hat{Y}-\hat{\nabla}_{\hat{Y}}\hat{X}-[\hat{X},\hat{Y}], \tag{2.1}$$

for any vector fields $\hat{X}$ and $\hat{Y}$ in M. The connection $\hat{\nabla}$ is *symmetric* if its torsion tensor $\hat{T}$ vansihes, otherwise it is *non-symmetric*. If there is a Riemann metric $\hat{g}$ in M, $(M,\hat{g})$ is called a *Riemann manifold*. If $\hat{g}$ is a metric in M, such that $\hat{\nabla}\hat{g}=0$, then the connection $\hat{\nabla}$ is a *metric connection*, otherwise it is *non-metric*. It is well known that a linear connection is symmetric and metric if and only if it is the *Riemann connection* [8].

Let $\bar{\nabla}$ be a metric connection in $(M,\hat{g})$, which is non-symmetric. In [4, 5], if torsion tensor $\bar{T}$ of $\bar{\nabla}$, defined by (2.1), satisfies

$$\bar{T}(\hat{X},\hat{Y})=\hat{w}(\hat{Y})\hat{X}-\hat{w}(\hat{X})\hat{Y}, \tag{2.2}$$

for a $\hat{w}\in\Im_1^0(M)$, then the connection $\bar{\nabla}$ is called *semi-symmetric metric connection* in $(M,\hat{g})$.

A semi-symmetric metric connection $\bar{\nabla}$ in $(M,\hat{g})$ is given by

$$\bar{\nabla}_{\hat{X}}\hat{Y}=\hat{\nabla}_{\hat{X}}\hat{Y}+\hat{w}(\hat{Y})\hat{X}-\hat{g}(\hat{X},\hat{Y})\hat{P}, \tag{2.3}$$

for arbitrary vector fields $\hat{X}$ ve $\hat{Y}$ in $(M,\hat{g})$, where $\hat{\nabla}$ is a Riemann connection in $(M,\hat{g})$ and $\hat{P}$ is a vector field defined by $\hat{g}(\hat{P},\hat{X})=\hat{w}(\hat{X})$ for any vector field $\hat{X}$ in $(M,\hat{g})$.

The tangent bundle of M is denoted by TM with the projection $\pi_M:TM\rightarrow M$. $\Im_s^r(M)$ is the space of tensor fields of type (r,s) in M. According to [6], using the complete lift and vertical lift operations we have the following equalities:

$$
\begin{aligned}
[\hat{X}^C, \hat{Y}^C] &= [X,Y]^C, \qquad (2.4)\\
\hat{w}^V(\hat{X}^C) &= \Big(\hat{w}(\hat{X})\Big)^V,\\
\hat{w}^C(\hat{X}^C) &= \Big(\hat{w}(\hat{X})\Big)^C,\\
\hat{F}^C(\hat{X}^C) &= \hat{F}(\hat{X}),\\
\hat{g}^C(\hat{X}^V, \hat{Y}^C) &= \hat{g}^C(\hat{X}^C, \hat{Y}^V) = \Big(\hat{g}(\hat{X},\hat{Y})\Big)^V,\\
\hat{g}^C(\hat{X}^V, \hat{Y}^C) &= \Big(\hat{g}(\hat{X},\hat{Y})\Big)^C,\\
\hat{\nabla}^C_{\hat{X}^C}\hat{Y}^C &= \Big(\hat{\nabla}_{\hat{X}}\hat{Y}\Big)^C,\\
\hat{\nabla}^C_{\hat{X}^C}\hat{Y}^V &= \Big(\hat{\nabla}_{\hat{X}}\hat{Y}\Big)^V,\\
\hat{T}^C\left(\hat{X}^C, \hat{Y}^C\right) &= \left(\hat{T}\left(\hat{X},\hat{Y}\right)\right)^C,\\
\hat{R}^C\left(\hat{X}^C, \hat{Y}^C\right)\hat{Z}^C &= \left(\hat{R}\left(\hat{X},\hat{Y}\right)\hat{Z}\right)^C,
\end{aligned}
$$

for any $\hat{X}$,$\hat{Y}$,$\hat{Z} \in \Im^1_0(M)$, $\hat{w} \in \Im^0_1(M)$, $\hat{F} \in \Im^1_1(M)$, $\hat{g} \in \Im^0_2(M)$, $\hat{T} \in \Im^1_2(M)$ and $\hat{R} \in \Im^1_3(M)$.

Let S be an $(m-1)-$dimensional manifold imbedded differentially as a submanifold in $(M, \hat{g})$ and denote by $\imath : S \longrightarrow M$ its imbedding [3, 7]. The differential mapping $d\imath$ is a mapping from TS into TM, which is called the tangent map of $\imath$, where TS and TM are the tangent bundles of S and M, respectively. The tangent map $d\imath$ is denoted by B. The tangent map of B is denoted by $\tilde{B} : T(TS) \longrightarrow T(TM)$.

The hypersurface S is also a Riemann manifold with the *induced metric* g defined by $g(X,Y) = \hat{g}(BX, BY)$ for arbitrary $X, Y \in \Im^1_0(S)$. Thus, ∇ is a Riemann connection with the *induced connection* on (S, g) from $\hat{\nabla}$ defined by [3]

$$\hat{\nabla}_{BX}BY = B(\nabla_X Y) + h(X,Y)N, \qquad (2.5)$$

for any $X, Y \in \Im^1_0(S)$, where N is unit normal vector field on (S, g) and h is the second fundamental tensor field of (S, g). Also, the following equality

$$h(X,Y) = g(HX, Y),$$

for any $X, Y \in \Im^1_0(S)$, where $H \in \Im^1_1(S)$.

If h vanishes, then S is called *totally geodesic* with respect to ∇ and if h is proportional to g, then S is called *totally umbilical* with respect to ∇ [3].

3 Tangent Bundle of Hypersurface with Semi-Symmmetric Metric Connection

In [5], $\mathring{\nabla}$ is a semi-symmetric metric connection induced on the hypersurface S from $\bar{\nabla}$, which satisfies the equation

$$\bar{\nabla}_{BX} BY = B(\mathring{\nabla}_X Y) + m(X,Y)N, \tag{3.1}$$

for $X, Y \in \Im_0^1(S)$, where m is a tensor field type of $(0,2)$ in S. Defining $M = H - \eta I$, we obtain the equality

$$m(X,Y) = g(MX,Y), \tag{3.2}$$

for any $X, Y \in \Im_0^1(S)$, where I is the unit tensor field of type $(1,1)$ in S.

If m vanishes, then S is called *totally geodesic* with respect to $\mathring{\nabla}$ and if m is proportional to g, then S is called *totally umbilical* with respect to $\mathring{\nabla}$.

Theorem 1 *The connection induced on a hypersurfaces of a Riemann manifold with a semi-symmetric metric connection with respect to the unit normal is also a semi-symmetric metric one [5].*

Then, we have

$$\mathring{\nabla}_X Y = \nabla_X Y + w(Y)X - g(X,Y)P,$$

for arbitrary $X, Y \in \Im_0^1(S)$. Here, P is a vector field in S determined by $\hat{P} = BP + \eta N$, where η is a function in S and w is a $1-$form in S determined by $w(X) = \hat{w}(BX)$.

For the Riemann metric $\hat{g}$ in M, the complete lift $\hat{g}^C$ of $\hat{g}$ is the *pseudo-Riemann metric* in TM.

Therefore, if we denote the induced metric on TS from $\hat{g}^C$ by $\tilde{g}$, then

$$\tilde{g}(X^C, Y^C) = \hat{g}^C(\tilde{B}X^C, \tilde{B}Y^C),$$

for arbitrary $X, Y \in \Im_0^1(S)$. Thus, the complete lift $\hat{\nabla}^C$ of the Riemann connection $\hat{\nabla}$ in $(M, \hat{g})$ is the Riemann connection in the *pseudo-Riemann manifold* $(TM, \hat{g}^C)$. Similarly, the complete lift ∇^C of the induced connection ∇ on (S, g) is also the Riemann connection in $(TS, \tilde{g})$.

Theorem 2 *If $\hat{T}$ is torsion tensor of $\hat{\nabla}$ in $(M, \hat{g})$, then $\hat{T}^C$ is torsion tensor of $\hat{\nabla}^C$ in $\left(TM, \hat{g}^C\right)$ [6].*

Now, the main theorem of this study follows.

Theorem 3 *Let $\bar{\nabla}$ be a semi-symmetric metric connection with respect to $\hat{\nabla}$ Riemann connection in $(M,\hat{g})$. Then, $\bar{\nabla}^C$ is also a semi-symmetric metric connection with respect to $\hat{\nabla}^C$ Riemann connection in $(TM,\hat{g}^C)$.*

Proof. Firstly, we shall show that $\hat{w}^V(\tilde{B}X^C)=(\hat{w}(BX))^{\bar{V}}$ and $\hat{w}^C(\tilde{B}X^C)=(\hat{w}(BX))^C$. In [3], using (3.10) we get

$$\begin{aligned}
\hat{w}^V(\tilde{B}X^C) &= \hat{w}^V(BX)^{\bar{C}} = \#\left(\hat{w}^V(\hat{X}^C)\right) = \#\left(\hat{w}(\hat{X})\right)^V = (\hat{w}(BX))^{\bar{V}}, \\
\hat{w}^C(\tilde{B}X^C) &= \hat{w}^C(BX)^{\bar{C}} = \#\left(\hat{w}^C(\hat{X}^C)\right) = \#\left(\hat{w}(\hat{X})\right)^C = (\hat{w}(BX))^{\bar{C}}
\end{aligned}$$

for arbitrary $X,Y\in\Im_0^1(S)$. Here, we denote the operation of restriction to $\pi_M^{-1}(\imath(S))$ by $\#$. Also, we denote the vertical and complete lift operations on $\pi_M^{-1}(\imath(S))$ by $\bar{V}$ and $\bar{C}$, respectively. Now taking the complete lift of both sides of the Equation (2.3) and using the Equation (1) we get

$$\begin{aligned}
\left(\bar{\nabla}_{BX}BY\right)^{\bar{C}} &= \left(\hat{\nabla}_{BX}BY\right)^{\bar{C}} + (\hat{w}(BY)BX)^{\bar{C}} - \left(\hat{g}\left(BX,BY\right)\hat{P}\right)^{\bar{C}}, \\
\left(\bar{\nabla}_{BX}BY\right)^{\bar{C}} &= \left(\hat{\nabla}_{BX}BY\right)^{\bar{C}} + (\hat{w}(BY))^{\bar{C}}(BX)^{\bar{V}} + (\hat{w}(BY))^{\bar{V}}(BX)^{\bar{C}} \\
&\quad - (\hat{g}\left(BX,BY\right))^{\bar{C}}\hat{P}^{\bar{V}} - (\hat{g}\left(BX,BY\right))^{\bar{V}}\hat{P}^{\bar{C}}, \\
\bar{\nabla}^C_{\tilde{B}X^C}\tilde{B}Y^C &= \hat{\nabla}^C_{\tilde{B}X^C}\tilde{B}Y^C + \hat{w}^C(\tilde{B}Y^C)\tilde{B}X^V + \hat{w}^V(\tilde{B}Y^C)\left(\tilde{B}X^C\right) \\
&\quad -\hat{g}^C\left(\tilde{B}X^C,\tilde{B}Y^C\right)\hat{P}^{\bar{V}} - \hat{g}^C\left(\tilde{B}X^V,\tilde{B}Y^C\right)\hat{P}^{\bar{C}}.
\end{aligned}$$

Then, we have

$$\begin{aligned}
\bar{\nabla}^C_{\tilde{B}X^C}\tilde{B}Y^C - \bar{\nabla}^C_{\tilde{B}Y^C}\tilde{B}X^C - [X^C,Y^C] &= \hat{w}^C(\tilde{B}Y^C)\tilde{B}X^V + \hat{w}^V(\tilde{B}Y^C)\left(\tilde{B}X^C\right) \\
&\quad -\hat{w}^C(\tilde{B}X^C)\tilde{B}Y^V - \hat{w}^V(\tilde{B}X^C)\left(\tilde{B}Y^C\right).
\end{aligned}$$

Therefore, from the Equation (2.1) and Theorem 2, we obtain

$$\begin{aligned}
\bar{T}^C\left(\tilde{B}X^C,\tilde{B}Y^C\right) &= \hat{w}^C(\tilde{B}Y^C)\tilde{B}X^V + \hat{w}^V(\tilde{B}Y^C)\left(\tilde{B}X^C\right) \\
&\quad -\hat{w}^C(\tilde{B}X^C)\tilde{B}Y^V - \hat{w}^V(\tilde{B}X^C)\left(\tilde{B}Y^C\right).
\end{aligned} \tag{3.3}$$

By computing

$$
\begin{aligned}
& \hat{g}^C(\bar{\nabla}^C_{\tilde{B}X^C}\tilde{B}Y^C,\tilde{B}Z^C)+\hat{g}^C(\tilde{B}Y^C,\bar{\nabla}^C_{\tilde{B}X^C}\tilde{B}Z^C) \\
= & \ \hat{g}^C\left(\begin{array}{c} \hat{\nabla}^C_{\tilde{B}X^C}\tilde{B}Y^C+\hat{w}^C(\tilde{B}Y^C)\tilde{B}X^V \\ +\hat{w}^V(\tilde{B}Y^C)\left(\tilde{B}X^C\right)-\hat{g}^C\left(\tilde{B}X^C,\tilde{B}Y^C\right)\hat{P}^{\bar{V}} \\ -\hat{g}^C\left(\tilde{B}X^V,\tilde{B}Y^C\right)\hat{P}^{\bar{C}},\tilde{B}Z^C \end{array}\right) \\
& +\hat{g}^C\left(\begin{array}{c} \tilde{B}Y^C,\hat{\nabla}^C_{\tilde{B}X^C}\tilde{B}Z^C+\hat{w}^C(\tilde{B}Z^C)\tilde{B}X^V \\ +\hat{w}^V(\tilde{B}Z^C)\left(\tilde{B}X^C\right)-\hat{g}^C\left(\tilde{B}X^C,\tilde{B}Z^C\right)\hat{P}^{\bar{V}} \\ -\hat{g}^C\left(\tilde{B}X^V,\tilde{B}Z^C\right)\hat{P}^{\bar{C}} \end{array}\right) \\
= & \ \hat{g}^C\left(\hat{\nabla}_{\tilde{B}X^C}\tilde{B}Y^C,\tilde{B}Z^C\right)+\hat{g}^C\left(\tilde{B}Z^C,\hat{\nabla}_{\tilde{B}X^C}\tilde{B}Z^C\right) \\
= & \left(\tilde{B}X^C\right)\hat{g}^C(\tilde{B}Y^C,\tilde{B}Z^C),
\end{aligned}
$$

we get

$$\left(\bar{\nabla}^C_{\tilde{B}X^C}\hat{g}^C\right)\left(\tilde{B}Y^C,\tilde{B}Z^C\right)=0 \tag{3.4}$$

The Equation (1) and the Equation (3.4) imply the desired result. ■

Corollary 4 *Let $\mathring{\nabla}$ be a semi-symmetric metric connection with respect to ∇ Riemann connection in (S,g). Then, $\mathring{\nabla}^C$ is also semi-symmetric metric connection with respect to ∇^C Riemann connection in $(TS,\tilde{g})$.*

Proof. We have

$$
\begin{aligned}
\left(\bar{\nabla}_{BX}BY\right)^{\bar{C}} &= \left(\hat{\nabla}_{BX}BY\right)^{\bar{C}}+(\hat{w}(BY)BX)^{\bar{C}}-\left(\hat{g}\left(BX,BY\right)\hat{P}\right)^{\bar{C}}, \\
\left(\bar{\nabla}_{BX}BY\right)^{\bar{C}} &= \left(\hat{\nabla}_{BX}BY\right)^{\bar{C}}+(\hat{w}(BY))^{\bar{C}}(BX)^{\bar{V}}+(\hat{w}(BY))^{\bar{V}}(BX)^{\bar{C}} \\
& \quad -(\hat{g}\left(BX,BY\right))^{\bar{C}}\hat{P}^{\bar{V}}-(\hat{g}\left(BX,BY\right))^{\bar{V}}\hat{P}^{\bar{C}}, \\
\bar{\nabla}^C_{\tilde{B}X^C}\tilde{B}Y^C &= \hat{\nabla}^C_{\tilde{B}X^C}\tilde{B}Y^C+\hat{w}^C(\tilde{B}Y^C)\tilde{B}X^V+\hat{w}^V(\tilde{B}Y^C)\left(\tilde{B}X^C\right) \\
& \quad -\hat{g}^C\left(\tilde{B}X^C,\tilde{B}Y^C\right)\hat{P}^{\bar{V}}-\hat{g}^C\left(\tilde{B}X^V,\tilde{B}Y^C\right)\hat{P}^{\bar{C}},
\end{aligned}
$$

for any $X,Y\in\Im^1_0(S)$. Hence, from the Equation (2.5) and the Equation (3.1) we obtain

$$
\begin{aligned}
\left(B\left(\mathring{\nabla}_XY\right)+m(X,Y)N\right)^{\bar{C}} = & \ (B\left(\nabla_XY\right)+h(X,Y)N)^{\bar{C}}+\hat{w}^C(\tilde{B}Y^C)\tilde{B}X^V \\
& +\hat{w}^V(\tilde{B}Y^C)\left(\tilde{B}X^C\right)-\hat{g}^C\left(\tilde{B}X^C,\tilde{B}Y^C\right)\left(\tilde{B}P^V+\eta^VN^{\bar{V}}\right) \\
& -\hat{g}^C\left(\tilde{B}X^V,\tilde{B}Y^C\right)\left(\tilde{B}P^C+\eta^VN^{\bar{C}}+\eta^CN^{\bar{V}}\right),
\end{aligned}
$$

$$
\begin{aligned}
&\tilde{B}\left(\mathring{\nabla}_X Y\right)^C + m^V(X^C,Y^C)N^{\bar{C}} + m^C(X^C,Y^C)N^{\bar{V}} \\
= \ &\tilde{B}\left(\nabla_X Y\right)^C + h^V(X^C,Y^C)N^{\bar{C}} + h^C(X^C,Y^C)N^{\bar{V}} \\
&+\hat{w}^C(\tilde{B}Y^C)\tilde{B}X^V + \hat{w}^V(\tilde{B}Y^C)\left(\tilde{B}X^C\right) \\
&-\hat{g}^C\left(\tilde{B}X^C,\tilde{B}Y^C\right)\tilde{B}P^V - \eta^V\hat{g}^C\left(\tilde{B}X^C,\tilde{B}Y^C\right)N^{\bar{V}} \\
&-\hat{g}^C\left(\tilde{B}X^V,\tilde{B}Y^C\right)\tilde{B}P^C - \eta^V\hat{g}^C\left(\tilde{B}X^V,\tilde{B}Y^C\right)N^{\bar{C}} \\
&-\eta^C\hat{g}^C\left(\tilde{B}X^V,\tilde{B}Y^C\right)N^{\bar{V}}.
\end{aligned}
$$

Moreover, we get

$$
\begin{aligned}
\tilde{B}\left(\mathring{\nabla}_X Y\right)^C = \ &\tilde{B}\left(\nabla_X Y\right)^C + \hat{w}^C(\tilde{B}Y^C)\tilde{B}X^V + \hat{w}^V(\tilde{B}Y^C)\left(\tilde{B}X^C\right) \\
&-\hat{g}^C\left(\tilde{B}X^C,\tilde{B}Y^C\right)\tilde{B}P^V - \hat{g}^C\left(\tilde{B}X^V,\tilde{B}Y^C\right)\tilde{B}P^C,
\end{aligned} \tag{3.5}
$$

and

$$
\begin{aligned}
m^V(X^C,Y^C)N^{\bar{C}} + m^C(X^C,Y^C)N^{\bar{V}} = \ &\left(h^V(X^C,Y^C) - \eta^V\hat{g}^C\left(\tilde{B}X^V,\tilde{B}Y^C\right)\right)N^{\bar{C}} \\
&+\begin{pmatrix} h^C(X^C,Y^C) - \eta^V\hat{g}^C\left(\tilde{B}X^C,\tilde{B}Y^C\right) \\ -\eta^C\hat{g}^C\left(\tilde{B}X^V,\tilde{B}Y^C\right) \end{pmatrix} N^{\bar{V}}.
\end{aligned} \tag{3.6}
$$

From the Equation (1), it follows that

$$
\left(\mathring{\nabla}_X Y\right)^C = \left(\nabla_X Y\right)^C + w^C(Y^C)X^V + w^V(Y^C)X^C - \tilde{g}(X^C,Y^C)P^V - \tilde{g}(X^V,Y^C)P^C,
$$

and, finally, we obtain

$$
\mathring{\nabla}^C_{X^C}Y^C = \nabla^C_{X^C}Y^C + w^C(Y^C)X^V + w^V(Y^C)X^C - \tilde{g}(X^C,Y^C)P^V - \tilde{g}(X^V,Y^C)P^C.
$$

Thus, we have

$$
\begin{aligned}
\mathring{\nabla}^C_{X^C}Y^C - \mathring{\nabla}^C_{Y^C}X^C - [X^C,Y^C] = \ &w^C(Y^C)X^V + w^V(Y^C)X^C \\
&-w^C(X^C)Y^V - w^V(X^C)Y^C,
\end{aligned}
$$

that is

$$
\mathring{T}^C(X^C,Y^C) = w^C(Y^C)X^V + w^V(Y^C)X^C - w^C(X^C)Y^V - w^V(X^C)Y^C. \tag{3.7}
$$

Similarly

$$
\tilde{g}\left(\mathring{\nabla}^C_{X^C}Y^C,Z^C\right) + \tilde{g}\left(Y^C,\mathring{\nabla}^C_{X^C}Y^C\right) = X^C\left(\tilde{g}\left(Y^C,Z^C\right)\right),
$$

we obtain

$$\left(\mathring{\nabla}^C_{X^C}\tilde{g}\right)(Y^C, Z^C) = 0. \tag{3.8}$$

The Equation (3.7) and the Equation (3.8) complete. ■

The semi-symmetric metric connection $\mathring{\nabla}^C$ on $(TS, \tilde{g})$ can be given by

$$\mathring{\nabla}^C_{X^C}Y^C = \nabla^C_{X^C}Y^C + w^C(Y^C)X^V + w^V(Y^C)X^C - \tilde{g}(X^C, Y^C)P^V - \tilde{g}(X^V, Y^C)P^C,$$

and taking the complete lift of both sides of the Equation (3.1) we obtain

$$\bar{\nabla}^C_{\tilde{B}X^C}\tilde{B}Y^C = \tilde{B}\left(\mathring{\nabla}^C_{X^C}Y^C\right) + m^V(X^C, Y^C)N^{\bar{C}} + m^C(X^C, Y^C)N^{\bar{V}}.$$

From the Equation (1), it follows that

$$\begin{aligned} m^V(X^C, Y^C) &= h^V(X^C, Y^C) - \eta^V\hat{g}^C(\tilde{B}X^V, \tilde{B}Y^C), \\ m^C(X^C, Y^C) &= h^C(X^C, Y^C) - \eta^V\hat{g}^C(\tilde{B}X^C, \tilde{B}Y^C) - \eta^C\hat{g}^C(\tilde{B}X^V, \tilde{B}Y^C). \end{aligned}$$

According to [3], TS is totally umbilical if and only if there exist differentiable functions λ and μ, such that

$$\begin{aligned} m^V(\tilde{X}, \tilde{Y}) &= \lambda\tilde{g}(\tilde{X}, \tilde{Y}), \\ m^C(\tilde{X}, \tilde{Y}) &= \mu\tilde{g}(\tilde{X}, \tilde{Y}), \end{aligned}$$

for any $\tilde{X}, \tilde{Y} \in \Im^1_0(TS)$. If both λ and μ vanish, then TS is totally geodesic.

It is trivial to prove the following theorems by using the Equation (1).

Theorem 5 *TS is totally umbilical with respect to the semi-symmetric metric connection $\mathring{\nabla}^C$ if and only if it is totally umbilical or totally geodesic with respect to the Riemann connection ∇^C.*

Theorem 6 *TS is totally umbilical with respect to the semi-symmetric metric connection $\mathring{\nabla}^C$ if and only if S is totally umbilical with respect to the semi-symmetric metric connection $\mathring{\nabla}$.*

Theorem 7 *TS is totally geodesic with respect to the semi-symmetric metric connection $\mathring{\nabla}^C$ if and only if it is totally geodesic with respect to the Riemann connection ∇^C and the vector field $\hat{P}$ is tangent to S.*

Theorem 8 *TS is totally geodesic with respect to the semi-symmetric metric connection $\mathring{\nabla}^C$ if and only if S is totally geodesic with respect to the semi-symmetric metric connection $\mathring{\nabla}$.*

4 The Structure Equations of Tangent Bundle with Semi-symmetric Metric Connection

Theorem 9 *In [5], the structure equations of S are given by:*

$$\begin{aligned} \bar{\nabla}_{BX}N &= -BMX, \\ g\left(\mathring{R}(X,Y)Z,W\right) &= \hat{g}\left(\bar{R}(BX,BY)BZ,BW\right) \\ &\quad +g\left((MX)\,m(Y,Z)-(MY)\,m(X,Z),W\right), \\ \hat{g}\left(\bar{R}(BX,BY)N,BZ\right) &= g\left(\mathring{\nabla}_Y MX-\mathring{\nabla}_X MY+M[X,Y],Z\right), \end{aligned}$$

for $X,Y,Z\in\Im_0^1(S)$.

Theorem 10 *If $\hat{R}$ is the curvature tensor field of the Riemann connection $\hat{\nabla}$ in $(M,\hat{g})$, then, the complete lift $\hat{R}^C$ of $\hat{R}$ is the curvature tensor field of the Riemann connection $\hat{\nabla}^C$ in $\left(TM,\hat{g}^C\right)$. Similarly, the complete lift R^C of R is the curvature tensor field of the Riemann connection ∇^C in $(TS,\tilde{g})$, where R is the curvature tensor field of the induced connection ∇ on (S,g) [6].*

Let $\bar{R}$ be a curvature tensor field of the semi-symmetric connection $\bar{\nabla}$ in $(M,\hat{g})$. Then, the curvature tensor field of the semi-symmetric connection $\bar{\nabla}^C$ is $\bar{R}^C$ in $(TM,\hat{g}^C)$. Similarly, the complete lift $\mathring{R}^C$ of $\mathring{R}$ is the curvature tensor field of the semi-symmetric metric connection $\mathring{\nabla}^C$ in $(TS,\tilde{g})$ where $\mathring{R}$ is the curvature tensor field of the induced connection $\mathring{\nabla}$ on (S,g).

Theorem 11 *The Weingarten equation of TS is obtained as:*

$$\begin{aligned} \bar{\nabla}^C_{\tilde{B}X^C}N^{\bar{V}} &= -\tilde{B}M^V X^C, \\ \bar{\nabla}^C_{\tilde{B}X^C}N^{\bar{C}} &= -\tilde{B}M^C X^C, \end{aligned}$$

for any $X\in\Im_0^1(S)$.

Proof. Using the Equation (1), Theorem 9 and by virtue of the Section 2 in [3], we get

$$\bar{\nabla}^C_{\tilde{B}X^C}N^{\bar{V}} = \left(\bar{\nabla}_{BX}N\right)^{\bar{V}} = (-BMX)^{\bar{V}} = -\tilde{B}\,(MX)^V = -\tilde{B}M^V X^C,$$
$$\bar{\nabla}^C_{\tilde{B}X^C}N^{\bar{C}} = \left(\bar{\nabla}_{BX}N\right)^{\bar{C}} = (-BMX)^{\bar{C}} = -\tilde{B}(MX)^C = -\tilde{B}M^C X^C. \quad \blacksquare$$

Theorem 12 *The Gauss equation of TS is obtained as:*

$$\begin{aligned} \tilde{g}(\mathring{R}^C\left(X^C,Y^C\right)Z^C,W^C) &= \hat{g}^C\left(\bar{R}^C\left(\tilde{B}X^C,\tilde{B}Y^C\right)\tilde{B}Z^C,\tilde{B}W^C\right) \\ &\quad +\tilde{g}\left(\left(M^CX^C\right)m^V\left(Y^C,Z^C\right)+\left(M^VX^C\right)m^C\left(Y^C,Z^C\right),W^C\right) \\ &\quad -\tilde{g}\left(\left(M^CY^C\right)m^V\left(X^C,Z^C\right)+\left(M^VY^C\right)m^C\left(X^C,Z^C\right),W^C\right), \end{aligned}$$

for any $X,Y,Z\in\Im_0^1(S)$.

Proof. Using the Equation (1), Theorem 9 and by virtue of the Section 2 and the Equation (6.4) in [3], we get

$$\begin{aligned}
\tilde{g}\left(\mathring{R}^{C}\left(X^{C},Y^{C}\right)Z^{C},W^{C}\right) &= \tilde{g}\left(\left(\mathring{R}\left(X,Y\right)Z\right)^{C},W^{C}\right) \\
&= \begin{pmatrix} \hat{g}\left(\bar{R}\left(BX,BY\right)BZ,BW\right) \\ +g\left(\left(MX\right)m(Y,Z)-\left(MY\right)m(X,Z),W\right) \end{pmatrix}^{C} \\
&= \begin{pmatrix} \hat{g}^{C}\left(\left(\bar{R}\left(BX,BY\right)BZ\right)^{\bar{C}},\left(BW\right)^{\bar{C}}\right) \\ +\tilde{g}\left(\left(MX\right)^{C}\left(m(Y,Z)\right)^{V}+\left(MX\right)^{V}\left(m(Y,Z)\right)^{C},W^{C}\right) \\ -\tilde{g}\left(\left(MY\right)^{C}\left(m(X,Z)\right)^{V}+\left(MY\right)^{V}\left(m(Y,Z)\right)^{C},W\right) \end{pmatrix} \\
&= \hat{g}^{C}\left(\bar{R}^{C}\left(\tilde{B}X^{C},\tilde{B}Y^{C}\right)\tilde{B}Z^{C},\tilde{B}W^{C}\right) \\
&\quad +\tilde{g}\left(\left(M^{C}X^{C}\right)m^{V}\left(Y^{C},Z^{C}\right)+\left(M^{V}X^{C}\right)m^{C}\left(Y^{C},Z^{C}\right),W^{C}\right) \\
&\quad -\tilde{g}\left(\left(M^{C}Y^{C}\right)m^{V}\left(X^{C},Z^{C}\right)+\left(M^{V}Y^{C}\right)m^{C}\left(X^{C},Z^{C}\right),W^{C}\right).
\end{aligned}$$

■

Theorem 13 *The Codazzi-Ricci equation of TS is obtained as:*

$$\begin{aligned}
\bar{R}^{C}\left(\tilde{B}X^{C},\tilde{B}Y^{C}\right)N^{\bar{V}} &= \tilde{B}\left(\mathring{\nabla}^{C}_{Y^{C}}M^{V}X^{C}-\mathring{\nabla}^{C}_{X^{C}}M^{V}Y^{C}+M^{V}[X^{C},Y^{C}]\right), \\
\bar{R}^{C}\left(\tilde{B}X^{C},\tilde{B}Y^{C}\right)N^{\bar{C}} &= \tilde{B}\left(\mathring{\nabla}^{C}_{Y^{C}}M^{C}X^{C}-\mathring{\nabla}^{C}_{X^{C}}M^{C}Y^{C}+M^{C}[X^{C},Y^{C}]\right), \\
\bar{R}^{C}\left(N^{\bar{V}},N^{\bar{C}}\right)\tilde{B}X^{C} &= 0,
\end{aligned}$$

for any $X,Y,Z\in\Im^{1}_{0}(S)$.

Proof. Using the Equation (1), Theorem 9 and by virtue of the Section 2 and the Equation (6.4) in [3], we get

$$\begin{aligned}
\bar{R}^{C}\left(\tilde{B}X^{C},\tilde{B}Y^{C}\right)N^{\bar{V}} &= \left(\bar{R}\left(BX,BY\right)N\right)^{\bar{V}} \\
&= \left(B\left(\mathring{\nabla}_{Y}MX-\mathring{\nabla}_{X}MY-M[X,Y]\right)\right)^{\bar{V}} \\
&= \tilde{B}\left(\mathring{\nabla}_{Y}MX-\mathring{\nabla}_{X}MY-M[X,Y]\right)^{V} \\
&= \tilde{B}\left(\mathring{\nabla}^{C}_{Y^{C}}M^{V}X^{C}-\mathring{\nabla}^{C}_{X^{C}}M^{V}Y^{C}+M^{V}[X^{C},Y^{C}]\right),
\end{aligned}$$

$$\begin{aligned}
\bar{R}^C\left(\tilde{B}X^C, \tilde{B}Y^C\right)N^{\bar{C}} &= \left(\bar{R}\left(BX, BY\right)N\right)^{\bar{C}} \\
&= \left(B\left(\mathring{\nabla}_Y MX - \mathring{\nabla}_X MY - M[X,Y]\right)\right)^{\bar{C}} \\
&= \tilde{B}\left(\mathring{\nabla}_Y MX - \mathring{\nabla}_X MY - M[X,Y]\right)^C \\
&= \tilde{B}\left(\mathring{\nabla}^C_{Y^C} M^C X^C - \mathring{\nabla}^C_{X^C} M^C Y^C + M^C[X^C, Y^C]\right),
\end{aligned}$$

and

$$\bar{R}^C\left(N^{\bar{V}}, N^{\bar{C}}\right)\tilde{B}X^C = \left(\bar{R}\left(N,N\right)BX\right)^{\bar{C}} = 0.$$

■

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