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Some Results on Cauchy's Proper Bound for the Zeros of Entire Functions

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Abstract

The aim of this paper is to deduce the bounds for the moduli of zeros of entire functions. Some examples are provided to clear the notions.

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1 Introduction, Definitions and Notations.

Let

$$P(z) = a_0 + a_1 z + a_2 z^2 + a_3 z^3 + ... + a_{n-1} z^{n-1} + a_n z^n; |a_n| \neq 0$$

be a polynomial of degree n. Datt and Govil[2]; Govil and Rahaman[4]; Marden[8]; Mohammad[9]; Chattopadhyay, Das, Jain and Konwer[1]; Joyal, Labelle and Rahaman[5]; Jain{[6],[7]}; Sun and Hsieh[10]; Zilovic, Roytman, Combettes and Swamy[12]; Das and Datta[3] etc. worked in the theory of the distribution of the zeros of polynomials and obtained some newly developed results.

In this paper we intend to establish some sharper results concerning the theory of distribution of zeros of entire functions.

2 Lemma.

In this section we present a lemma which will be needed in the sequel.

Lemma 1 *[11] If $f(z)$ is an entire function of order ρ then for every $\epsilon > 0$ and for all sufficiently large r the inequality $N(r) \leq r^{\rho+\epsilon}$ holds where $N(r)$ is the number of zeros of $f(z)$ in $|z| \leq r$.*

3 Theorems.

Theorem 1 *Let $P(z)$ be an entire function defined as*

$$P(z) = a_0 + a_1 z + ... + a_n z^n + ...$$

whose order ρ is finite. Also for all suffiently large r in the disc $|z| \leq r$, $a_0 \neq 0$ and $a_{N(r)} \neq 0$. Also $a_n \to 0$ as $n > N(r)$. Then all the zeros of $P(z)$ lie in the ring shaped region

$$\frac{1}{t_0'} \leq |z| \leq t_0$$

where t_0 and t_0' are the positive roots of the equations

$$g(t) \equiv \left|a_{N(r)}\right| t^{N(r)} - \left|a_{N(r)-1}\right| t^{N(r)-1} - .... - |a_0| = 0$$

and

$$h(t) \equiv |a_0| t^{N(r)} - |a_1| t^{N(r)-1} - .... - \left|a_{N(r)}\right| = 0$$

respectively in $|z| \leq r$ and $N(r)$ denotes the number of zeros of $P(z)$ in $|z| \leq r$ for sufficiently large r.

Proof. Since $P(z)$ is an entire function of finite order ρ then from Lemma 1 we have for sufficiently large r in the disc $|z| \leq r$,

$$N(r) \leq r^{\rho+\epsilon} \text{ for } \epsilon > 0.$$

Also $a_0 \neq 0$ and $a_{N(r)} \neq 0$. Further $a_n \to 0$ as $n > N(r)$.
Hence we have

$$\begin{aligned} P(z) &= a_0 + a_1 z + \dots\dots\dots\dots + a_n z^n + \dots\dots \\ &\approx a_0 + a_1 z + \dots\dots\dots + a_{N(r)} z^{N(r)}. \end{aligned}$$

Therefore

$$\begin{aligned} |P(z)| &\approx \left| a_0 + a_1 z + \dots\dots\dots\dots + a_{N(r)} z^{N(r)} \right| \\ &\geq \left| a_{N(r)} \right| |z|^{N(r)} - \left| a_{N(r)-1} \right| |z|^{N(r)-1} \dots\dots\dots\dots - |a_0| \end{aligned} \tag{1}$$

in the disc $|z| \leq r$ for sufficiently large r. In fact (1) can be deduced in the following way

$$\begin{aligned} \left| a_0 + \dots\dots + a_{N(r)-1} z^{N(r)-1} \right| &\leq |a_0| + \dots\dots + \left| a_{N(r)-1} \right| |z|^{N(r)-1} \\ \text{i.e.,} \quad - |a_0| \dots\dots - \left| a_{N(r)-1} \right| |z|^{N(r)-1} &\leq - \left| a_0 + \dots\dots + a_{N(r)-1} z^{N(r)-1} \right|. \end{aligned}$$

Hence

$$\begin{aligned} &\left| a_{N(r)} z^{N(r)} + a_{N(r)-1} z^{N(r)-1} + \dots\dots + a_0 \right| \\ &\geq \left| a_{N(r)} \right| |z|^{N(r)} - \left| a_{N(r)-1} z^{N(r)-1} + \dots\dots + a_0 \right| \\ &\geq \left| a_{N(r)} \right| |z|^{N(r)} - \left| a_{N(r)-1} \right| |z|^{N(r)-1} - \dots\dots - |a_0|. \end{aligned}$$

Now let us write

$$g(t) \equiv \left| a_{N(r)} \right| t^{N(r)} - \left| a_{N(r)-1} \right| t^{N(r)-1} - \dots\dots\dots\dots - |a_0|. \tag{2}$$

Since (2) has one change of sign, by Descartes' rule of sign, the maximum number of positive root of (2) is one. Moreover

$$g(0) = - |a_0| < 0$$

and $g(\infty)$ is a positive quantity.
Clearly $t > t_0$ implies $g(t) > 0$.
If not, let for some $t_1 > t_0$, $g(t_1) < 0$.
Then $g(t) = 0$ has another positive root in (t_1, ∞) which gives a contradiction.

Hence $g(t) > 0$ for $t > t_0$.
Therefore $|P(z)| > 0$ for $|z| > t_0$. So $P(z)$ does not vanish in $|z| > t_0$ and therefore all the zeros of $P(z)$ lie in $|z| \leq t_0$ where t_0 is the positive root of

$$g(t) \equiv \left|a_{N(r)}\right| t^{N(r)} - \left|a_{N(r)-1}\right| t^{N(r)-1} - \ldots\ldots\ldots\ldots - |a_0| = 0.$$

Now we give the proof of the other part of the theorem.
Let us consider

$$Q(z) = z^{N(r)} P\left(\frac{1}{z}\right) \tag{3}$$

for sufficently large r in the disc $|z| \leq r$. Now

$$\begin{aligned} Q(z) &= z^{N(r)} P\left(\frac{1}{z}\right) \\ &\approx z^{N(r)} \left[a_0 + \frac{a_1}{z} + \ldots\ldots\ldots\ldots\ldots\ldots + a_{N(r)} \frac{1}{z^{N(r)}}\right] \\ &= a_0 z^{N(r)} + a_1 z^{N(r)-1} + \ldots\ldots\ldots\ldots\ldots + a_{N(r)}. \end{aligned} \tag{4}$$

Again we have

$$\left|a_1 z^{N(r)-1} + \ldots\ldots. + a_{N(r)}\right| \leq |a_1| \, |z|^{N(r)-1} + |a_2| \, |z|^{N(r)-2} + \ldots\ldots + \left|a_{N(r)}\right|$$

i.e.,

$$- |a_1| \, |z|^{N(r)-1} - \ldots\ldots\ldots\ldots - \left|a_{N(r)}\right| \leq - \left|a_1 z^{N(r)-1} + \ldots\ldots\ldots. + a_{N(r)}\right|.$$

So we get that

$$\begin{aligned} \left|a_0 z^{N(r)} + \ldots\ldots + a_{N(r)}\right| &\geq |a_0| \, |z|^{N(r)} - \left|a_1 z^{N(r)-1} + \ldots\ldots. + a_{N(r)}\right| \\ &\geq |a_0| \, |z|^{N(r)} - |a_1| \, |z|^{N(r)-1} - \ldots.. - \left|a_{N(r)}\right|. \end{aligned} \tag{5}$$

Let us consider the equation

$$h(t) \equiv |a_0| \, |t|^{N(r)} - |a_1| \, |t|^{N(r)-1} - \ldots\ldots - \left|a_{N(r)}\right| = 0. \tag{6}$$

Since (6) has one change of sign, by Descartes' rule of sign the maximum number of positive root of (6) is one. Moreover

$$h(0) = - \left|a_{N(r)}\right| < 0$$

and $h(\infty)$ is a positive quantity. So $h(t)$ has exactly one positive root.

Let $t_0^{'}$ be the positive root of $h(t)=0$. Clearly for $t>t_0^{'}$ we get $h(t)>0$. If not, let $t_1^{'}>t_0^{'}$. Then $h(t_1^{'})<0$. Hence $h(t)=0$ has another positive root in $(t_1^{'},\infty)$ which gives a contradiction.
Therefore $h(t)>0$ for $t>t_0^{'}$ and $|Q(z)|>0$ for $|z|>t_0^{'}$.
So $Q(z)$ does not vanish in $|z|>t_0^{'}$ and therefore all the zeros of $Q(z)$ lie in $|z|\leq t_0^{'}$. Let $z=z_0$ be any zero of $P(z)=0$. Clearly $z_0\neq 0$ as $|a_0|\neq 0$.

Putting $z=\frac{1}{z_0}$ in $Q(z)$ we get that

$$Q\left(\frac{1}{z_0}\right)=\left(\frac{1}{z_0}\right)^{N(r)}P(z_{0)}=\left(\frac{1}{z_0}\right)^{N(r)}.0=0.$$

So $\frac{1}{z_0}$is a zero of $Q(z)$. Therefore$\left|\frac{1}{z_0}\right|\leq t_0^{'}$ i.e, $|z_0|\geq\frac{1}{t_0^{'}}$. Since z_0 is any arbitrary zero of $P(z)$, all the zeros of $P(z)$ lie in $|z|\geq\frac{1}{t_0^{'}}$.

Hence all the zeros of $P(z)$ lie in the ring shaped region

$$\frac{1}{t_0^{'}}\leq|z|\leq t_0$$

where t_0 and $t_0^{'}$ are the positive roots of

$$g(t)\equiv\left|a_{N(r)}\right|t^{N(r)}-\left|a_{N(r)-1}\right|t^{N(r)-1}-........-|a_0|=0$$

and

$$h(t)\equiv|a_0|\,t^{N(r)}-|a_1|\,t^{N(r)-1}-............\left|a_{N(r)}\right|=0$$

respectively for sufficiently large r in the disc $|z|\leq r$. ∎

Remark 2 *The limit in Theorem 1 is attained by $P(z)=nz^2+(n-1)z-1$ for any positive real number $n\geq 2$. It can be easily seen that $M(r)=|n|\,r^2=nr^2$ for large r in $|z|=r$.*
So

$$\begin{aligned}\rho=\limsup_{r\to\infty}\frac{\log\log M(r)}{\log r}&=\limsup_{r\to\infty}\frac{\log\log(nr^2)}{\log r}\\&=\limsup_{r\to\infty}\frac{\frac{1}{\log(nr^2)}\cdot\frac{1}{nr^2}.n.2r}{\frac{1}{r}}\\&=\limsup_{r\to\infty}\frac{2}{\log(nr^2)}=0.\end{aligned}$$

Hence the order of the polynomial is 0. Also here $N(r) = 2 \leq r^{0+\epsilon}$ *for* $\epsilon > 0$ *and* r *be sufficiently large in* $|z| \leq r$*. The zeros of* $P(z)$ *is given by solving* $P(z) = 0$ *and* $a_{N(r)} = n \neq 0$, $a_{N(r)+1} = a_{N(r)+2} = ... = 0$*. Now*

$$\begin{aligned} nz^2 + (n-1)z - 1 &= 0 \\ i.e., if\ (nz-1)(z+1) &= 0 \\ i.e,. if\ z &= \frac{1}{n}, -1. \end{aligned}$$

Let $z_1 = \frac{1}{n}$ *and* $z_2 = -1$*. Then* z_1 *and* z_2 *are the zeros of*

$$P(z) = nz^2 + (n-1)z - 1 = 0.$$

Here $a_0 = 1$, $a_1 = n - 1$ *and* $a_2 = n$*. Therefore* $|a_0| = 1$, $|a_1| = n - 1$, $|a_2| = n$ *and so*

$$\begin{aligned} f(t) &\equiv |a_2|\, t^2 - |a_1|\, t - |a_0| \\ &= nt^2 - (n-1)t - 1 = 0. \end{aligned}$$

Hence $t = 1$ *and* $-\frac{1}{n}$.
Thus the positive root of $f(t) = 0$ *is* $t = t_0 = 1$.

Again to find the positive root of

$$g(t) \equiv |a_0|\, t^2 - |a_1|\, t - |a_2| = 0$$

we get that

$$t^2 - (n-1)t - n = 0.$$

which implies $t = n$ *and* $t = -1$*. Therefore the positive root of* $g(t) = 0$ *is* $t_0' = n$.
Hence according to Theorem 1 all the zeros of $P(z)$ *lie in*

$$\begin{aligned} \frac{1}{t_0'} &\leq |z| \leq t_0 \\ i.e., in\ \ \frac{1}{n} &\leq |z| \leq 1. \end{aligned}$$

Theorem 3 *Let* $P(z)$ *be an entire function defined by*

$$P(z) = a_0 + a_1 z + ... + a_n z^n + ...$$

whose order ρ is finite. Also for all sufficiently large r in the disc $|z| \leq r$, $a_{N(r)} \neq 0$ and $a_0 \neq 0$. Further $a_n \to 0$ as $n > N(r)$.
Then all the zeros of $P(z)$ lie in the ring shaped region

$$\frac{1}{1+M'} < |z| < 1+M$$

$$\text{where } M = \max_{0 \leq k \leq N(r)-1} \left| \frac{a_k}{a_{N(r)}} \right| \text{ and } M' = \max_{0 \leq k \leq N(r)-1} \left| \frac{a_k}{a_0} \right|.$$

Proof. Since $P(z)$ is an entire functuion of order ρ, then by Lemma 1 for sufficiently large values of r in $|z| \leq r$ we have $N(r) \leq r^{\rho+\epsilon}$ for $\epsilon > 0$. Also $a_0 \neq 0$, $a_{N(r)} \neq 0$, and $a_n \to 0$ as $n > N(r)$. Hence we may write

$$\begin{aligned} P(z) &= a_0 + a_1 z + ... + a_n z^n + ... \\ &\approx a_0 + a_1 z + ... + a_{N(r)} z^{N(r)}. \end{aligned}$$

Now

$$\begin{aligned} &\left|a_0 + a_1 z + ... + a_{N(r)-1} z^{N(r)-1}\right| \\ &\leq |a_0| + ... + \left|a_{N(r)-1}\right| |z|^{N(r)-1} \\ &= \left|a_{N(r)}\right| \left\{ \frac{|a_0|}{\left|a_{N(r)}\right|} + ... + \frac{\left|a_{N(r)-1}\right|}{\left|a_{N(r)}\right|} |z|^{N(r)-1} \right\} \end{aligned}$$

i.e.,

$$\begin{aligned} &\left|a_0 + a_1 z + ... + a_{N(r)-1} z^{N(r)-1}\right| \\ &\leq \left|a_{N(r)}\right| M \left(|z|^{N(r)-1} + |z|^{N(r)-2} + ... + 1 \right) \\ &= \left|a_{N(r)}\right| M |z|^{N(r)} \left\{ \frac{1}{|z|} + \frac{1}{|z|^2} + ... + \frac{1}{|z|^{N(r)}} \right\} \end{aligned}$$

where $|z| \neq 0$. Therefore when $|z| \neq 0$,

$$\begin{aligned} &- \left|a_0 + a_1 z + ... + a_{N(r)-1} z^{N(r)-1}\right| \\ &\geq - \left|a_{N(r)}\right| M |z|^{N(r)} \left\{ \frac{1}{|z|} + \frac{1}{|z|^2} + ... + \frac{1}{|z|^{N(r)}} \right\}. \end{aligned}$$

So for $|z| \neq 0$

$$\begin{aligned} &\left|a_{N(r)}\right| |z|^{N(r)} - \left|a_0 + a_1 z + ... + a_{N(r)-1} z^{N(r)-1}\right| \\ &\geq \left|a_{N(r)}\right| |z|^{N(r)} - \left|a_{N(r)}\right| |z|^{N(r)} M \left\{ \frac{1}{|z|} + \frac{1}{|z|^2} + ... + \frac{1}{|z|^{N(r)}} \right\}. \qquad (7) \end{aligned}$$

Now

$$\begin{aligned}|P(z)| &\approx \left|a_0 + a_1 z + ... + a_{N(r)} z^{N(r)}\right| \\ &\geq \left|a_{N(r)}\right| |z|^{N(r)} - \left|a_0 + a_1 z + ... + a_{N(r)-1} z^{N(r)-1}\right|.\end{aligned}$$

Using (7) we have

$$\begin{aligned}|P(z)| &\geq \left|a_{N(r)}\right| |z|^{N(r)} - \left|a_{N(r)}\right| |z|^{N(r)} M \left\{\frac{1}{|z|} + \frac{1}{|z|^2} + ... + \frac{1}{|z|^{N(r)}}\right\} \\ &= \left|a_{N(r)}\right| |z|^{N(r)} \left\{1 - M\left(\frac{1}{|z|} + \frac{1}{|z|^2} + ... + \frac{1}{|z|^{N(r)}}\right)\right\} \text{ for } |z| \neq 0.\end{aligned}$$

i.e., when $|z| \neq 0$

$$|P(z)| > \left|a_{N(r)}\right| |z|^{N(r)} \left\{1 - M\left(\frac{1}{|z|} + \frac{1}{|z|^2} + ... + \frac{1}{|z|^{N(r)}} + ...\right)\right\}.$$

Therefore

$$|P(z)| > \left|a_{N(r)}\right| |z|^{N(r)} \left\{1 - M \sum_{j=1}^{\infty} \frac{1}{|z|^j}\right\} \text{ for } |z| \neq 0.$$

Now the geometric series $\sum_{j=1}^{\infty} \frac{1}{|z|^j}$ is convergent when $\frac{1}{|z|} < 1$ i.e., when $|z| > 1$ and is equal to

$$\frac{1}{|z|} \frac{1}{1 - \frac{1}{|z|}} = \frac{1}{|z| - 1}.$$

On $|z| > 1$ we can write

$$|P(z)| > \left|a_{N(r)}\right| |z|^{N(r)} \left(1 - \frac{M}{|z| - 1}\right).$$

Now on $|z| > 1$,

$$\begin{aligned}|P(z)| > 0 \text{ if } &\left|a_{N(r)}\right| |z|^{N(r)} \left(1 - \frac{M}{|z| - 1}\right) \geq 0 \\ \text{i.e, if } \quad & 1 - \frac{M}{|z| - 1} \geq 0 \\ \text{i.e, if } \quad & |z| - 1 \geq M \\ \text{i.e, if } \quad & |z| \geq M + 1.\end{aligned}$$

Therefore

$$|z| \geq M + 1 > 1 \text{ as } M > 0.$$

Hence

$$|P(z)| > 0 \text{ if } |z| \geq M + 1.$$

Therefore all the zeros of $P(z)$ lie in $|z| < M + 1$.
Secondly, we give the proof of the lower bound. Let us consider

$$Q(z) = z^{N(r)} P\left(\frac{1}{z}\right).$$

Therefore

$$\begin{aligned} Q(z) &= |z|^{N(r)} \left\{ a_0 + \frac{a_1}{|z|} + \dots\dots\dots + \frac{a_{N(r)}}{|z|^{N(r)}} \right\} \\ &= a_0 |z|^{N(r)} + a_1 |z|^{N(r)-1} + \dots\dots. + a_{N(r)}. \end{aligned}$$

Now

$$\begin{aligned} \left| a_1 |z|^{N(r)-1} + \dots\dots. + a_{N(r)} \right| &\leq |a_1| |z|^{N(r)-1} + \dots\dots.. + \left| a_{N(r)} \right| \\ &= |a_0| \left(\frac{|a_1|}{|a_0|} |z|^{N(r)-1} + \dots\dots\dots + \frac{\left| a_{N(r)} \right|}{|a_0|} \right) \\ &\leq |a_0| M' \left(|z|^{N(r)-1} + \dots\dots\dots + 1 \right) \\ &= |a_0| M' |z|^{N(r)} \left(\frac{1}{|z|} + \dots\dots\dots + \frac{1}{|z|^{N(r)}} \right). \end{aligned}$$

Therefore

$$-\left| a_1 |z|^{N(r)-1} + \dots\dots. + a_{N(r)} \right| \geq -|a_0| M' |z|^{N(r)} \left(\frac{1}{|z|} + \dots\dots\dots + \frac{1}{|z|^{N(r)}} \right).$$

So

$$\begin{aligned} |Q(z)| &\geq |a_0|\,|z|^{N(r)} - \left|a_1 z^{N(r)-1} + ... + a_{N(r)}\right| \\ &\geq |a_0|\,|z|^{N(r)} - |a_0|\,M'\,|z|^{N(r)}\left(\frac{1}{|z|} + ... + \frac{1}{|z|^{N(r)}}\right) \\ &= |a_0|\,|z|^{N(r)}\left\{1 - M'\left(\frac{1}{|z|} + ... + \frac{1}{|z|^{N(r)}}\right)\right\} \\ &\geq |a_0|\,|z|^{N(r)}\left\{1 - M'\left(\frac{1}{|z|} + ... + \frac{1}{|z|^{N(r)}} + ...\right)\right\}. \end{aligned}$$

Hence using above we get that

$$|Q(z)| > |a_0|\,|z|^{N(r)}\left\{1 - M'\sum_{j=1}^{\infty}\frac{1}{|z|^j}\right\}.$$

Now the geometric series $\sum\limits_{j=1}^{\infty}\frac{1}{|z|^j}$ is convergent when $\frac{1}{|z|} < 1$ i.e., $|z| > 1$ and is equal to

$$\frac{1}{|z|}\frac{1}{1-\frac{1}{|z|}} = \frac{1}{|z|-1}.$$

On $|z| > 1$ we may write

$$|Q(z)| \geq |a_0|\,|z|^{N(r)}\left(1 - \frac{M'}{|z|-1}\right).$$

Now for $|z| > 1$,

$$\begin{aligned} |Q(z)| > 0 \text{ if} &\quad |a_0|\,|z|^{N(r)}\left(1 - \frac{M'}{|z|-1}\right) \geq 0 \\ \text{i.e., if} &\quad 1 - \frac{M'}{|z|-1} \geq 0 \\ \text{i.e., if} &\quad |z| \geq 1 + M'. \end{aligned}$$

Therefore $|z| \geq 1+ M' > 1$ as $M' > 0$.
Hence $|Q(z)| > 0$ for $|z| \geq 1+ M'$.

So all the zeros of $Q(z)$ lie in $|z| < 1+ M'$.
Let $z = z_0$ be any zero of $P(z)$. Therefore $P(z_0) = 0$. Clearly $z_0 \neq 0$ as $|a_0| \neq 0$.
Putting $z = \frac{1}{z_0}$ in $|Q(z)|$ we have

$$\left|Q\left(\frac{1}{z_0}\right)\right| = \left(\frac{1}{z_0}\right)^n P(z_0) = 0.$$

Therefore $z = \frac{1}{z_0}$is a root of $Q(z)$. So

$$\left|\frac{1}{z_0}\right| < 1 + M',$$

which implies that

$$|z_0| > \left|\frac{1}{1+M'}\right|.$$

As z_0 is an arbitrary root of $P(z) = 0$, all the zeros of $P(z)$ lie in $|z| > \left|\frac{1}{1+M'}\right|$. Hence all the zeros of $P(z)$ lie in the ring shaped region

$$\frac{1}{1+M'} < |z| < 1 + M.$$

This proves the theorem. ■

Remark 4 *Let us consider the polynomial*

$$P(z) = nz^2 + (n-1)z - 1, n \geq 2.$$

$$\textit{Here} \qquad a_0 = -1, a_1 = n-1, a_2 = n.$$

Therefore

$$|a_0| = 1, |a_1| = n-1, |a_2| = n.$$

Also order ρ *of* $P(z)$ *is* 0 *and so* $N(r) = 2 \leq r^{\rho+\epsilon} = r^{\epsilon}$ *for sufficiently large* r.

$$\textit{Again} \quad M = \max\left\{\frac{|a_0|}{|a_1|}, \frac{|a_1|}{|a_2|}\right\} = \max\left\{\frac{1}{n}, \frac{n-1}{n}\right\} = \frac{n-1}{n}$$

$$\textit{and } M' = \max\left\{\frac{|a_1|}{|a_0|}, \frac{|a_2|}{|a_0|}\right\} = \max\left\{\frac{n-1}{n}, \frac{n}{1}\right\} = n.$$

The roots of $P(z) = 0$ *are* $z_1 = \frac{1}{n}$ *and* $z_2 = -1$. *So by Theorem 2 the roots of* $P(z)$ *lies in*

$$\frac{1}{1+M'} < |z| < 1 + M.$$

$$\textit{Hence} \qquad \frac{1}{1+n} < |z| < 1 + \frac{n-1}{n}$$

$$\textit{i.e.,} \qquad \frac{1}{1+n} < |z| < 2 - \frac{1}{n}.$$

Theorem 5 *Let $P(z)$ be an entire function defined by*

$$P(z) = a_0 + a_1 z + a_2 z^2 + ... + a_n z^n + ...$$

whose order ρ is finite. Also for sufficiently large r in the disc $|z| \leq r, a_{N(r)} \neq 0, a_0 \neq 0$ and $a_n \to 0$ as $n > N(r)$. For any p, q with $p > 1, q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, all the zeros of $P(z)$ lie in the annular region

$$\frac{1}{\left[1 + \left(\sum_{j=0}^{N(r)-1} \left|\frac{a_{N(r)-j}}{a_0}\right|^p\right)^{\frac{q}{p}}\right]^{\frac{1}{q}}} < |z| < \left[1 + \left(\sum_{j=0}^{N(r)-1} \left|\frac{a_j}{a_{N(r)}}\right|^p\right)^p\right]^{\frac{q}{p}}.$$

Proof. Given that $a_0 \neq 0$, $a_{N(r)} \neq 0$ and $a_n \to 0$ as $n > N(r)$. Therefore for sufficiently large r in the disc $|z| \leq r$ the existence of $N(r)$ implies that

$$\begin{aligned} P(z) &= a_0 + a_1 z + a_2 z^2 + ... + a_n z^n + ... \\ &\approx a_0 + a_1 z + a_2 z^2 + ... + a_{N(r)} z^{N(r)}. \end{aligned}$$

Now

$$\begin{aligned} &\left|a_0 + a_1 z + a_2 z^2 + ... + a_{N(r)-1} z^{N(r)-1}\right| \\ &\leq |a_0| + |a_1| |z| + ... \left|a_{N(r)-1}\right| |z|^{N(r)-1} \\ &= \left|a_{N(r)}\right| \left\{\frac{|a_0|}{\left|a_{N(r)}\right|} + ... + \frac{\left|a_{N(r)-1}\right|}{\left|a_{N(r)}\right|} |z|^{N(r)-1}\right\} \\ &= \left|a_{N(r)}\right| \sum_{j=0}^{N(r)-1} \left|\frac{a_j}{a_{N(r)}}\right| |z|^j . \end{aligned} \tag{8}$$

Therefore using (8) we get that

$$\begin{aligned} |P(z)| &\approx \left|a_0 + a_1 z + a_2 z^2 + ... + a_{N(r)} z^{N(r)}\right| \\ &\geq \left|a_{N(r)}\right| |z|^{N(r)} - \left|a_0 + a_1 z + a_2 z^2 + ... + a_{N(r)-1} z^{N(r)-1}\right| \end{aligned}$$

$$\geq \left|a_{N(r)}\right| |z|^{N(r)} - \left|a_{N(r)}\right| \sum_{j=0}^{N(r)-1} \left|\frac{a_j}{a_{N(r)}}\right| |z|^j$$

$$\text{i.e.,} \quad |P(z)| \geq \left|a_{N(r)}\right| \left\{|z|^{N(r)} - \sum_{j=0}^{N(r)-1} \left|\frac{a_j}{a_{N(r)}}\right| |z|^j\right\}.$$

By Holder's inequality we have

$$\sum_{j=0}^{N(r)-1}\left|\frac{a_j}{a_{N(r)}}\right||z|^j \le \left(\sum_{j=0}^{N(r)-1}\left|\frac{a_j}{a_{N(r)}}\right|^p\right)^{\frac{1}{p}}\left(\sum_{j=0}^{N(r)-1}\left(|z|^j\right)^q\right)^{\frac{1}{q}}. \tag{9}$$

In view of (9) we obtain that

$$\begin{aligned}
|P(z)| &\ge \left|a_{N(r)}\right|\left\{|z|^{N(r)} - \left(\sum_{j=0}^{N(r)-1}\left|\frac{a_j}{a_{N(r)}}\right|^p\right)^{\frac{1}{p}}\left(\sum_{j=0}^{N(r)-1}\left(|z|^j\right)^q\right)^{\frac{1}{q}}\right\}\\
&= \left|a_{N(r)}\right|\left\{|z|^{N(r)} - |z|^{N(r)}\left(\sum_{j=0}^{N(r)-1}\left|\frac{a_j}{a_{N(r)}}\right|^p\right)^{\frac{1}{p}}\left(\sum_{j=0}^{N(r)-1}\frac{|z|^{jq}}{|z|^{N(r)q}}\right)^{\frac{1}{q}}\right\}\\
&= \left|a_{N(r)}\right||z|^{N(r)}\left\{1 - \left(\sum_{j=0}^{N(r)-1}\left|\frac{a_j}{a_{N(r)}}\right|^p\right)^{\frac{1}{p}}\left(\sum_{j=0}^{N(r)-1}\left(\frac{1}{|z|^q}\right)^{N(r)-j}\right)^{\frac{1}{q}}\right\}\\
&= \left|a_{N(r)}\right||z|^{N(r)}\left\{1 - \left(\sum_{j=0}^{N(r)-1}\left|\frac{a_j}{a_{N(r)}}\right|^p\right)^{\frac{1}{p}}\left(\sum_{j=1}^{N(r)}\left(\frac{1}{|z|^q}\right)^{j}\right)^{\frac{1}{q}}\right\}.
\end{aligned}$$

Now the geometric series $\sum\limits_{j=1}^{N(r)}\left(\frac{1}{|z|^q}\right)^j$ is convergent for

$$\begin{aligned}
\left|\frac{1}{|z|^q}\right| &< 1\\
\text{i.e., for } |z|^q &> 1\\
\text{i.e., for } |z| &> 1
\end{aligned}$$

and is convergent to

$$\left|\frac{1}{|z|^q}\right|.\frac{1}{1-\frac{1}{|z|^q}} = \frac{1}{|z|^q - 1}.$$

So

$$\left(\sum_{j=1}^{N(r)}\left(\frac{1}{|z|^q}\right)^j\right)^{\frac{1}{q}} \text{ converges to } \left(\frac{1}{|z|^q-1}\right)^{\frac{1}{q}} \text{ for } |z|>1.$$

Therefore on $|z| > 1$

$$|P(z)| > \left|a_{N(r)}\right| |z|^{N(r)} \left\{ 1 - \left(\sum_{j=0}^{N(r)-1} \left| \frac{a_j}{a_{N(r)}} \right|^p \right)^{\frac{1}{p}} \left(\frac{1}{|z|^q - 1} \right)^{\frac{1}{q}} \right\}.$$

Now if $|P(z)| > 0$ then we have

$$\left|a_{N(r)}\right| |z|^{N(r)} \left\{ 1 - \left(\sum_{j=0}^{N(r)-1} \left| \frac{a_j}{a_{N(r)}} \right|^p \right)^{\frac{1}{p}} \left(\frac{1}{|z|^q - 1} \right)^{\frac{1}{q}} \right\} \geq 0$$

$$\text{i.e,} \qquad 1 - \left(\sum_{j=0}^{N(r)-1} \left| \frac{a_j}{a_{N(r)}} \right|^p \right)^{\frac{1}{p}} \left(\frac{1}{|z|^q - 1} \right)^{\frac{1}{q}} \geq 0$$

$$\text{i.e,} \qquad (|z|^q - 1)^{\frac{1}{q}} \geq \left(\sum_{j=0}^{N(r)-1} \left| \frac{a_j}{a_{N(r)}} \right|^p \right)^{\frac{1}{p}}$$

$$\text{i.e.,} \qquad |z|^q - 1 \geq \left(\sum_{j=0}^{N(r)-1} \left| \frac{a_j}{a_{N(r)}} \right|^p \right)^{\frac{q}{p}}$$

$$\text{i.e.,} \qquad |z| \geq \left[1 + \left(\sum_{j=0}^{N(r)-1} \left| \frac{a_j}{a_{N(r)}} \right|^p \right)^{\frac{q}{p}} \right]^{\frac{1}{q}}.$$

Clearly

$$\left[1 + \left(\sum_{j=0}^{N(r)-1} \left| \frac{a_j}{a_{N(r)}} \right|^p \right)^{\frac{q}{p}} \right]^{\frac{1}{q}} > 1.$$

Therefore $|P(z)| > 0$ for

$$|z| \geq \left[1 + \left(\sum_{j=0}^{N(r)-1} \left| \frac{a_j}{a_{N(r)}} \right|^p \right)^{\frac{q}{p}} \right]^{\frac{1}{q}}.$$

Therefore all the zeros of $P(z)$ lie in

$$|z| < \left[1 + \left(\sum_{j=0}^{N(r)-1} \left| \frac{a_j}{a_{N(r)}} \right|^p \right)^{\frac{q}{p}} \right]^{\frac{1}{q}}. \tag{10}$$

For the lower bound let us take $Q(z) = z^{N(r)} P(\frac{1}{z})$. Therefore

$$\begin{aligned} Q(z) &= z^{N(r)} P\left(\frac{1}{z}\right) \\ &\approx z^{N(r)} \left\{a_0 + \frac{a_1}{z} + ... + \frac{a_{N(r)}}{z^{N(r)}}\right\} \\ &= a_0 z^{N(r)} + a_1 z^{N(r)-1} + ... + a_{N(r)}. \end{aligned}$$

Therefore

$$|Q(z)| \approx \left|a_0 z^{N(r)} + a_1 z^{N(r)-1} + ... + a_{N(r)}\right|.$$

Now

$$\begin{aligned} \left|a_1 z^{N(r)-1} + ... + a_{N(r)}\right| &\leq |a_1| \, |z|^{N(r)-1} + ... + \left|a_{N(r)}\right| \\ &= |a_0| \left\{\frac{|a_1|}{|a_0|} |z|^{N(r)-1} + ... + \frac{\left|a_{N(r)}\right|}{|a_0|}\right\} \\ &= |a_0| \sum_{j=0}^{N(r)-1} \left|\frac{a_{N(r)-j}}{a_0}\right| |z|^j . \end{aligned} \tag{11}$$

Therefore using (11) we get that

$$\begin{aligned} |Q(z)| &\geq |a_0| \, |z|^{N(r)} - \left|a_1 z^{N(r)-1} + ... + a_{N(r)}\right| \\ &\geq |a_0| \, |z|^{N(r)} - |a_0| \sum_{j=0}^{N(r)-1} \left|\frac{a_{N(r)-j}}{a_0}\right| |z|^j \\ &= |a_0| \left\{ |z|^{N(r)} - \sum_{j=0}^{N(r)-1} \left|\frac{a_{N(r)-j}}{a_0}\right| |z|^j \right\}. \end{aligned}$$

Now by Holder's inequality we have

$$\sum_{j=0}^{N(r)-1} \left|\frac{a_{N(r)-j}}{a_0}\right| |z|^j \leq \left(\sum_{j=0}^{N(r)-1} \left|\frac{a_{N(r)-j}}{a_0}\right|^p\right)^{\frac{1}{p}} \left(\sum_{j=0}^{N(r)-1} \left(|z|^j\right)^q\right)^{\frac{1}{q}}. \tag{12}$$

Using (12) we obtain from above that

$$\begin{aligned} |Q(z)| \;\; &\geq \;\; |a_0| \left\{ |z|^{N(r)} - \left(\sum_{j=0}^{N(r)-1} \left|\frac{a_{N(r)-j}}{a_0}\right|^p\right)^{\frac{1}{p}} \left(\sum_{j=0}^{N(r)-1} \left(|z|^j\right)^q\right)^{\frac{1}{q}} \right\} \\ &= \;\; |a_0| \left\{ |z|^{N(r)} - |z|^{N(r)} \left(\sum_{j=0}^{N(r)-1} \left|\frac{a_{N(r)-j}}{a_0}\right|^p\right)^{\frac{1}{p}} \left(\sum_{j=0}^{N(r)-1} \frac{\left(|z|^j\right)^q}{|z|^{N(r)q}}\right)^{\frac{1}{q}} \right\} \end{aligned}$$

$$= |a_0|\,|z|^{N(r)}\left\{1-\left(\sum_{j=0}^{N(r)-1}\left|\frac{a_{N(r)-j}}{a_0}\right|^p\right)^{\frac{1}{p}}\left(\sum_{j=0}^{N(r)-1}\frac{\left(|z|^j\right)^q}{|z|^{N(r)q}}\right)^{\frac{1}{q}}\right\}$$

$$= |a_0|\,|z|^{N(r)}\left\{1-\left(\sum_{j=0}^{N(r)-1}\left|\frac{a_{N(r)-j}}{a_0}\right|^p\right)^{\frac{1}{p}}\left(\sum_{j=0}^{N(r)-1}\frac{1}{|z|^{q(N(r)-j)}}\right)^{\frac{1}{q}}\right\}$$

$$= |a_0|\,|z|^{N(r)}\left\{1-\left(\sum_{j=0}^{N(r)-1}\left|\frac{a_{N(r)-j}}{a_0}\right|^p\right)^{\frac{1}{p}}\left(\sum_{j=0}^{N(r)-1}\left(\frac{1}{|z|^q}\right)^{(N(r)-j)}\right)^{\frac{1}{q}}\right\}$$

$$= |a_0|\,|z|^{N(r)}\left\{1-\left(\sum_{j=0}^{N(r)-1}\left|\frac{a_{N(r)-j}}{a_0}\right|^p\right)^{\frac{1}{p}}\left(\sum_{j=0}^{N(r)}\left(\frac{1}{|z|^q}\right)^{j}\right)^{\frac{1}{q}}\right\}.$$

Therefore

$$|Q(z)| > |a_0|\,|z|^{N(r)}\left\{1-\left(\sum_{j=0}^{N(r)-1}\left|\frac{a_{N(r)-j}}{a_0}\right|^p\right)^{\frac{1}{p}}\left(\sum_{j=1}^{N(r)}\left(\frac{1}{|z|^q}\right)^{j}\right)^{\frac{1}{q}}\right\}.$$

Now the geometric series $\sum_{j=1}^{\infty}\left(\frac{1}{|z|^q}\right)^j$ is convergent for

$$\left|\frac{1}{|z|^q}\right| < 1$$

$$\text{i.e., for} \qquad |z|^q > 1.$$

Therefore for $|z| > 1$ and the series is convergent to

$$\frac{1}{|z|^q}\frac{1}{1-\frac{1}{|z|^q}} = \frac{1}{|z|^q - 1}.$$

So

$$\left(\sum_{j=1}^{\infty}\left(\frac{1}{|z|^q}\right)^j\right)^{\frac{1}{q}} \text{ is convergent to } \left(\frac{1}{|z|^q-1}\right)^{\frac{1}{q}} \text{ for } |z| > 1.$$

Therefore on $|z| > 1$,

$$|Q(z)| > |a_0|\,|z|^{N(r)}\left\{1-\left(\sum_{j=0}^{N(r)-1}\left|\frac{a_{N(r)-j}}{a_0}\right|^p\right)^{\frac{1}{p}}\left(\frac{1}{|z|^q-1}\right)^{\frac{1}{q}}\right\}.$$

Now if $|Q(z)| > 0$ then

$$|a_0|\,|z|^{N(r)}\left\{1-\left(\sum_{j=0}^{N(r)-1}\left|\frac{a_{N(r)-j}}{a_0}\right|^p\right)^{\frac{1}{p}}\left(\frac{1}{|z|^q-1}\right)^{\frac{1}{q}}\right\}\geq 0$$

$$\text{i.e,}\qquad 1-\left(\sum_{j=0}^{N(r)-1}\left|\frac{a_{N(r)-j}}{a_0}\right|^p\right)^{\frac{1}{p}}\left(\frac{1}{|z|^q-1}\right)^{\frac{1}{q}}\geq 0$$

$$\text{i.e.,}\qquad 1\geq\left(\sum_{j=0}^{N(r)-1}\left|\frac{a_{N(r)-j}}{a_0}\right|^p\right)^{\frac{1}{p}}\left(\frac{1}{|z|^q-1}\right)^{\frac{1}{q}}$$

$$\text{i.e.,}\qquad (|z|^q-1)^{\frac{1}{q}}\geq\left(\sum_{j=0}^{N(r)-1}\left|\frac{a_{N(r)-j}}{a_0}\right|^p\right)^{\frac{1}{p}}$$

$$\text{i.e.,}\qquad |z|^q-1\geq\left(\sum_{j=0}^{N(r)-1}\left|\frac{a_{N(r)-j}}{a_0}\right|^p\right)^{\frac{q}{p}}$$

$$\text{i.e.,}\qquad |z|\geq\left[1+\left(\sum_{j=0}^{N(r)-1}\left|\frac{a_{N(r)-j}}{a_0}\right|^p\right)^{\frac{q}{p}}\right]^{\frac{1}{q}}.$$

Clearly

$$\left[1+\left(\sum_{j=0}^{N(r)-1}\left|\frac{a_{N(r)-j}}{a_0}\right|^p\right)^{\frac{q}{p}}\right]^{\frac{1}{q}}>1.$$

Therefore $|Q(z)| > 0$ if

$$|z|\geq\left[1+\left(\sum_{j=0}^{N(r)-1}\left|\frac{a_{N(r)-j}}{a_0}\right|^p\right)^{\frac{q}{p}}\right]^{\frac{1}{q}}.$$

Therefore all the zeros $Q(z)$ lie in

$$|z|<\left[1+\left(\sum_{j=0}^{N(r)-1}\left|\frac{a_{N(r)-j}}{a_0}\right|^p\right)^{\frac{q}{p}}\right]^{\frac{1}{q}}.$$

Let $z = z_0$ be any other zero of $P(z)$. Therefore $P(z_0) = 0$. Clearly $z_0 \neq 0$ as $a_0 \neq 0$.
Putting $z = \frac{1}{z_0}$in $Q(z)$ we have

$$Q(z_0) = \left(\frac{1}{z_0}\right)^{N(r)} P(z_0) = 0.$$

Therefore $z = \frac{1}{z_0}$ is a zero of $Q(z)$. So

$$\left|\frac{1}{z_0}\right| < \left[1 + \left(\sum_{j=0}^{N(r)-1} \left|\frac{a_{N(r)-j}}{a_0}\right|^p\right)^{\frac{q}{p}}\right]^{\frac{1}{q}}$$
$$\text{i.e., } |z_0| > \frac{1}{\left[1 + \left(\sum_{j=0}^{N(r)-1} \left|\frac{a_{N(r)-j}}{a_0}\right|^p\right)^{\frac{q}{p}}\right]^{\frac{1}{q}}}.$$

As z_0 is an arbitrary zero of $P(z)$ so all the zeros of $P(z)$ lie in

$$|z| > \frac{1}{\left[1 + \left(\sum_{j=0}^{N(r)-1} \left|\frac{a_{N(r)-j}}{a_0}\right|^p\right)^{\frac{q}{p}}\right]^{\frac{1}{q}}}. \tag{13}$$

Hence combining (10) and (13) we may say that all the zeros of $P(z)$ lie in the ring shaped region

$$\frac{1}{\left[1 + \left(\sum_{j=0}^{N(r)-1} \left|\frac{a_{N(r)-j}}{a_0}\right|^p\right)^{\frac{q}{p}}\right]^{\frac{1}{q}}} < |z| < \left[1 + \left(\sum_{j=0}^{N(r)-1} \left|\frac{a_j}{a_{N(r)}}\right|^p\right)^{\frac{q}{p}}\right]^{\frac{1}{q}}.$$

This proves the theorem. ■

Corollary 6 *In particular if we take $p = 2, q = 2$ in Theorem 3 then we get that all the zeros of the polynomial*

$$P(z) = a_0 + a_1 z + a_2 z^2 + ... + a_n z^n$$

lie in the ring shaped region

$$\frac{1}{\left[1+\left(\sum_{j=0}^{n}\left|\frac{a_n}{a_0}\right|^2\right)\right]^{\frac{1}{2}}}<|z|<\left[1+\left(\sum_{j=0}^{n}\left|\frac{a_j}{a_n}\right|^2\right)\right]^{\frac{1}{2}}.$$

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