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A Note on the Definition of L-Type of a Meromorphic Function of L-Order Zero or L-Order Infinity

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Abstract

In this paper we introduce the definition of L-type of a meromorphic function of L-order zero or L-order infinity and obtain its integral representation.

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1 Introduction, Definitions and Notations.

Let f be a meromorphic function defined in the open complex plane $\mathbb{C}$. We use the standard notations and definitions in the theory of entire and meromorphic functions which are available in [5] and [1]. In the sequel we use the following notations:

$$\log^{[k]} x = \log(\log^{[k-1]} x) \text{ for } k = 1, 2, 3, ... \text{ and } \log^{[0]} x = x$$

and

$$\exp^{[k]} x = \exp[\exp^{[k-1]} x] \text{ for } k = 1, 2, 3, ...\text{and } \exp^{[0]} x = x.$$

Somasundaram and Thamizharasi [4] introduced the notion of L-order and L-type for entire functions where $L \equiv L(r)$ is a positive continuous function increasing slowly *i.e.*, $L(ar) \sim L(r)$ as $r \to \infty$ for every positive constant a.

The L-order and L-type of a meromorphic function f are defined in the following way:

Definition 1 *The L-order ρ_f^L of a meromorphic function f is defined as*

$$\rho_f^L = \limsup_{r\to\infty} \frac{\log T(r, f)}{\log[rL(r)]}.$$

If f is entire then one can easily verify that

$$\rho_f^L = \limsup_{r\to\infty} \frac{\log^{[2]} M(r, f)}{\log[rL(r)]}.$$

Definition 2 *The L-type σ_f^L of a meromorphic function f is defined as follows*

$$\sigma_f^L = \limsup_{r\to\infty} \frac{T(r, f)}{[rL(r)]^{\rho_f^L}}, \quad 0 < \rho_f^L < \infty.$$

When f is entire then

$$\sigma_f^L = \limsup_{r\to\infty} \frac{\log M(r, f)}{[rL(r)]^{\rho_f^L}}, \quad 0 < \rho_f^L < \infty.$$

But when a meromorphic function f is of L-order zero or L-order infinity, the L-type of f can not be defined. In this paper we introduce the defininition of L-type of meromoromorphic function of L-order zero or L-order infinity and deduce its integral representation. In order to do this we just recall the definition of zero L-order (i.e., alternatively L-order zero) of a merommorphic function. In the line of Liao and Yang [3] we may give the following definition.

Definition 3 *Let f be a meromorphic function of L-order zero. Then the quantitiy ρ_f^{*L} is defined as*

$$\rho_f^{*L} = \limsup_{r\to\infty} \frac{\log T(r,f)}{\log^{[2]}[rL(r)]}.$$

If f is entire then clearly,

$$\rho_f^{*L} = \limsup_{r\to\infty} \frac{\log^{[2]} M(r,f)}{\log^{[2]}[rL(r)]}.$$

The following definition is also well known.

Definition 4 *The hyper L-order $\overline{\rho}_f^L$ of a meromorphic function f is defined as followes:*

$$\overline{\rho}_f^L = \limsup_{r\to\infty} \frac{\log^{[2]} T(r,f)}{\log[rL(r)]}.$$

If f is entire then

$$\overline{\rho}_f^L = \limsup_{r\to\infty} \frac{\log^{[3]} M(r,f)}{\log[rL(r)]}.$$

In this paper we introduce the following definitions.

Definition A *The L-type σ_f^{*L} of a meromorphic function of L-order zero is defined by*

$$\sigma_f^{*L} = \limsup_{r\to\infty} \frac{T(r,f)}{[\log\{rL(r)\}]^{\rho_f^{*L}}} \ , \quad 0 < \rho_f^{*L} < \infty.$$

Definition B *A meromorphic function f of L-order zero is said to be of L-type σ_f^{*L} if the integral*

$$\int_{r_0}^{\infty} \frac{\exp\left[T(r,f)\right] dr}{\left[\exp\left\{\log\left(rL(r)\right)\right\}^{\rho_f^{*L}}\right]^{k+1}} \quad (r_0 > 0)$$

is convergent for $k > \sigma_f^{*L}$ and divergent for $k < \sigma_f^{*L}$ where $0 < \rho_f^{*L} < \infty$.

Definition C *The L-type $\overline{\sigma}_f^L$ of a meromorphic function of L-order infinity is defined as follows*

$$\overline{\sigma}_f^L = \limsup_{r\to\infty} \frac{\log T(r,f)}{\{rL(r)\}^{\overline{\rho}_f^L}}, \text{ where } 0 < \overline{\rho}_f^L < \infty.$$

Definition D *A meromorphic function f of L-order infinity is said to be of L-type $\overline{\sigma}_f^L$ if the integral*

$$\int_{r_0}^{\infty} \frac{T(r,f) dr}{\left[\exp\left\{rL(r)\right\}^{\overline{\rho}_f^L}\right]^{k+1}} \quad (r_0 > 0)$$

converges for $k > \overline{\sigma}_f^L$ and diverges for $k < \overline{\sigma}_f^L$.

2 Lemmas.

In this section we present some lemmas which will be needed in the sequel.

Lemma 1 *Let the integral*

$$\int_{r_0}^{\infty} \frac{\exp\left[T(r,f)\right] dr}{\left[\exp\left\{\log\left(rL(r)\right)\right\}^{\rho_f^{*L}}\right]^{k+1}} \quad (r_0 > 0) \tag{A}$$

converges for $0 < k < \infty$. *Then*

$$\lim_{r\to\infty} \frac{\exp\left[T(r,f)\right]}{\left[\exp\left\{\log\left(rL(r)\right)\right\}^{\rho_f^{*L}}\right]^{k}} = 0.$$

Proof. Since the integral

$$\int_{r_0}^{\infty} \frac{\exp\left[T(r,f)\right] dr}{\left[\exp\left\{\log\left(rL(r)\right)\right\}^{\rho_f^{*L}}\right]^{k+1}}$$

is convergent for $0 < k < \infty$, given $\epsilon\,(> 0)$ there exists a number $R = R(\epsilon)$ such that

$$\int_{r_0}^{\infty} \frac{\exp\left[T(r,f)\right] dr}{\left[\exp\left\{\log\left(rL(r)\right)\right\}^{\rho_f^{*L}}\right]^{k+1}} < \epsilon \text{ for } \quad r_0 > R$$

i.e., for $r_0 > R$,

$$\int_{r_0}^{r_0+\exp[\log\{r_0L(r_0)\}]^{\rho_f^{*L}}} \frac{\exp\left[T(r,f)\right] dr}{\left[\exp\left\{\log\left(rL(r)\right)\right\}^{\rho_f^{*L}}\right]^{k+1}} < \epsilon.$$

As $\exp\left[T(r,f)\right]$ is an increasing function of r, so

$$\begin{aligned}
&\int_{r_0}^{r_0+\exp[\log\{r_0L(r_0)\}]^{\rho_f^{*L}}} \frac{\exp\left[T(r,f)\right] dr}{\left[\exp\left\{\log\left(rL(r)\right)\right\}^{\rho_f^{*L}}\right]^{k+1}} \\
\geq\ & \frac{\exp\left[T(r_0,f)\right]}{\left[\exp\left\{\log\left(r_0L(r_0)\right)\right\}^{\rho_f^{*L}}\right]^{k+1}} . \exp\left[\log\left\{r_0L(r_0)\right\}\right]^{\rho_f^{*L}} \\
=\ & \frac{\exp\left[T(r_0,f)\right]}{\left[\exp\left\{\log\left(r_0L(r_0)\right)\right\}^{\rho_f^{*L}}\right]^{k}}
\end{aligned}$$

$$i.e., \qquad \frac{\exp\left[T(r_0, f)\right]}{\left[\exp\left\{\log\left(r_0 L(r_0)\right)\right\}^{\rho_f^{*L}}\right]^k} < \epsilon \quad \text{for} \quad r_0 > R,$$

from which it follows that

$$\limsup_{r\to\infty} \frac{\exp\left[T(r, f)\right]}{\left[\exp\left\{\log\left(r L(r)\right)\right\}^{\rho_f^{*L}}\right]^k} = 0.$$

This proves the lemma. ■

Lemma 2 *If the integral*

$$\int_{r_0}^{\infty} \frac{T(r, f)dr}{\left[\exp\left\{rL(r)\right\}^{\overline{\rho}_f^L}\right]^{k+1}} \quad (r_0 > 0)$$

is convergent for $0 < k < \infty$ *then*

$$\limsup_{r\to\infty} \frac{T(r, f)}{\left[\exp\left\{rL(r)\right\}^{\overline{\rho}_f^L}\right]^{k}} = 0.$$

Proof. Since the integral

$$\int_{r_0}^{\infty} \frac{T(r, f)dr}{\left[\exp\left\{rL(r)\right\}^{\overline{\rho}_f^L}\right]^{k+1}}$$

converges for $0 < k < \infty$, given $\epsilon\,(> 0)$ there exists a number $R = R(\epsilon)$ such that

$$\int_{r_0}^{\infty} \frac{T(r, f)dr}{\left[\exp\left\{rL(r)\right\}^{\overline{\rho}_f^L}\right]^{k+1}} < \epsilon \quad \text{for} \quad r > R.$$

$$i.e., \qquad \int_{r_0}^{r_0+\exp\left[\left\{r_0L(r_0)\right\}^{\overline{\rho}_f^L}\right]} \frac{T(r, f)dr}{\left[\exp\left\{rL(r)\right\}^{\overline{\rho}_f^L}\right]^{k+1}} \geq \frac{T(r_0, f).\exp\left[\left\{r_0L(r_0)\right\}^{\overline{\rho}_f^L}\right]}{\exp\left[\left\{r_0L(r_0)\right\}^{\overline{\rho}_f^L}\right]^{k+1}}$$

$$= \frac{T(r_0, f)}{\exp\left[\left\{r_0L(r_0)\right\}^{\overline{\rho}_f^L}\right]^{k}}$$

$$i.e., \qquad \frac{T(r_0, f)}{\exp\left[\left\{r_0 L(r_0)\right\}^{\overline{\rho}_f^L}\right]^k} < \epsilon \quad \text{for} \quad r_0 > R.$$

Now from the above it follows that

$$\limsup_{r\to\infty} \frac{T(r, f)}{\exp\left[\left\{r L(r)\right\}^{\overline{\rho}_f^L}\right]^k} = 0.$$

Thus the lemma is established. ■

Lemma 3 *[2]If f is a non constant entire function then*

$$T(r, f) \leq \log M(r, f) \leq \log T(2r, f) + o(1) \quad as \quad r \to \infty.$$

3 Theorems.

In this section we present the main results of the paper.

Theorem 1 *Let f be meromorphic with L-order zero. Also let $0 < \rho_f^{*L} < \infty$. Then Definition A and Definition B are equivalent.*

Proof. Case I: $\sigma_f^{*L} = \infty$.

Definition A ⇒ Definition B.

As $\sigma_f^{*L} = \infty$, from Definition A we obtain for arbitrary positive G and for a sequence of values of r tending to infinity that

$$\begin{aligned} T(r, f) &> G\left\{\log\left(rL(r)\right)\right\}^{\rho_f^{*L}} \\ i.e., \qquad \exp\left[T(r, f)\right] &> \left[\exp\left\{\log\left(rL(r)\right)\right\}^{\rho_f^{*L}}\right]^G. \end{aligned} \tag{1}$$

If possible, let the integral

$$\int_{r_0}^{\infty} \frac{\exp\left[T(r, f)\right] dr}{\left[\exp\left\{\log\left(rL(r)\right)\right\}^{\rho_f^{*L}}\right]^{G+1}} \quad (r_0 > 0)$$

be converge. Then by Lemma 1 we get that

$$\limsup_{r\to\infty} \frac{\exp\left[T(r, f)\right]}{\left[\exp\left\{\log\left(rL(r)\right)\right\}^{\rho_f^{*L}}\right]^G} = 0.$$

So for all sufficiently large values of r,

$$\exp\left[T(r, f)\right] < \left[\exp\left\{\log\left(rL(r)\right)\right\}^{\rho_f^{*L}}\right]^G. \tag{2}$$

Now from (1) and (2) we arrive at a contradiction. Hence

$$\int_{r_0}^{\infty} \frac{\exp\left[T(r,f)\right] dr}{\left[\exp\left\{\log\left(rL(r)\right)\right\}^{\rho_f^{*L}}\right]^{G+1}} \quad (r_0 > 0)$$

diverges whenever G is finite which is Definition B.

Definition B ⇒ Definition A.

Let G be any positive number. Since $\sigma_f^{*L} = \infty$, from Definition B, the divergence of the integral,

$$\int_{r_0}^{\infty} \frac{\exp\left[T(r,f)\right] dr}{\left[\exp\left\{\log\left(rL(r)\right)\right\}^{\rho_f^{*L}}\right]^{G+1}} \ (r_0 > 0)$$

gives for arbitrary positive ϵ and for a sequence of values of r tending to infinity

$$\begin{aligned} \exp\left[T(r,f)\right] &> \left[\exp\left\{\log\left(rL(r)\right)\right\}^{\rho_f^{*L}}\right]^{G-\epsilon} \\ \text{i.e., } T(r,f) &> (G-\epsilon)\left\{\log\left(rL(r)\right)\right\}^{\rho_f^{*L}}. \end{aligned}$$

This gives that

$$\limsup_{r\to\infty} \frac{T(r,f)}{\left\{\log\left(rL(r)\right)\right\}^{\rho_f^{*L}}} \geq G-\epsilon.$$

Since $G > 0$ is arbitrary, it follows that

$$\limsup_{r\to\infty} \frac{T(r,f)}{\left\{\log\left(rL(r)\right)\right\}^{\rho_f^{*L}}} = \infty.$$

Thus Definition A follows.

Case II: $0 \leq \sigma_f^{*L} < \infty$.

Definition A ⇒ Definition B.

Subcase (a): Let f be of L-type σ_f^{*L}, where $0 < \sigma_f^{*L} < \infty$. Then for arbitrary $\epsilon\,(> 0)$ and for all sufficiently large values of r,

$$\frac{T(r,f)}{\{\log(rL(r))\}^{\rho_f^{*L}}} < \sigma_f^{*L} + \epsilon$$

$$i.e., \quad T(r,f) < (\sigma_f^{*L} + \epsilon)\,\{\log(rL(r))\}^{\rho_f^{*L}}$$

$$i.e., \quad \exp[T(r,f)] < e^{(\sigma_f^{*L}+\epsilon)\{\log(rL(r))\}^{\rho_f^{*L}}}$$

$$i.e., \quad \exp[T(r,f)] < \left[\exp\{\log(rL(r))\}^{\rho_f^{*L}}\right]^{(\sigma_f^{*L}+\epsilon)}$$

$$i.e., \quad \frac{\exp[T(r,f)]}{\left[\exp\{\log(rL(r))\}^{\rho_f^{*L}}\right]^{k}} < \frac{\left[\exp\{\log(rL(r))\}^{\rho_f^{*L}}\right]^{(\sigma_f^{*L}+\epsilon)}}{\left[\exp\{\log(rL(r))\}^{\rho_f^{*L}}\right]^{k}}$$

$$i.e., \quad \frac{\exp[T(r,f)]}{\left[\exp\{\log(rL(r))\}^{\rho_f^{*L}}\right]^{k}} < \frac{1}{\left[\exp\{\log(rL(r))\}^{\rho_f^{*L}}\right]^{k-(\sigma_f^{*L}+\epsilon)}}.$$

Therefore

$$\int_{r_0}^{\infty} \frac{\exp[T(r,f)]\,dr}{\left[\exp\{\log(rL(r))\}^{\rho_f^{*L}}\right]^{k+1}} \ (r_0 > 0)$$

converges if $k > \sigma_f^{*L}$ and diverges if $k < \sigma_f^{*L}$.

Subcase (b): When f is of L-type $\sigma_f^{*L} = 0$. Definition A gives for all sufficiently large values of r that

$$\frac{T(r,f)}{\{\log(rL(r))\}^{\rho_f^{*L}}} < \epsilon.$$

Then as before we obtain that

$$\int_{r_0}^{\infty} \frac{\exp[T(r,f)]\,dr}{\left[\exp\{\log(rL(r))\}^{\rho_f^{*L}}\right]^{k+1}} \quad (r_0 > 0)$$

converges for $k > 0$ and diverges for $k < 0$.

Thus combining Subcase (a) and Subcase (b) Definition B follows.

Definition B $\Rightarrow$ Definition A.

Since f be of L-type σ_f^{*L}, by Definition B for arbitrary $\epsilon > 0$, the integral

$$\int_{r_0}^{\infty} \frac{\exp[T(r,f)]\,dr}{\left[\exp\{\log(rL(r))\}^{\rho_f^{*L}}\right]^{\sigma_f^{*L}+1+\epsilon}}$$

converges. Then by Lemma 1

$$\limsup_{r\to\infty}\frac{\exp\left[T(r,f)\right]}{\left[\exp\left\{\log\left(rL(r)\right)\right\}^{\rho_f^{*L}}\right]^{\sigma_f^{*L}+\epsilon}}=0$$

i.e., for all sufficiently large values of r,

$$\frac{\exp\left[T(r,f)\right]}{\left[\exp\left\{\log\left(rL(r)\right)\right\}^{\rho_f^{*L}}\right]^{\sigma_f^{*L}+\epsilon}}<\epsilon$$

$$\text{i.e.,}\qquad \exp\left[T(r,f)\right]<\epsilon\left[\exp\left\{\log\left(rL(r)\right)\right\}^{\rho_f^{*L}}\right]^{\sigma_f^{*L}+\epsilon}$$

$$\text{i.e.,}\qquad T(r,f)<\log\epsilon+\left(\sigma_f^{*L}+\epsilon\right)\left\{\log\left(rL(r)\right)\right\}^{\rho_f^{*L}}$$

$$\text{i.e.,}\qquad \frac{T(r,f)}{\left\{\log\left(rL(r)\right)\right\}^{\rho_f^{*L}}}<\frac{\log\epsilon}{\log\left[\left(rL(r)\right)\right]^{\rho_f^{*L}}}+\left(\sigma_f^{*L}+\epsilon\right)$$

$$\text{i.e.,}\qquad \limsup\frac{T(r,f)}{\left\{\log\left(rL(r)\right)\right\}^{\rho_f^{*L}}}\le\sigma_f^{*L}+\epsilon.$$

Since $\epsilon>0$ is arbitrary, it follows from above that

$$\limsup_{r\to\infty}\frac{T(r,f)}{\left\{\log\left(rL(r)\right)\right\}^{\rho_f^{*L}}}\le\sigma_f^{*L}. \tag{3}$$

Again by Definition B, the divergence of the integral

$$\int_{r_0}^{\infty}\frac{\exp\left[T(r,f)\right]dr}{\left[\exp\left\{\log\left(rL(r)\right)\right\}^{\rho_f^{*L}}\right]^{\sigma_f^{*L}+1-\epsilon}}$$

implies that there exists a sequence of values of r tending to infinity such that

$$\frac{\exp\left[T(r,f)\right]}{\left[\exp\left\{\log\left(rL(r)\right)\right\}^{\rho_f^{*L}}\right]^{\sigma_f^{*L}+1-\epsilon}}>\frac{1}{\left[\exp\left\{\log\left(rL(r)\right)\right\}^{\rho_f^{*L}}\right]^{1+\epsilon}}$$

$$\text{i.e.,}\qquad \exp\left[T(r,f)\right]>\left[\exp\left\{\log\left(rL(r)\right)\right\}^{\rho_f^{*L}}\right]^{\sigma_f^{*L}-2\epsilon}$$

$$\text{i.e.,}\qquad T(r,f)>\left(\sigma_f^{*L}-2\epsilon\right)\left\{\log\left(rL(r)\right)\right\}^{\rho_f^{*L}}$$

$$\text{i.e.,}\qquad \frac{T(r,f)}{\left\{\log\left(rL(r)\right)\right\}^{\rho_f^{*L}}}>\sigma_f^{*L}-2\epsilon.$$

As $\epsilon\,(>0)$ is arbitrary we get that

$$\limsup_{r\to\infty}\frac{T(r,f)}{\left\{\log\left(rL(r)\right)\right\}^{\rho_f^{*L}}}\ge\sigma_f^{*L}. \tag{4}$$

Therefore from (3) and (4) it follows that

$$\limsup_{r\to\infty}\frac{T(r,f)}{\{\log(rL(r))\}^{\rho_f^{*L}}}=\sigma_f^{*L}.$$

Thus we obtain Definition A.

Now combining Case I and Case II, the theorem follows. ∎

Theorem 2 *The integral*

$$\int_{r_0}^{\infty}\frac{\exp\{T(r,f)\}\,dr}{\left[\exp\{\log(rL(r))\}^{\rho_f^{*L}}\right]^{\sigma_f^{*L}+1}}\quad(r_0>0)$$

follows if and only if the integral

$$\int_{r_0}^{\infty}\frac{M(r,f)dr}{\left[\exp\{\log(rL(r))\}^{\rho_f^{*L}}\right]^{\sigma_f^{*L}+1}}\quad\textit{converges.}$$

Proof. *Let*

$$\int_{r_0}^{\infty}\frac{M(r,f)dr}{\left[\exp\{\log(rL(r))\}^{\rho_f^{*L}}\right]^{\sigma_f^{*L}+1}}\quad(r_0>0)$$

converges. Then by the first part of Lemma 3 we obtain that

$$\int_{r_0}^{\infty}\frac{\exp\{T(r,f)\}\,dr}{\left[\exp\{\log(rL(r))\}^{\rho_f^{*L}}\right]^{\sigma_f^{*L}+1}}\le\int_{r_0}^{\infty}\frac{M(r,f)dr}{\left[\exp\{\log(rL(r))\}^{\rho_f^{*L}}\right]^{\sigma_f^{*L}+1}}$$

$$\textit{i.e.,}\ \int_{r_0}^{\infty}\frac{\exp\{T(r,f)\}\,dr}{\left[\exp\{\log(rL(r))\}^{\rho_f^{*L}}\right]^{\sigma_f^{*L}+1}}\ \textit{converges.}$$

Next let

$$\int_{r_0}^{\infty}\frac{\exp\{T(r,f)\}\,dr}{\left[\exp\{\log(rL(r))\}^{\rho_f^{*L}}\right]^{\sigma_f^{*L}+1}}\quad(r_0>0)$$

be convergent. Then by the second part of Lemma 3, we get that

$$\begin{aligned}&\int_{r_0}^{\infty}\frac{M(r,f)dr}{\left[\exp\{\log(rL(r))\}^{\rho_f^{*L}}\right]^{\sigma_f^{*L}+1}}\\<&\int_{r_0}^{\infty}\frac{\exp\{T(2r,f)\}\,dr}{\left[\exp\{\log(rL(r))\}^{\rho_f^{*L}}\right]^{\sigma_f^{*L}+1}}+\int_{r_0}^{\infty}\frac{o(1)dr}{\left[\exp\{\log(rL(r))\}^{\rho_f^{*L}}\right]^{\sigma_f^{*L}+1}}\\=&\frac{1}{2\left[\exp\left(\frac{1}{2}\rho_f^{*L}\right)\right]}\int_{r_0}^{\infty}\frac{\exp\{T(r,f)\}\,dr}{\left[\exp\left\{\log\left(rL\left(\frac{r}{2}\right)\right)\right\}^{\rho_f^{*L}}\right]^{\sigma_f^{*L}+1}}+o(1).\end{aligned}$$

Thus

$$\int_{r_0}^{\infty} \frac{M(r,f)dr}{\left[\exp\left\{\log\left(rL(r)\right)\right\}^{\rho_f^{*L}}\right]^{\sigma_f^{*L}+1}}$$

converges. This proves the theorem. ∎

Now in view of Theorem 1 and Theorem 2 we may give an alternative definition of L-type σ_f^{*L} of an entire function f with L-order zero as follows:

An entire function f with L-order zero is said to be of L-type σ_f^{*L} if the integral

$$\int_{r_0}^{\infty} \frac{M(r,f)dr}{\left[\exp\left\{\log\left(rL(r)\right)\right\}^{\rho_f^{*L}}\right]^{k+1}} \quad (r_0 > 0)$$

converges for $k > \sigma_f^{*L}$ and diverges for $k < \sigma_f^{*L}$.

Theorem 3 *If f be a meromorphic function of infinite order and $0 < \overline{\rho}_f^L < \infty$ then Definition C and Definition D are equivalant.*

Proof. Case I: $\overline{\sigma}_f^L = \infty$.

Definition C⇒ Definition D.

As $\overline{\sigma}_f^L = \infty$, from Definition C, we obtain for arbitrary positive G and for a sequence of values of r tending to infinity that

$$\begin{aligned} \log T(r,f) &> G\left[rL(r)\right]^{\overline{\rho}_f^L} \\ \textit{i.e.,} \qquad T(r,f) &> \left[\exp\left\{\left(rL(r)\right)^{\overline{\rho}_f^L}\right\}\right]^G. \end{aligned} \tag{5}$$

If possible, let the integral

$$\int_{r_0}^{\infty} \frac{T(r,f)dr}{\left[\exp\left\{\left(rL(r)\right)^{\overline{\rho}_f^L}\right\}\right]^{G+1}} \quad (r_0 > 0)$$

be convergent. Then by Lemma 2

$$\limsup_{r\to\infty} \frac{T(r,f)}{\left[\exp\left\{\left(rL(r)\right)^{\overline{\rho}_f^L}\right\}\right]^G} = 0.$$

So for all sufficiently large values of r,

$$T(r,f) < \left[\exp\left\{\left(rL(r)\right)^{\overline{\rho}_f^L}\right\}\right]^G. \tag{6}$$

Now from (5) and (6) we arrive at a contradiction. Hence

$$\int_{r_0}^{\infty} \frac{T(r,f)dr}{\left[\exp\left\{(rL(r))^{\overline{\rho}_f^L}\right\}\right]^{G+1}} \quad (r_0 > 0)$$

diverges whenever G is finite, which is Definition D.

Definition D⇒ Definition C.

Let G be any positive number. Since $\overline{\sigma}_f^L = \infty$, from Definition D, the divergence of the integral

$$\int_{r_0}^{\infty} \frac{T(r,f)dr}{\left[\exp\left\{(rL(r))^{\overline{\rho}_f^L}\right\}\right]^{G+1}} \quad (r_0 > 0)$$

gives for arbitrary positive ϵ and for a sequence of values of r tending to infinity,

$$\begin{aligned} T(r,f) &> \left[\exp\left\{(rL(r))^{\overline{\rho}_f^L}\right\}\right]^{G-\epsilon} \\ i.e., \quad \log T(r,f) &> (G-\epsilon)\,(rL(r))^{\overline{\rho}_f^L} \end{aligned}$$

This gives that

$$\limsup_{r\to\infty} \frac{\log T(r,f)}{[rL(r)]^{\overline{\rho}_f^L}} \geq G - \epsilon.$$

Since G is arbitrary, this shows that

$$\limsup_{r\to\infty} \frac{\log T(r,f)}{[rL(r)]^{\overline{\rho}_f^L}} = \infty.$$

Thus Definition C follows.

Case II: $0 \leq \overline{\sigma}_f^L < \infty$.

Definition C⇒ Definition D.

Subcase(a): Let f be of L-type $\overline{\sigma}_f^L$ where $0 \leq \overline{\sigma}_f^L < \infty$. Then according to Definition C, for arbitrary positive ϵ and for all sufficiently large values of r,

$$\begin{aligned} & \log T(r,f) < \left(\overline{\sigma}_f^L + \epsilon\right)[rL(r)]^{\overline{\rho}_f^L} \\ i.e., \quad & T(r,f) < \exp\left[\left(\overline{\sigma}_f^L + \epsilon\right)(rL(r))^{\overline{\rho}_f^L}\right] \\ i.e., \quad & T(r,f) < \left[\exp\{rL(r)\}^{\overline{\rho}_f^L}\right]^{\left(\overline{\sigma}_f^L+\epsilon\right)} \\ i.e., \quad & \frac{T(r,f)}{\left[\exp\{rL(r)\}^{\overline{\rho}_f^L}\right]^{k'}} < \frac{\left[\exp\{rL(r)\}^{\overline{\rho}_f^L}\right]^{\left(\overline{\sigma}_f^L+\epsilon\right)}}{\left[\exp\{rL(r)\}^{\overline{\rho}_f^L}\right]^{k'}} \\ i.e., \quad & \frac{T(r,f)}{\left[\exp\{rL(r)\}^{\overline{\rho}_f^L}\right]^{k'}} < \frac{1}{\left[\exp\{rL(r)\}^{\overline{\rho}_f^L}\right]^{k'-\left(\overline{\sigma}_f^L+\epsilon\right)}}. \end{aligned}$$

Therefore

$$\int_{r_0}^{\infty} \frac{T(r,f)dr}{\left[\exp\{rL(r)\}^{\overline{\rho}_f^L}\right]^{k'}} \quad (r_0 > 0)$$

converges if $k' > \overline{\sigma}_f^L$ and diverges if $k' < \overline{\sigma}_f^L$.

$$i.e., \qquad \int_{r_0}^{\infty} \frac{T(r,f)dr}{\left[\exp\{rL(r)\}^{\overline{\rho}_f^L}\right]^{k'+1}} \quad (r_0 > 0)$$

converges if $k' > \overline{\sigma}_f^L$ and diverges if $k' < \overline{\sigma}_f^L$.

Subcase (b): When f is of L-type $\overline{\sigma}_f^L = 0$, Definition C gives for all sufficiently large values of r,

$$\frac{\log T(r,f)}{\{rL(r)\}^{\overline{\rho}_f^L}} < \epsilon$$

Then as before, we obtain that

$$\int_{r_0}^{\infty} \frac{T(r,f)dr}{\left[\exp\{rL(r)\}^{\overline{\rho}_f^L}\right]^{k'+1}} \quad (r_0 > 0)$$

converges if $k' > 0$ and diverges if $k' < 0$. Thus combining Subcase (a) and Subcase (b), Definition D follows.

Definition D⇒ Definition C.

Since f is of L-type $\overline{\sigma}_f^L$, by Definition D, for arbitrary $\epsilon(> 0)$ the integral

$$\int_{r_0}^{\infty} \frac{T(r,f)dr}{\left[\exp\{rL(r)\}^{\overline{\rho}_f^L}\right]^{\overline{\sigma}_f^L+1+\epsilon}}$$

converges. Then by Lemma 2, we obtain that

$$\limsup_{r\to\infty} \frac{T(r,f)}{\left[\exp\{rL(r)\}^{\overline{\rho}_f^L}\right]^{\overline{\sigma}_f^L+\epsilon}} = 0$$

i.e. for all sufficiently large values of r,

$$\frac{T(r,f)}{\left[\exp\{rL(r)\}^{\overline{\rho}_f^L}\right]^{\overline{\sigma}_f^L+\epsilon}} < \epsilon$$

$$i.e., \qquad T(r,f) < \epsilon\left[\exp\{rL(r)\}^{\overline{\rho}_f^L}\right]^{\overline{\sigma}_f^L+\epsilon}$$

$$i.e., \qquad \log T(r,f) < \log\epsilon + \left(\overline{\sigma}_f^L + \epsilon\right)\{rL(r)\}^{\overline{\rho}_f^L}$$

$$i.e., \qquad \frac{\log T(r,f)}{\{rL(r)\}^{\overline{\rho}_f^L}} < \frac{\log\epsilon}{\{rL(r)\}^{\overline{\rho}_f^L}} + (\overline{\sigma}_f^L + \epsilon).$$

Since $\epsilon\,(>0)$ is arbitrary it follows from above that

$$\limsup_{r\to\infty}\frac{\log T(r,f)}{\{rL(r)\}^{\overline{\rho}_f^L}}\leq\overline{\sigma}_f^L. \tag{7}$$

Again by Definition D, for arbitrary positive ϵ, the divergence of the integral

$$\int_{r_0}^{\infty}\frac{T(r,f)dr}{\left[\exp\{rL(r)\}^{\overline{\rho}_f^L}\right]^{\overline{\sigma}_f^L+1-\epsilon}}$$

implies that there exist a sequence of values of r tending to infinity such that

$$\frac{T(r,f)}{\left[\exp\{rL(r)\}^{\overline{\rho}_f^L}\right]^{\overline{\sigma}_f^L+1-\epsilon}}>\frac{1}{\left[\exp\{rL(r)\}^{\overline{\rho}_f^L}\right]^{1+\epsilon}}$$

$$i.e.,\qquad T(r,f)>\left[\exp\{rL(r)\}^{\overline{\rho}_f^L}\right]^{\overline{\sigma}_f^L-2\epsilon}$$

$$i.e.,\qquad \log T(r,f)>\left(\overline{\sigma}_f^L-2\epsilon\right)\{rL(r)\}^{\overline{\rho}_f^L}$$

$$i.e.,\qquad \frac{\log T(r,f)}{\{rL(r)\}^{\overline{\rho}_f^L}}\geq\overline{\sigma}_f^L-2\epsilon$$

$$i.e.,\qquad \limsup\frac{\log T(r,f)}{\{rL(r)\}^{\overline{\rho}_f^L}}\geq\overline{\sigma}_f^L-2\epsilon.$$

As $\epsilon\,(>0)$ is arbitrary we obtain from above that

$$\limsup_{r\to\infty}\frac{\log T(r,f)}{\{rL(r)\}^{\overline{\rho}_f^L}}\geq\overline{\sigma}_f^L. \tag{8}$$

Now from (7) and (8) it follows that ,

$$\limsup_{r\to\infty}\frac{\log T(r,f)}{\{rL(r)\}^{\overline{\rho}_f^L}}=\overline{\sigma}_f^L.$$

Thus we get Definition C.
Hence combining Case I and Case II, the theorem follows. ■

Theorem 4 *The integral*

$$\int_{r_0}^{\infty}\frac{T(r,f)dr}{\left[\exp\{rL(r)\}^{\overline{\rho}_f^L}\right]^{\overline{\sigma}_f^L+1}}\qquad(r_0>0)$$

converges if and only if the integral

$$\int_{r_0}^{\infty}\frac{\log M(r,f)dr}{\left[\exp\{rL(r)\}^{\overline{\rho}_f^L}\right]^{\overline{\sigma}_f^L+1}}\qquad(r_0>0)\quad\textit{converges.}$$

Proof. Let

$$\int_{r_0}^{\infty} \frac{\log M(r,f)dr}{\left[\exp\{rL(r)\}^{\overline{\rho}_f^L}\right]^{\overline{\sigma}_f^L+1}} \qquad (r_0 > 0)$$

be convergent. Then by the first part of Lemma 3, we obtain that

$$\int_{r_0}^{\infty} \frac{T(r,f)dr}{\left[\exp\{rL(r)\}^{\overline{\rho}_f^L}\right]^{\overline{\sigma}_f^L+1}} \leq \int_{r_0}^{\infty} \frac{\log M(r,f)dr}{\left[\exp\{rL(r)\}^{\overline{\rho}_f^L}\right]^{\overline{\sigma}_f^L+1}}$$

i.e., $$\int_{r_0}^{\infty} \frac{T(r,f)dr}{\left[\exp\{rL(r)\}^{\overline{\rho}_f^L}\right]^{\overline{\sigma}_f^L+1}} \quad (r_0 > 0) \ \textit{converges.}$$

Next let

$$\int_{r_0}^{\infty} \frac{T(r,f)dr}{\left[\exp\{rL(r)\}^{\overline{\rho}_f^L}\right]^{\overline{\sigma}_f^L+1}} \qquad (r_0 > 0)$$

be convergent. Then by the second part of Lemma 3, we get that

$$\begin{aligned} & \int_{r_0}^{\infty} \frac{\log M(r,f)dr}{\left[\exp\{rL(r)\}^{\overline{\rho}_f^L}\right]^{\overline{\sigma}_f^L+1}} \\ \leq \ & \int_{r_0}^{\infty} \frac{T(2r,f)dr}{\left[\exp\{rL(r)\}^{\overline{\rho}_f^L}\right]^{\overline{\sigma}_f^L+1}} + \int_{r_0}^{\infty} \frac{o(1)dr}{\left[\exp\{rL(r)\}^{\overline{\rho}_f^L}\right]^{\overline{\sigma}_f^L+1}} \\ = \ & \frac{1}{2\left[\exp\left(\frac{1}{2}\overline{\rho}_f^L\right)\right]} \int_{r_0}^{\infty} \frac{T(r,f)dr}{\left[\exp\left\{rL\left(\frac{r}{2}\right)\right\}^{\overline{\rho}_f^L}\right]^{\overline{\sigma}_f^L+1}} + o(1). \end{aligned}$$

Thus

$$\int_{r_0}^{\infty} \frac{\log M(r,f)dr}{\left[\exp\{rL(r)\}^{\overline{\rho}_f^L}\right]^{\overline{\sigma}_f^L+1}} \qquad (r_0 > 0)$$

is convergent. This proves the theorem. ∎

Now in view of Theorem 3 and Theorem 4, we may give an alternative definition of L-type $\overline{\sigma}_f^L$ of an entire function f with L-infinite order as follows:

An entire function f with L-infinite order is said to be of L-type $\overline{\sigma}_f^L$ if the integral

$$\int_{r_0}^{\infty} \frac{\log M(r,f)dr}{\left[\exp\{rL(r)\}^{\overline{\rho}_f^L}\right]^{k+1}} \qquad (r_0 > 0)$$

converges for $k > \overline{\sigma}_f^L$ and diverges for $k < \overline{\sigma}_f^L$.

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References

[1] Hayman, W.K.: Meromorphic functions, The Clarendon Press, Oxford (1964).

[2] Lahiri, I.: Generalised order of meromorphic functions, Soochow J. Math., Vol. 17, No.1 (1991) pp. 67-82.

[3] Liao, L. and Yang, C.C.: On the growth of composite entire functions, Yokohama Math. J., Vol. 46 (1999), pp. 97-107.

[4] Somasundaram, D. and Thamizharasi, R.: A note on the entire functions of L-bounded index and L-type, Indian J. Pure and Appl. Math., Vol. 19, No. 3 (March 1988), pp. 284-293.

[5] Valiron, G., Lectures on the general theory of integral functions, Chelsea Publishing Company (1949).

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