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On Intra-Regular Γ-Semigroups

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Abstract

In this note, we characterize when a Γ-semigroup is an intra-regular Γ-semigroup based on bi-ideals, quasi-ideals and left (right) ideals.

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1 Preliminaries

In 1986, Sen and Saha [4] defined Γ-semigroup as a generalization of semigroup as follows:

Definition 1.1 *Let S and Γ be two nonempty sets. Then S is called a Γ-semigroup if there is a mapping $S \times \Gamma \times S \to S$, written as $(x, \gamma, y) \mapsto x\gamma y$, such that $(x\gamma y)\beta z = x\gamma(y\beta z)$ for all $x, y, z \in S$ and all $\gamma, \beta \in \Gamma$.*

Let $(S, \cdot)$ be a semigroup and Γ a nonempty set. For $x, y \in S$ and $\gamma \in \Gamma$, let $x\gamma y$ be defined by $x\gamma y = x \cdot y$. Then S is a Γ-semigroup.

Let S be a Γ-semigroup. For $A, B \subseteq S$, let

$$A\Gamma B = \{a\gamma b \mid a \in A, b \in B, \gamma \in \Gamma\}.$$

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For $x \in S$, let $A\Gamma x = A\Gamma\{x\}$ and $x\Gamma A = \{x\}\Gamma A$.

Definition 1.2 *Let S be a Γ-semigroup. A nonempty subset A of S is called a left (respectively, right) ideal of S if $S\Gamma A \subseteq A$ (respectively, $A\Gamma S \subseteq A$). If A is both left and right ideal of S, then A is called an ideal of S.*

Definition 1.3 *Let S be a Γ-semigroup. A nonempty subset Q of S is called a quasi-ideal of S if $Q\Gamma S \cap S\Gamma Q \subseteq Q$.*

Definition 1.4 *Let S be a Γ-semigroup. A nonempty subset B of S is called a bi-ideal of S if $B\Gamma S\Gamma B \subseteq B$.*

Let S be a Γ-semigroup and $a \in S$. The right (respectively, left) ideal of S generated by a, denoted by $R(a)$ (respectively, $L(a)$) is of the form:

$$R(a) = a \cup a\Gamma S \text{ and } L(a) = a \cup S\Gamma a.$$

The ideal of S generated by a is of the form

$$a \cup S\Gamma a \cup a\Gamma S \cup S\Gamma a\Gamma S.$$

In [1] and [2], the authors proved that the quasi- (respectively, bi-) ideal of S generated by a, denoted by $Q(a)$ (respectively, $B(a)$) is of the form:

$$Q(a) = a \cup (a\Gamma S \cap S\Gamma a) \text{ and } B(a) = a \cup a\Gamma a \cup a\Gamma S\Gamma a.$$

Definition 1.5 *A Γ-semigroup S is said to be intra-regular if $a \in S\Gamma a\Gamma a\Gamma S$ for all $a \in S$.*

Intra-regular semigroups have been studied in [3, 4, 5, 6, 7]. In this note, we characterize when a Γ-semigroup S is an intra-regular semigroup based on bi-ideals, quasi-ideals and left (right) ideals of S. The similar results on semigroups and ordered semigroup have been done by the authors in [3, 4, 5, 6, 7].

2 Main Results

These are the main results of the paper.

Theorem 2.1 *Let S be a Γ-semigroup. Then:*

(1) *S is intra-regular if and only if for a bi-ideal B and a quasi-ideal Q of S, we have $B \cap Q \subseteq S\Gamma B\Gamma Q\Gamma S$.*

(2) *S is intra-regular if and only if for a bi-ideal B and a quasi-ideal Q of S, we have $B \cap Q \subseteq S\Gamma Q\Gamma B\Gamma S$.*

Proof. (1) Assume that S is intra-regular. Let B be a bi-ideal of S and Q a quasi-ideal of S. Let $a \in B \cap Q$. By assumption, $a = x\gamma a\gamma' a\gamma'' y$ for some $x, y \in S, \gamma, \gamma', \gamma'' \in \Gamma$. Then $a = x\gamma a\gamma' a\gamma'' y = x\gamma a\gamma'(x\gamma a\gamma' a\gamma'' y)\gamma'' y = x\gamma(a\gamma' x\gamma a)\gamma' a\gamma'' y\gamma'' y \in S\Gamma B\Gamma Q\Gamma S$. Hence $B \cap Q \subseteq S\Gamma B\Gamma Q\Gamma S$.

Conversely, assume that for a bi-ideal B and a quasi-ideal Q of S, we have $B \cap Q \subseteq S\Gamma B\Gamma Q\Gamma S$. Let $a \in S$. Consider:

$$\begin{aligned} a &\in B(a) \cap Q(a) \\ &\subseteq S\Gamma B(a)\Gamma Q(a)\Gamma S \\ &= S\Gamma(a \cup a\Gamma a \cup a\Gamma S\Gamma a)\Gamma(a \cup (a\Gamma S \cap S\Gamma a))\Gamma S \\ &\subseteq (S\Gamma a \cup S\Gamma a\Gamma a \cup S\Gamma a\Gamma S\Gamma a)\Gamma(a \cup a\Gamma S)\Gamma S \\ &\subseteq S\Gamma a\Gamma(a \cup a\Gamma S)\Gamma S \\ &\subseteq S\Gamma a\Gamma(a\Gamma S \cup a\Gamma S\Gamma S) \\ &\subseteq S\Gamma a\Gamma a\Gamma S \cup S\Gamma a\Gamma a\Gamma S\Gamma S \\ &\subseteq S\Gamma a\Gamma a\Gamma S. \end{aligned}$$

This proves that S is intra-regular.

(2) Assume that S is intra-regular. Let B be a bi-ideal of S and Q a quasi-ideal of S. Let $a \in B \cap Q$. By assumption, $a = x\gamma a\gamma' a\gamma'' y$ for some $x, y \in S, \gamma, \gamma', \gamma'' \in \Gamma$. Since $a = x\gamma a\gamma' a\gamma'' y = x\gamma(x\gamma a\gamma' a\gamma'' y)\gamma' a\gamma'' y = x\gamma x\gamma a\gamma'(a\gamma'' y\gamma' a)\gamma'' y \in S\Gamma Q\Gamma B\Gamma S$, we have $B \cap Q \subseteq S\Gamma Q\Gamma B\Gamma S$.

Conversely, assume that for a bi-ideal B and a quasi-ideal Q of S, we have $B \cap Q \subseteq S\Gamma Q\Gamma B\Gamma S$. Let $a \in S$. Consider:

$$\begin{aligned} a &\in B(a) \cap Q(a) \\ &\subseteq S\Gamma Q(a)\Gamma B(a)\Gamma S \\ &= S\Gamma(a \cup (a\Gamma S \cap S\Gamma a))\Gamma(a \cup a\Gamma a \cup a\Gamma S\Gamma a)\Gamma S \\ &\subseteq S\Gamma(a \cup a\Gamma S)\Gamma(a\Gamma S \cup a\Gamma a\Gamma S \cup a\Gamma S\Gamma a\Gamma S) \\ &\subseteq (S\Gamma a \cup S\Gamma a\Gamma S)\Gamma(a\Gamma S \cup a\Gamma a\Gamma S \cup a\Gamma S\Gamma a\Gamma S) \\ &\subseteq (S\Gamma a \cup S\Gamma a\Gamma S)\Gamma(a\Gamma S) \\ &\subseteq (S\Gamma a)\Gamma(a\Gamma S). \end{aligned}$$

Therefore, S is intra-regular.

Theorem 2.2 *Let S be a Γ-semigroup. Then:*

(1) *S is intra-regular if and only if for a left ideal L and a bi-ideal B of S, we have $L \cap B \subseteq L\Gamma B\Gamma S$.*

(2) *S is intra-regular if and only if for a right ideal R and a bi-ideal B of S, we have $B \cap R \subseteq S\Gamma B\Gamma R$.*

Proof. (1) Assume that S is intra-regular. Let L be a left ideal of S and B a bi-ideal of S. Let $a \in L \cap B$. By assumption, $a = x\gamma a\gamma' a\gamma'' y$ for some $x, y \in S, \gamma, \gamma', \gamma'' \in \Gamma$. Since $a = (x\gamma a)\gamma' a\gamma'' y$, we have $a \in L\Gamma B\Gamma S$.

Conversely, for a left ideal L and a bi-ideal B of S, we have $L \cap B \subseteq L\Gamma B\Gamma S$. For $a \in S$, we have

$$\begin{aligned} a &\in L(a) \cap B(a) \\ &\subseteq L(a)\Gamma B(a)\Gamma S \\ &= (a \cup S\Gamma a)\Gamma(a \cup a\Gamma a \cup a\Gamma S\Gamma a)\Gamma S \\ &\subseteq (a \cup S\Gamma a)\Gamma(a\Gamma S \cup a\Gamma a\Gamma S \cup a\Gamma S\Gamma a\Gamma S) \\ &\subseteq (a \cup S\Gamma a)\Gamma(a\Gamma S) \\ &\subseteq a\Gamma a\Gamma S \cup S\Gamma a\Gamma a\Gamma S. \end{aligned}$$

If $a \in S\Gamma a\Gamma a\Gamma S$, then a is intra-regular. If $a \in a\Gamma a\Gamma S$, then $a = a\gamma a\gamma' x$ for some $x \in S, \gamma, \gamma' \in \Gamma$. Since $a = a\gamma a\gamma' x = a\gamma a\gamma a\gamma' x\gamma' x = a\gamma(a\gamma a)\gamma' x\gamma' x$, $a \in S\Gamma a\Gamma a\Gamma S$. Then a is intra-regular.

(2) Assume that S is intra-regular. Let R be a right ideal of S and B a bi-ideal of S. Let $a \in R \cap B$. By assumption, $a = x\gamma a\gamma' a\gamma'' y$ for some $x, y \in S, \gamma, \gamma', \gamma'' \in \Gamma$. Since $a = x\gamma a\gamma'(a\gamma'' y)$, we get $a \in S\Gamma B\Gamma R$.

Conversely, for a right ideal R and a bi-ideal B of S, we have $B \cap R \subseteq S\Gamma B\Gamma R$. For $a \in S$, we have

$$\begin{aligned} a &\in B(a) \cap R(a) \\ &\subseteq S\Gamma B(a)\Gamma R(a) \\ &= S\Gamma(a \cup a\Gamma a \cup a\Gamma S\Gamma a)\Gamma(a \cup a\Gamma S) \\ &= (S\Gamma a \cup S\Gamma a\Gamma a \cup S\Gamma a\Gamma S\Gamma a)\Gamma(a \cup a\Gamma S) \\ &\subseteq (S\Gamma a)\Gamma(a \cup a\Gamma S) \\ &\subseteq S\Gamma a\Gamma a \cup S\Gamma a\Gamma a\Gamma S. \end{aligned}$$

If $a \in S\Gamma a\Gamma a\Gamma S$, then a is intra-regular. If $a \in S\Gamma a\Gamma a$, then $a = x\gamma a\gamma' a$ for some $x \in S, \gamma, \gamma' \in \Gamma$. Since $a = x\gamma a\gamma' a = x\gamma x\gamma a\gamma' a\gamma' a = x\gamma x\gamma(a\gamma' a)\gamma' a$, $a \in S\Gamma a\Gamma a\Gamma S$. Then a is intra-regular.

Using Theorem 2.1 and Theorem 2.2, we obtain:

Theorem 2.3 *Let S be a Γ-semigroup. Then following are equivalent:*

(i) *S is intra-regular.*

(ii) *$B(a) \cap Q(a) \subseteq S\Gamma B(a)\Gamma Q(a)\Gamma S$.*

(iii) *$B(a) \cap Q(a) \subseteq S\Gamma Q(a)\Gamma B(a)\Gamma S$.*

(iv) $L(a) \cap B(a) \subseteq L(a)\Gamma B(a)\Gamma S$.

(v) $B(a) \cap R(a) \subseteq S\Gamma B(a)\Gamma R(a)$.

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