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On Generalized Doubly Stochastic Lumping Markov Chains

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Abstract

In this paper, the problem of aggregation of Markov chains has been considered. The necessary and sufficient conditions of the transition probability matrix of the original Markov chain to be a doubly stochastic matrix assuming that the transition probability matrix of the lumped chain was doubly stochastic, have been introduced for a general certain partitions within isomorphic. Also, special cases have been considered for each case, for each partition.

Keywords: Markov Chains – Aggregation – Identifiability Problem –Lumpability – Isomorphic – Doubly Stochastic Matrix.

1- Introduction

A.Moneim and F.Leysieffer (1982), completely solved the identifiability problem in a special case when $(m = 2\ ,\ n = 3)$ then A.B.Volidis (1985), completely solved the problem in a special case when $(m = 3 , n = 4)$, and for $(m = 2 , n = 4)$ with respect to the partition $A = \{\{1,2\}, \{3,4\}\}$ within isomorphic. U.Sumita and M.Rieders (1989), discussed the lumpability problem for large Markov Chain and introduced a new algorithm for large m and n. A.Moneim and A.A.El-sheikh (1996), generalized the solution of the identifiability problem for all partitions within isomorphic with respect to the cases $(m = k , n = k + 1)$ and $(m = k , n = k + 2)$ for both lumped and weakly lumped situations. Also, they generalized the solution when $(m = k , n = k + r)$ for a certain partitions for the lumped situation.

Ledoux and Leguesdron (2000) considered the question of whether a function of a finite state Markov Chain is also Markovian. They explored how an initial distribution with respect to which the chain is lumpable may differ from a pseudo-stationary initial distribution.

Rubino and Sericola (2004) analyzed under which conditions the aggregated process constructed from a homogeneous Markov Chain over a given partition of the state space is also Markov homogeneous.

Jacobi and Gornerup (2007) presented the necessary and sufficient conditions for identifying strong lumpability in Markov Chains. They showed that the states in a lump necessarily correspond to identical elements in eigenvectors of the dual transition matrix.

A.A.El-sheikh (2008), gave the necessary and sufficient conditions of the transition probability matrix(t.p.m) of the original Markov chain $X(t)$ to be a doubly stochastic matrix if t.p.m of the lumped chain $Y(t)$ was doubly stochastic for all (certain) partitions within isomorphic with respect to the cases $(k, k+1)$, $(k, k+2)$ and $(2, k)$.

Assume that $Y(t)$ is the result of an aggregation finite Markov Chain $X(t)$ of higher dimension under given partition $A = \{A_1, A_2, \ldots, A_m\}$ and unknown t.p.m P with t.p.m P^* which is doubly stochastic matrix (i.e. $\sum_k p_{ik} = \sum_k p_{kj} = 1, \quad p_{ij} \geq 0, \quad \forall i, j$). The t.p.m of the lumped chain $Y(t)$ will be,

$$P^* = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1k} \\ a_{21} & a_{22} & \dots & a_{2k} \\ \downarrow & \downarrow & \dots & \downarrow \\ a_{k1} & a_{k2} & \dots & a_{kk} \end{bmatrix}_{(k \times k)} \tag{1.1}$$

Where, $\sum_k a_{ik} = \sum_k a_{kj} = 1, \quad a_{ij} \geq 0, \quad \forall i, j$

J.Kemeny and J.Snell (1978), have given the sufficient condition for lumpability,

$$\ell(P, A) = P^* \Leftrightarrow WP^* = PW \ , \tag{1.2}$$

where P and P^* are defined as above and W is a matrix of order $(n \times m)$ with the j[th] column vector with 1's in the components corresponding to states in A_j and 0's otherwise.

So, in the next sections, the necessary and the sufficient conditions and the form of all t.p.m P where, $\ell(P, A) = P^*$ to be a doubly stochastic matrix for generalized certain partitions within isomorphic will be introduced.

2 − The Partition, $A = \{\{1\}, \{2\}, \dots, \{k, k+1, \dots, k+r\}\}$

The form of all transition matrices P of the Markov Chain $X(t)$ of order $(k+r) \times (k+r)$, where, $k \geq 2$ and $r \geq 1$, such that $\ell(P, A) = P^*$ will be,

$$P = \begin{bmatrix} a_{11} & a_{12} & \dots & \lambda_{1k} a_{1k} & \dots & \lambda_{1(k+r)} a_{1k} \\ a_{21} & a_{22} & \dots & \lambda_{2k} a_{2k} & \dots & \lambda_{2(k+r)} a_{2k} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ a_{k1} & a_{k2} & \dots & \lambda_{kk} a_{kk} & \dots & \lambda_{k(k+r)} a_{kk} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{k1} & a_{k2} & \dots & \lambda_{(k+r)k} a_{kk} & \dots & \lambda_{(k+r)(k+r)} a_{kk} \end{bmatrix}_{(k+r)\times(k+r)} , \qquad (2.1)$$

Where,

$$\sum_{j=k}^{k+r} \lambda_{ij} = 1, \quad \forall i = 1,2,\dots,k+r \ ,$$

Since, the t.p.m P^* is a double stochastic matrix, then after very simple steps the necessary and sufficient conditions for the t.p.m P to be a double stochastic matrix will be:

$$\left.\begin{array}{ll} \text{1- } \sum_s a_{is} = \sum_s a_{sj} = 1, & \forall i, j \ \& \ s = 1,2,\dots,k-1 \\ \text{2- } a_{ik} = a_{kj} = 0, & \forall i, j = 1,2,\dots,k-1 \\ \text{3- } \sum_t \lambda_{it} = \sum_t \lambda_{tj} = 1, & \forall i, j = k,\dots,k+r \end{array}\right\} \qquad (2.2)$$

Therefore, the form of all t.p.m P where P is a double stochastic matrix can be written in the form:

$$P = \begin{bmatrix} R & O \\ O^t & T \end{bmatrix} \qquad (2.3)$$

Where,

$$\left.\begin{array}{l}
1)\ R=\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1(k-1)} \\ a_{21} & a_{22} & \cdots & a_{2(k-1)} \\ \cdots & \cdots & \cdots & \cdots \\ a_{(k-1)1} & a_{(k-1)2} & \cdots & a_{(k-1)(k-1)} \end{bmatrix}_{(k-1)(k-1)} \\[2em]
2)\ O \text{ is the zero matrix of order } (k-1)(r+1) \\[2em]
3)\ T=\begin{bmatrix} \lambda_{kk} & \lambda_{k(k+1)} & \cdots & \lambda_{k(k+r)} \\ \lambda_{(k+1)k} & \lambda_{(k+1)(k+1)} & \cdots & \lambda_{(k+1)(k+r)} \\ \vdots & \vdots & \vdots & \vdots \\ \lambda_{(k+r)k} & \lambda_{(k+r)(k+1)} & \cdots & \lambda_{(k+r)(k+r)} \end{bmatrix}_{(r+1)(r+1)}
\end{array}\right\} \quad (2.4)$$

<u>*Special Cases*</u>

Let, $r=1(2)$ (with respect to the partition $A=\{\{1\},\{2\},\ldots,\{k,k+1,k+2\}\}$), then the results of A.A.El-Sheikh (2008) will be obtained.

3− The Partition, $A=\{\{1\},\{2\},\ldots,\{k-1,k\},\{k+1,\ldots,k+r\}\}$

The form of all t.p.m P of the original Markov Chain $X(t)$ of order $(k+r)\times(k+r)$, where, $k\geq 3$ and $r\geq 2$, such that $\ell(P,A)=P^{*}$ will be,

$$P=\begin{bmatrix} a_{11} & a_{12} & \cdots & \lambda_{1(k-1)}a_{1(k-1)} & \lambda_{1k}a_{1(k-1)} & \lambda_{1(k+1)}a_{1k}\rightarrow & \lambda_{1(k+r)}a_{1k} \\ a_{21} & a_{22} & \cdots & \lambda_{2(k-1)}a_{2(k-1)} & \lambda_{2k}a_{2(k-1)} & \lambda_{2(k+1)}a_{2k}\rightarrow & \lambda_{2(k+r)}a_{2k} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{(k-1)1} & a_{(k-1)2} & \cdots & \lambda_{(k-1)(k-1)}a_{(k-1)(k-1)} & \lambda_{(k-1)k}a_{(k-1)(k-1)} & \lambda_{(k-1)(k+1)}a_{(k-1)k}\rightarrow & \lambda_{(k-1)(k+r)}a_{(k-1)k} \\ a_{(k-1)1} & a_{(k-1)2} & \cdots & \lambda_{k(k-1)}a_{(k-1)(k-1)} & \lambda_{kk}a_{(k-1)(k-1)} & \lambda_{k(k+1)}a_{(k-1)k}\rightarrow & \lambda_{k(k+r)}a_{(k-1)k} \\ a_{k1} & a_{k2} & \cdots & \lambda_{(k+1)(k-1)}a_{k(k-1)} & \lambda_{(k+1)k}a_{k(k-1)} & \lambda_{(k+1)(k+1)}a_{kk}\rightarrow & \lambda_{(k+1)(k+r)}a_{kk} \\ \downarrow & \downarrow & & \downarrow & \downarrow & \downarrow & \downarrow \\ a_{k1} & a_{k2} & \cdots & \lambda_{(k+r)(k-1)}a_{k(k-1)} & \lambda_{(k+r)k}a_{k(k-1)} & \lambda_{(k+r)(k+1)}a_{kk}\rightarrow & \lambda_{(k+r)(k+r)}a_{kk} \end{bmatrix}$$

Where,

$$\sum_{j=k-1}^{k}\lambda_{ij} = \sum_{j=k+1}^{k+r}\lambda_{ij} = 1, \qquad \forall i = 1,2,\ldots,k+r \tag{3.1}$$

The necessary and sufficient conditions for the t.p.m P to be a double stochastic matrix with respect to the partition A within isomorphic will be:

$$\left.\begin{array}{l} 1\text{- } \sum_{s} a_{is} = \sum_{s} a_{sj} = 1, \quad \forall i, j \ \& \ s = 1,2,\ldots,k-2 \\ 2\text{- } a_{ij} = a_{ji} = 0, \quad \forall i = 1,2,\ldots,k-2 \ \ \& \ \ \forall j = k-1, k \\ 3\text{- } \sum_{i=k-1}^{k}\lambda_{ij} = \sum_{i=k+1}^{k+r}\lambda_{ij} = 1, \quad \forall j = k-1, k, k+1, \ldots, k+r \end{array}\right\} \tag{3.2}$$

Therefore, the form of all t.p.m P where P is a double stochastic matrix can be written in the form:

$$P = \begin{bmatrix} R & O \\ O^t & T \end{bmatrix}$$

Where,

$$\left.\begin{array}{l}
1)\; R=\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1(k-2)} \\ a_{21} & a_{22} & \cdots & a_{2(k-2)} \\ \cdots & \cdots & \cdots & \cdots \\ a_{(k-2)1} & a_{(k-2)2} & \cdots & a_{(k-2)(k-2)} \end{bmatrix}_{(k-2)(k-2)} \\
2)\; O \text{ is the zero matrix of order } (k-2)\times(r+2) \\
3)\; T=\begin{bmatrix} \lambda_{(k-1)(k-1)}a_{(k-1)(k-1)} & \lambda_{(k-1)k}a_{(k-1)(k-1)} & \cdots & \lambda_{(k-1)(k+r)}a_{(k-1)k} \\ \lambda_{k(k-1)}a_{(k-1)(k-1)} & \lambda_{kk}a_{(k-1)(k-1)} & \cdots & \lambda_{k(k+r)}a_{(k-1)k} \\ \lambda_{(k+1)(k-1)}a_{k(k-1)} & \lambda_{(k+1)k}a_{k(k-1)} & \cdots & \lambda_{(k+1)(k+r)}a_{kk} \\ \downarrow & \downarrow & \downarrow & \downarrow \\ \lambda_{(k+r)(k-1)}a_{k(k-1)} & \lambda_{(k+r)k}a_{k(k-1)} & \lambda_{(k+r)(k+1)}a_{kk} & \lambda_{(k+r)(k+r)}a_{kk} \end{bmatrix}_{(r+2)(r+2)}
\end{array}\right\} \quad (3.3)$$

<u>*Special Case*</u>

Let, $r=2$ ($A_2=\{\{1\},\{2\},\ldots,\{k-1,k\},\{k+1,k+2\}\}$, $k\geq 3$) then the results of A.A.El-Sheikh (2008) will be obtained.

4− The Partition, $A=\{\{1,2\},\{3,4\},\ldots,\{2k-1,2k,2k+1\ldots,2k+r\}\}$

In this section, the form of all transition matrices P of the Markov Chain $X(t)$ of order $(2k+r)\times(2k+r)$, $k\geq 2$ and $r\geq 0$, where $\ell(P,A)=P^*$ will be,

$$P=\begin{bmatrix} \Re_{(k-1)(k-1)} & W_{(k-1)(1)} \\ V_{(1)(k-1)} & Z_{(1\times 1)} \end{bmatrix}_{(2k+r)\times(2k+r)}, \qquad (4.1)$$

Where,

$$\left.\begin{array}{lll}
1. & \Re = a_{ij} \otimes B_{ij}, & i, j = 1,2,\dots,k-1 \\
 & B_{ij}, \text{ is a matrix of order } (2\times 2), & \forall i, j = 1,2,\dots,k-1 \\
2. & W = a_{ik} \otimes B_{ik}, & i = 1,2,\dots,k-1 \\
 & B_{ik}, \text{ is a matrix of order } (2\times (r+2)) & \\
3. & V = a_{kj} \otimes B_{kj}, & j = 1,2,\dots,k-1 \\
 & B_{kj}, \text{ is a matrix of order } ((r+2)\times 2) & \\
4. & Z = a_{kk} \otimes B_{(r+2)(r+2)}, & r = 0,1,2,\dots
\end{array}\right\} \quad (4.2)$$

The necessary and sufficient conditions for the t.p.m P to be a double stochastic matrix with respect to the partition A within isomorphic will be:

1. B_{ij}, is a double stochastic matrix (From (1) , 4.2)

2. $B_{ik} = \left[B \middle| \mathrm{O}\right]$, (From (2) , 4.2)

$B_{(2\times 2)}$ is a double stochastic matrix and $\mathrm{O}_{(2\times r)}$ is a zero matrix

3. $B_{kj} = \begin{bmatrix} H \\ \hline \mathrm{O} \end{bmatrix}$, (From (3) , 4.2)

$H_{(2\times 2)}$ is a double stochastic matrix and $\mathrm{O}_{(r\times 2)}$ is a zero matrix

4. $B_{(r+2)(r+2)}$, is a double stochastic matrix (From (4) , 4.2)

4.1-Special Case: $r = 0, \ (A = \{\{1,2\},\{3,4\},\dots,\{2k-1,2k\}\})$

The form of all transition matrices P of the Markov Chain $X(t)$ of order $(2k)\times(2k)$, $k \ge 2$ where $\ell(P, A) = P^{*}$ will be,

$$P=\begin{bmatrix} \lambda_{11}a_{11} & \lambda_{12}a_{11} & \lambda_{13}a_{12} & \lambda_{14}a_{12} & \cdots & \lambda_{1(2k-1)}a_{1k} & \lambda_{1(2k)}a_{1k} \\ \lambda_{21}a_{11} & \lambda_{22}a_{11} & \lambda_{23}a_{12} & \lambda_{24}a_{12} & \cdots & \lambda_{2(2k-1)}a_{1k} & \lambda_{2(2k)}a_{1k} \\ \lambda_{31}a_{21} & \lambda_{32}a_{21} & \lambda_{33}a_{22} & \lambda_{34}a_{22} & \cdots & \lambda_{3(2k-1)}a_{2k} & \lambda_{3(2k)}a_{2k} \\ \lambda_{41}a_{21} & \lambda_{42}a_{21} & \lambda_{43}a_{22} & \lambda_{44}a_{22} & \cdots & \lambda_{4(2k-1)}a_{2k} & \lambda_{4(2k)}a_{2k} \\ \downarrow & \downarrow & \downarrow & \downarrow & \cdots & \downarrow & \downarrow \\ \lambda_{(2k-1)1}a_{k1} & \lambda_{(2k-1)2}a_{k1} & \lambda_{(2k-1)3}a_{k2} & \lambda_{(2k-1)4}a_{k2} & \cdots & \lambda_{(2k-1)(2k-1)}a_{kk} & \lambda_{(2k-1)2k}a_{kk} \\ \lambda_{(2k)1}a_{k1} & \lambda_{(2k)2}a_{k1} & \lambda_{(2k)3}a_{k2} & \lambda_{(2k)4}a_{k2} & \cdots & \lambda_{(2k)(2k-1)}a_{kk} & \lambda_{(2k)(2k)}a_{kk} \end{bmatrix}_{(2k\times 2k)}$$

Where,

$$\sum_{j=1}^{2}\lambda_{ij} = \sum_{j=3}^{4}\lambda_{ij} = \ldots = \sum_{j=2k-1}^{2k}\lambda_{ij} = 1, \qquad \forall i = 1,2,\ldots,2k \qquad (4.3)$$

The necessary and sufficient conditions for the t.p.m P to be a double stochastic matrix with respect to the partition A within isomorphic will be:

$$\sum_{i=1}^{2}\lambda_{ij} = \sum_{i=3}^{4}\lambda_{ij} = \ldots = \sum_{i=2k-1}^{2k}\lambda_{ij} = 1, \quad \forall j = 1,2,\ldots,2k \qquad (4.4)$$

Therefore, the form of all t.p.m P where P is a double stochastic matrix can be written in the form:

$$P=\begin{bmatrix} a_{11}\otimes B_{11} & a_{12}\otimes B_{12} & \cdots & a_{1k}\otimes B_{1k} \\ \downarrow & \downarrow & \cdots & \downarrow \\ a_{k1}\otimes B_{k1} & a_{k2}\otimes B_{k2} & \cdots & a_{kk}\otimes B_{kk} \end{bmatrix}_{(2k\times 2k)}$$

Where, B_{ij}, is a double stochastic matrix of order (2×2), $\forall i, j = 1,2,\ldots,2k$

5. Summary

In this paper, the transition probability matrix of the lumped Markov Chain $Y(t)$ has been concerned assuming that it was a double stochastic matrix.

The form of all transition matrices P of the original Markov chain $X(t)$ according to partition A within isomorphic has been obtained where $\ell(P, A) = P^*$.

The necessary and the sufficient conditions such that P will be again a double stochastic matrix have been obtained for each partition within isomorphic. These conditions depend mainly on two aspects:

- The probabilities of the transition probability matrix P^* of the lumped Markov Chain $Y(t)$.
- The weights ($\lambda's$) of the transition probability matrix P of the original Markov Chain $X(t)$.
- The partition that $Y(t)$ was lumpable with respect to.

The transition probability matrix P of $X(t)$ has been represented by a partition double stochastic matrices with different dimensions according to the partition of the states within isomorphic.

A future work plan is to discuss the necessary and the sufficient conditions of the transition probability matrix of $X(t)$ to be a doubly stochastic matrix if the transition probability matrix of the weakly lumped Markov chain was doubly stochastic matrix.

References

1- Abdel Moneim,A.M. and Leysieffer,F.w. " Weak Lumpability In Finite Markov Chains". J.App Probability, 19 (1982), 685-691.

2- Abdel Moneim, A.M. and El-Sheikh, A.A."On Combining States of Markov Chains". The Journal of the Egyptian Statistical Society, Vol 12, No I, (1996), 37-46.

3- El-Sheikh, A.A. "On Double Stochastic Lumped Markov Chains", The Egyptian Statistical Journal, Egypt, (2008).

4- Jacobi, M.N. and Gornerup.O. " A Dual Eigenvector Condition for Strong Lumpability of Markov Chains". Http: //arxiv .org/ /PS_cache/arxiv/pdf/0710/0710.1986v1.pdf, 2007.

5- Kemeny, J.G. and Snell, J.L. "Finite Markov Chains". Springer- Verlag, New York, viii +210, 1978.

6- Ledoux, J.A. and Leguesdron, P. "Weak Lumpability and Pseudo-Stationary of Finite Markov Chains". Commun.Statist.-Stochastic Models, 16(1) (2000), 49-67.

7- Rubino,G. and Sericola,B. x, " On Weak Lumpability in Markov Chains". http:// www. Irisa .fr/ armor / lesmembres / Rubino/Para%20ev/Markov/weakLump.pdf, 2000.

8- Sumita,U. and Rieders,M."Lumpability and Time Reversibility in the Aggregation-Disaggregation Method for Large Markov Chains". Commun-Statis-Stochastic Models, 5(1) (1989), 63-81.

9- Volidis,A.B."On the Properties of Functions of Markov Chains". An M.Sc.Thesis at Institute of Statistical Studies and Research, Cairo University, 1985.

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